Vector Spaces of Generalized Euler Integrals

joint with Daniele Agostini, Anna-Laura Sattelberger, and Simon Telen

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MAX-PLANCK-GESELLSCHAFT

Generalized Euler integrals [GKZ]:

$$\int_\Gamma f^{s+a}\, x^{\nu+b}\, \frac{\mathrm{d}x}{x} \; = \; \int_\Gamma \left(\prod_{j=1}^{\ell} f_j^{s_j+a_j} \right) \cdot \left(\prod_{i=1}^{n} x_i^{\nu_i+b_i} \right) \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

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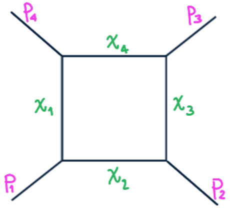
Denote

$$X \;:= \{\, x \in (\mathbb{C}^*)^n \mid f_1(x) \cdots f_\ell(x) \neq 0 \,\} \;=\; (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell) \;\subset\; (\mathbb{C}^*)^n$$

Motivation: Feynman integrals in Particle Physics



external edges

internal edges

loops

$$n = 4,\ E = 4,\ L = 1$$

$$p_i \in \mathbb{R}^{1,D-1}, \quad p \cdot q = p^{(0)}q^{(0)} - p^{(1)}q^{(1)} - \cdots - p^{(D-1)}q^{(D-1)}$$

$$x_e \in \mathbb{C}^*, m_e \in \mathbb{R}_{\geq 0}, \quad e = 1, \ldots, E$$

$$\mathcal{U} = x_1 + x_2 + x_3 + x_4$$

$$\mathcal{F} = p_1^2 x_1 x_2 + p_2^2 x_2 x_3 + p_3^2 x_3 x_4 + p_4^2 x_1 x_4 + (p_1 + p_2)^2 x_1 x_3 + (p_2 + p_3)^2 x_2 x_4 \\ - (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4)\mathcal{U}$$

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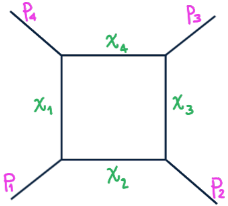

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Generalized Euler Integrals arise as Feynman Integrals in the Lee–Pomeransky representation:

$$I_\nu = \frac{\Gamma(d/2)}{\Gamma\left((L+1)d/2 - \sum_{i=1}^n \nu_i\right) \prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty x^\nu \mathcal{G}^{-d/2} \frac{\mathrm{d}x}{x},$$

where $\nu \in \mathbb{C}^n$, $d = E - LD/2$, and $\mathcal{G} = \mathcal{F} + \mathcal{U}$

Vector Spaces of Generalized Euler Integrals

$$V_{\Gamma} \ := \ \mathrm{Span}_{\mathbb{C}} \left\{ [\Gamma] \longmapsto \int_{\Gamma} f^{s+a}\, x^{\nu+b}\, \frac{\mathrm{d}x}{x} \right\}_{(a,b)\,\in\,\mathbb{Z}^{\ell}\times\mathbb{Z}^{n}}$$

Mizera,Mastrolia

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Bitoun, Bogner, Klausen, Panzer

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$$V_{c^*} \ := \ \mathrm{Span}_{\mathbb{C}} \left\{ c \longmapsto \int_{\Gamma} f(x;c)^{s} \, x^{\nu} \, \frac{\mathrm{d}x}{x} \right\}_{[\Gamma] \in H_n(X,\omega)}$$

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$s \in \mathbb{C}^{\ell}, \nu \in \mathbb{C}^{n}$ generic

Matsubara-Heo, Chestnov et al.

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Twisted de Rham
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Differential and difference
operators

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GKZ systems

Theorem

Let $X = (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell) \subset (\mathbb{C}^*)^n$, *where* f_j *are Laurent polynomials with fixed monomial supports and generic coefficients. Consider* $V_\Gamma, V_{s,\nu}, V_{c^*}$ *with generic choices of parameters each. Then*

$$\dim_{\mathbb{C}}(V_\Gamma) = \dim_{\mathbb{C}(s,\nu)}(V_{s,\nu}) = \dim_{\mathbb{C}}(V_{c^*}) = (-1)^n \cdot \chi(X),$$

where $\chi(X)$ *denotes the topological Euler characteristic of* X.

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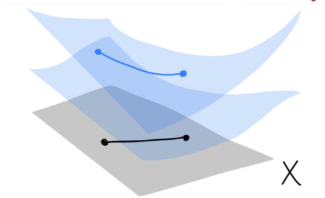
3. One slide on Numerical Nonlinear Algebra

1. Twisted deRham Cohomology

$$I_{a,b}(\Gamma) := \int_\Gamma f^{s+a}\, x^{\nu+b}\, \frac{\mathrm{d}x}{x} = \int_\Gamma f^s\, x^\nu \cdot f^a x^b\, \frac{\mathrm{d}x}{x}$$

CHOOSE A BRANCH



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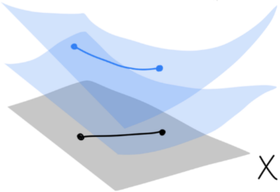

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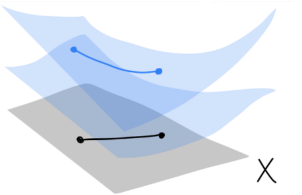

- $\Gamma = \Delta \otimes \phi \in C_n(X, \mathcal{L}_\omega^\vee)$
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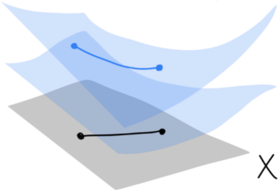

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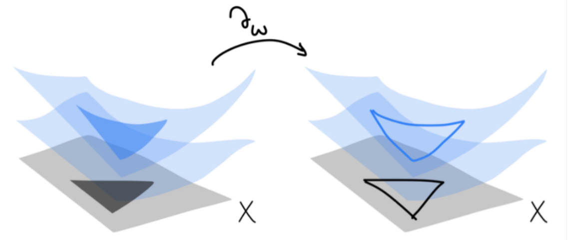
- $(d - \omega\wedge)\phi = 0$ solution sheaf $\mathcal{L}_\omega^\vee$

Twisted boundaries and chain complexes

$$\partial_\omega(\Delta \otimes_{\mathbb{C}} \phi) = \partial\Delta \otimes_{\mathbb{C}} \phi$$



$\partial_\omega : C_k(X, \mathcal{L}_\omega^\vee) \to C_{k-1}(X, \mathcal{L}_\omega^\vee)$ Kernel: twisted k-cycles

$H_k(X, \mathcal{L}_\omega^\vee) = \ker \partial_\omega^k / \operatorname{im} \partial_\omega^{k+1}$ k-th twisted homology

$\parallel$

$H_k(X, \omega)$

$$V_\Gamma := \operatorname{Span}_{\mathbb{C}} \left\{ [\Gamma] \longmapsto \int_\Gamma f^{s+a}\, x^{\nu+b}\, \frac{\mathrm{d}x}{x} \right\}_{(a,b)\,\in\,\mathbb{Z}^\ell \times \mathbb{Z}^n} \subseteq \operatorname{Hom}_{\mathbb{C}}(\, H_n(X,\omega)\, ,\, \mathbb{C}\,).$$

Twisted cohomology

- Regular functions $\Omega^0_X(X) = \sum_{a,b} \mathbb{C} \cdot f^a x^b$
- Regular k-forms $\Omega^k_X(X) = \sum_{i_1<\cdots<i_k} \Omega^0_X(X) \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$

$$(\Omega^\bullet_X(X), \nabla_\omega) : \ 0 \longrightarrow \Omega^0_X(X) \xrightarrow{\nabla_\omega} \Omega^1_X(X) \xrightarrow{\nabla_\omega} \cdots \xrightarrow{\nabla_\omega} \Omega^n_X(X) \xrightarrow{\nabla_\omega} 0,$$

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Twisted de Rham cohomology:

$$H^k(X,\omega) := \ker\,(\Omega^k_X(X) \xrightarrow{\nabla_\omega} \Omega^{k+1}_X(X)) \,/\, \mathrm{im}\,(\Omega^{k-1}_X(X) \xrightarrow{\nabla_\omega} \Omega^k_X(X)).$$

$$\langle [\phi], [\Gamma] \rangle := \int_\Gamma f^s x^\nu \phi$$

Proposition

The $\mathbb{C}$-vector spaces V_Γ, $H^n(X, \omega)$, and $H^n_{\mathrm{dR}}(X, \omega)$ are isomorphic.

algebraic analytic

Vanishing Theorem of twisted de Rham cohomology (Matsubara-Heo):

Fix $f \in \mathbb{C}[x, x^{-1}]^\ell$ and let $X = (\mathbb{C}^*)^n \setminus V(f_1, \ldots, f_\ell)$. For general $(s, \nu) \in \mathbb{C}^{\ell+n}$, we have that

$$H^\star(X, \omega) = 0 \qquad \text{whenever } \star \neq n.$$

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Theorem

The Euler characteristic $\chi(X)$ equals the alternating sum

$$\sum_{k=0}^{2n} (-1)^k \dim_{\mathbb{C}} H^k(X,\omega).$$

Corollary

$\dim_{\mathbb{C}} V_\Gamma = \dim_{\mathbb{C}} H^n(X,\omega) = |\chi(X)|.$

Relations in cohomology

$$H^n(X,\omega) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\, H_n(X,\omega), \mathbb{C}\,), \qquad \left[f^a x^b \frac{\mathrm{d}x}{x}\right] \mapsto I_{a,b}$$

For all complex constants $C_{a,b}$

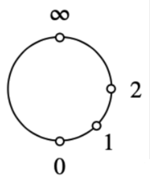
$$\begin{aligned}\sum_{a,b} C_{a,b} \cdot I_{a,b} \,=\, 0 \text{ in } V_\Gamma &\iff \left[\sum_{a,b} C_{a,b} \cdot f^a x^b \,\frac{\mathrm{d}x}{x}\right] \,=\, 0 \text{ in } H^n(X,\omega) \\ &\iff \sum_{a,b} C_{a,b} \cdot f^a x^b \,\frac{\mathrm{d}x}{x} \,\in\, \mathrm{im}\,(\nabla_\omega)\,.\end{aligned}$$

Hence for any $\phi \in \Omega_X^{n-1}(X)$, we find linear relations between the generators $I_{a,b}$ by expanding

$$\nabla_\omega(\phi) \,=\, \sum_{a,b} C_{a,b}(\phi) \cdot f^a\, x^b \,\frac{\mathrm{d}x}{x}.$$

Example $(n = 1, \ell = 2)$

Let $f = (x-1, x-2) \in \mathbb{C}[x, x^{-1}]^2$, so $X = \mathbb{P}^1 \setminus \{0, 1, 2, \infty\}$.



$$I_{a,b}(\Gamma) \;=\; \int_\Gamma \sqrt{(x-1)(x-2)x}\;\frac{\mathrm{d}x}{x}$$

Take $s = \left(\frac{1}{2}, \frac{1}{2}\right), \nu = \frac{1}{2}$ and set $\phi = 1$:

$$\nabla_\omega(1) \;=\; \frac{1}{2} f_1^{-1} f_2^0 x^1 \,\frac{\mathrm{d}x}{x} \;+\; \frac{1}{2} f_1^0 f_2^{-1} x^1 \,\frac{\mathrm{d}x}{x} \;+\; \frac{1}{2} f_1^0 f_2^0 x^0 \,\frac{\mathrm{d}x}{x}.$$

Hence, $C_{(-1,0),1}(1) = C_{(0,-1),1}(1) = C_{(0,0),0}(1) = \frac{1}{2}$.

For every choice of the twisted cycle Γ, we have

$$\int_\Gamma \frac{\sqrt{(x-1)(x-2)x}}{x-1}\,\mathrm{d}x + \int_\Gamma \frac{\sqrt{(x-1)(x-2)x}}{x-2}\,\mathrm{d}x + \int_\Gamma \frac{\sqrt{(x-1)(x-2)x}}{x}\,\mathrm{d}x \;=\; 0.$$

2. Mellin transform

Let $f \in \mathbb{C}[x_1, \ldots, x_n]^\ell$, fix $s \in \mathbb{C}^\ell$

$$\mathfrak{M}\{f^s\}(\nu) := \int_\Gamma f^s \, x^\nu \, \frac{\mathrm{d}x}{x},$$

where $\Gamma := \Gamma(s, \nu) \in H_n(X, \omega(s, \nu))$.

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$$\boxed{\mathfrak{M}\{f^{s+a} x^{b}\}(\nu) = \int_{\Gamma} f^{s+a} x^{\nu+b} \frac{dx}{x}}$$

Properties:

① $\mathfrak{M}\{x_i \cdot f^s\}(\nu) = \mathfrak{M}\{f^s\}(\nu + e_i)$,

② $\mathfrak{M}\left\{x_i \cdot \frac{\partial f^s}{\partial x_i}\right\}(\nu) = -\nu_i \cdot \mathfrak{M}\{f^s\}(\nu)$.

$$D_n := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle,$$

n-th Weyl Algebra

- $[\partial_i, x_i] := \partial_i x_i - x_i \partial_i = 1 \neq 0$, where $i = 1, \ldots, n$.
- $\partial_i \cdot x_i = x_i \cdot \partial_i + 1 \in D_n$, whereas $\partial_i \bullet x_i = 1 \in \mathbb{C}[x_1, \ldots, x_n]$.

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$$S_n := \mathbb{C}[\nu_1, \ldots, \nu_n]\langle \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1} \rangle$$

n-th Shift Algebra

- $\sigma_i^{\pm 1} \nu_i = (\nu_i \pm 1)\sigma_i^{\pm 1}$, where $i = 1 \ldots, n$.

Algebraic Mellin transform

$$\begin{aligned} \mathfrak{M}\{\cdot\}\colon\ D_{\mathbb{G}_m^n}[s_1,\ldots,s_\ell] &\longrightarrow S_n[s_1,\ldots,s_\ell] \\ x_i^{\pm 1} &\longmapsto \sigma_i^{\pm 1} \\ \theta_i &\longmapsto -\nu_i \\ s_i &\longmapsto s_i \end{aligned}$$

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Example ($n = \ell = 1$):

Let $P = x^2\partial_x - 3x\partial_x + 2\partial_x - 2xs + 3s \in \mathbb{C}[x, x^{-1}, s]\langle\partial_x\rangle$,

$$\begin{aligned} \mathfrak{M}\{P\} &= -\nu\sigma + 3\nu - 2\sigma^{-1}\nu - 2s\sigma + 3s \\ &= -(\nu+1)\sigma + 3\nu - 2(\nu-1)\sigma^{-1} - 2s\sigma + 3s \end{aligned}$$

$\sigma^{\pm 1}\nu = (\nu \pm 1)\sigma^{\pm 1}$

Computing linear relations: $\ell = 1$

Let $f \in \mathbb{C}[x_1, \dots, x_n]$, the **Bernstein–Sato polynomial** of f is the unique monic polynomial $b_f \in \mathbb{C}[s]$ of smallest degree for which there exists $P_f \in D_n[s]$ such that

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lowering in s

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$$\mathfrak{M}\{f^{s+1}\} = \mathfrak{M}\{f \cdot f^s\} = \underbrace{\mathfrak{M}\{f\}}_{\in S_n} \bullet \mathfrak{M}\{f^s\}$$

raising in s

Remark

It is enough to consider just shifts in the vector $b \in \mathbb{Z}^n$.

Example $(n = \ell = 1)$

Let $f = (x-1)(x-2) \in \mathbb{C}[x]$

$$\mathrm{Ann}_{D_n[s]}(f^s) := \{P \in D_n[s] \mid P \bullet f^s = 0\}$$

We can compute it using the library `dmod_lib` in `Singular`:

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LIB "dmod.lib";
ring r = 0,x,dp; poly f = (x-1)*(x-2);
def A = operatorBM(f); setring A;
LD; // s-parametric annihilator of f^s
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$$\mathrm{Ann}_{D_n[s]}(f^s) = \langle\, f\partial_x - s\partial_x \bullet f \,\rangle$$

A linear relation in $V_{s,\nu}$ among Euler integrals can be attained by expanding

$$\mathfrak{M}\{f\partial_x - s\partial_x \bullet f\} \bullet \mathfrak{M}\{f^s\} = (-(\nu+1)\sigma + 3\nu - 2(\nu-1)\sigma^{-1} - 2s\sigma + 3s)) \bullet \int_\Gamma f^s\, x^\nu\, \frac{\mathrm{d}x}{x}$$

Theorem [BBKP19]:

When $\ell = 1$, $\Gamma = \mathbb{R}_{\geq 0}$, the dimension of $V_{s,\nu}$ is given be the signed topological Euler characteristic of the hypersurface complement $(\mathbb{C}^*)^n \setminus V(f)$, i.e.,

$$\dim_{\mathbb{C}(s,\nu)}(V_{s,\nu}) \;=\; (-1)^n \cdot \chi\left((\mathbb{C}^*)^n \setminus V(f)\right).$$

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Definition

Let $\ell > 1$ and $a \in \mathbb{N}^\ell$. The a-**Bernstein–Sato ideal** of $(f_1, \ldots, f_\ell)$ is the ideal $B^a_{(f_1,\ldots,f_\ell)} \lhd \mathbb{C}[s_1, \ldots, s_\ell]$ consisting of all polynomials $p \in \mathbb{C}[s_1, \ldots, s_\ell]$ for which there exists $P \in D_n[s_1, \ldots, s_\ell]$ such that

$$P \bullet \left(f_1^{s_1+a_1} \cdots f_\ell^{s_\ell+a_\ell}\right) \;=\; p \cdot f_1^{s_1} \cdots f_\ell^{s_\ell}.$$

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Thorem [AFST22]

Let $f \in \mathbb{C}[x_1, \ldots, x_n]^\ell$. Then $\dim_{\mathbb{C}(s,\nu)}(V_{s,\nu}) \; = \; (-1)^n \cdot \chi\left((\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell)\right)$.

(Co)homology and Mellin transform

Proposition [AFST22]:

Let $\ell = 1$ and consider a differential operator $P \in \operatorname{Ann}_{D_n[s]}(f^s)$ which is of degree at most 1 in $\partial_1, \ldots, \partial_n$, i.e.,

$$P = \sum_{i=1}^{n} p_i(x,s) \cdot \partial_i + q(x,s), \quad \text{where } p_1, \ldots, p_n, q \in \mathbb{C}[x_1, \ldots, x_n, s].$$

Then the equalities $\mathfrak{M}\{P\} \bullet \mathfrak{M}\{f^s\} = 0$ and $\nabla_\omega(\phi) = 0$ in $H^n(X, \omega)$ with

$$\phi = \sum_{i=1}^{n} (-1)^{i-1} \frac{p_i}{x_i} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_{i-1}}{x_{i-1}} \wedge \frac{\mathrm{d}x_{i+1}}{x_{i+1}} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

lead to the same linear relation of integrals $I_{a,b}$.

Computing Euler Characteristics

Theorem (Huh):

The Euler characteristic $\chi(X)$ equals the number of complex critical points of

$$L = \log(f^s x^\nu) = \sum_{i=1}^{\ell} s_j \log f_j + \sum_{i=1}^{n} \nu_i \log x_i$$

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for general s, ν.

Solving rational function equations:

```
using HomotopyContinuation
@var x y s ν[1:2]
f = -x*y^2 + 2*x*y^3 + 3*x^2*y - x^2*y^3 - 2*x^3*y + 3*x^3*y^2
L = s*log(f) + ν[1]*log(x) + ν[2]*log(y)
F = System(differentiate(L,[x;y]), parameters = [s;ν])
```

MATHEMATICAL
SOFTWARE
REPOSITORY

MAX PLANCK INSTITUTE
FOR MATHEMATICS
IN THE SCIENCES

https://mathrepo.mis.mpg.de/EulerIntegrals

Thank you!

Theorem

Let $X = (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell) \subset (\mathbb{C}^*)^n$, *where* f_j *are Laurent polynomials with fixed monomial supports and generic coefficients. Consider* $V_\Gamma, V_{s,\nu}, V_{c^*}, H_A(\kappa)$ *with generic choices of parameters each. Then*

$$\dim_{\mathbb{C}}(V_\Gamma) = \dim_{\mathbb{C}(s,\nu)}(V_{s,\nu}) = \dim_{\mathbb{C}}(V_{c^*}) = \dim_{\mathbb{C}(c)}(R_A/(R_A \cdot H_A(\kappa))) = (-1)^n \cdot \chi(X),$$

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Outline

1. Twisted (co)homology;
2. Differential operators and Mellin transform;

2. GKZ systems

Let $\nu \in \mathbb{C}^n$, $s \in \mathbb{C}^\ell$, and the integers $a = b = 0$ to be fixed,

$$V_{c^*} := \operatorname{Span}_{\mathbb{C}} \left\{ c \longmapsto \int_\Gamma f(x;c)^s \, x^\nu \, \frac{\mathrm{d}x}{x} \right\}_{[\Gamma] \in H_n(X,\omega)}$$

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We fix finite subsets $\{A_j\}_{j=1,\ldots,\ell}$ of $\mathbb{Z}^n$ representing the monomial supports of the f_j:

$$f_j(x;c_j) \;=\; \sum_{\alpha \in A_j} c_{\alpha,j} \, x^\alpha .$$

$$A = \left(\begin{array}{ccc|ccc|c|ccc} & A_1 & & & A_2 & & & & A_\ell & \\ \\ 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \end{array}\right)$$

2. GKZ systems

Consider the Weyl algebra $D_A = \mathbb{C}[c_\alpha \,|\, \alpha \in A]\langle \partial_\alpha \,|\, \alpha \in A \rangle$.

Definition

$$I_A := \langle \partial^u - \partial^v \,|\, u - v \in \ker(A),\ u, v \in \mathbb{N}^A \rangle \lhd \mathbb{C}[\partial_\alpha \,|\, \alpha \in A].$$

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Let $\kappa = (-\nu, s)^\top \in \mathbb{C}^{n+\ell}$,

$$J_{A,\kappa} = \langle\, (A\theta - \kappa)_i,\; i = 1, \ldots, n + \ell \,\rangle,$$

where $\theta := (\theta_\alpha)_{\alpha \in A}$ and $\theta_\alpha = c_\alpha \partial_\alpha$.

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$$H_A(k) \;:=\; I_A + J_{A,\kappa} \;\lhd\; D_A$$

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Remark

The Newton polytope of the polynomial

$$h \;:=\; \sum_{j=1}^{\ell} x_{n+j}\, f_j(x, c_j) \;\in\; \mathbb{C}[x_1, \ldots x_{n+\ell}, x_1^{-1}, \ldots, x_{n+\ell}^{-1}]$$

coincides with the convex hull of the columns of A.

Example ($n = 2, \ell = 1$):

We consider the polynomial $f = -xy^2 + 2xy^3 + 3x^2y - x^2y^3 - 2x^3y + 3x^3y^2 \in \mathbb{C}[x, y]$, but replace its coefficients by indeterminates $c_1, \ldots, c_6$:

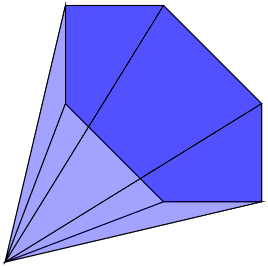
$$f = c_1xy^2 + c_2xy^3 + c_3x^2y + c_4x^2y^3 + c_5x^3y + c_6x^3y^2$$

Then

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$\mathrm{vol}(\mathrm{NP}(h)) = \mathrm{vol}(A) = 6,$$

where $h = z \cdot f$



Example ($n = 2, \ell = 1$):

Using the `Macaulay2` package `Dmodules`, one computes that the toric ideal I_A is generated by

$$\begin{aligned} I_A \ = \ &\langle \partial_2\partial_5 - \partial_1\partial_6, \partial_3\partial_4 - \partial_1\partial_6,\ \partial_4\partial_5^2 - \partial_3\partial_6^2,\ \partial_1\partial_5^2 - \partial_3^2\partial_6, \partial_4^2\partial_5 - \partial_2\partial_6^2, \\ &\partial_1\partial_4\partial_5 - \partial_2\partial_3\partial_6,\ \partial_1\partial_4^2 - \partial_2^2\partial_6,\ \partial_2\partial_3^2 - \partial_1^2\partial_5, \partial_2^2\partial_3 - \partial_1^2\partial_4\rangle. \end{aligned}$$

The ideal $J_{A,\kappa}$ is generated by the 3 operators

$$\theta_1+\theta_2+2\theta_3+2\theta_4+3\theta_5+3\theta_6+\nu_1,\ 2\theta_1+3\theta_2+\theta_3+3\theta_4+\theta_5+2\theta_6+\nu_2,\ \theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6-s.$$

Together, these 12 operators generate $H_A(\kappa)$.

```
G = gkz(A,k)
holonomicRank G
```

Theorem [Cauchy, Kovalevskaya, and Kashiwara]:

The dimension of the space of solutions of a D-ideal I on any simply connected domain U outside the *singular locus* $\mathrm{Sing}(I)$ is equal to the *holonomic rank* of I.

The singular locus of our GKZ system is the principal A-determinant $\{E_A(c) = 0\}$.

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Theorem

Let $c^* \in \mathbb{C}^A$ *be such that* $E_A(c^*) \neq 0$ *and let* κ *be non-resonant. For any simply connected domain* $U_{c^*} \ni c^*$ *such that* $U_{c^*} \cap \{E_A(c) = 0\} = \emptyset$*, we have that*

$$\dim_{\mathbb{C}}(V_{c^*}) = \dim_{\mathbb{C}(c)}(R_A/(R_A \cdot H_A(\kappa))) = |\chi(X(c^*))| = \operatorname{vol}(\operatorname{NP}(h)),$$

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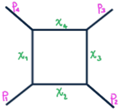
Remark

For any choice of parameters $\kappa = (-\nu, s)$, the following inequality holds

$$\dim_{\mathbb{C}(c)}(R_A/(R_A \cdot H_A(\kappa))) \geq \mathrm{vol}(\mathrm{NP}(h)).$$

Motivation: Feynman Integrals in Particle Physics

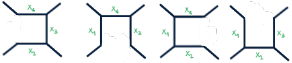
Let G be a connected Feynman diagram.



Definition

The first Symanzik polynomial $\mathcal{U}_G$ is

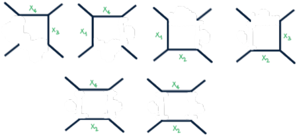
$$\mathcal{U}_G := \sum_{T \in \mathcal{T}_G} \prod_{x_e \notin T} x_e$$



Spanning trees of G

For $S \subset \{1, \ldots, n_G\}$, define

$$\mathcal{F}_{G,S} := \sum_{T_S \sqcup T_{\bar{S}} \in \mathcal{T}_G} \prod_{e \notin T_S, T_{\bar{S}}} x_e$$



The second Symanzik polynomial $\mathcal{F}_G$

$$\mathcal{F}_G := \sum_{S, \bar{S} \in \mathcal{P}_G} \left(\sum_{i \in S} p_i \right)^2 \mathcal{F}_{G,S} - \left(\sum_{e=1}^{E_G} m_e^2 x_e \right) \mathcal{U}_G$$

PARTITIONS OF n EXTERNAL LEGS
INTO 2 DISJOINT SUBSETS