Guessing with Little Data

Manuel Kauers and Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

25 November 2022
MATHEXP-PolSys seminar, GT: Transcendance et Combinatoire, LIP6, France

Motivating Example: OEIS A172671

$a_n := \#\big\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \;\big|\; \text{row sums} = 2,\ \text{column sums} = n\big\}$

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n \times 6} \mid \text{row sums} = 2, \text{column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n \times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n \times 6} \mid \text{row sums} = 2, \text{column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	1	0	0	0	1

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	1	0	0	0	1
0	0	0	1	1	0

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	1	0	0	0	1
0	0	0	1	1	0

Sequence: $1, 90, 202410, 747558000, 3536978063850, \ldots$

Motivating Example: OEIS A172671

$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	1	0	0	0	1
0	0	0	1	1	0

Sequence: $1, 90, 202410, 747558000, 3536978063850, \ldots$

Questions:

- Recurrence? Efficient method for computing next terms?

Motivating Example: OEIS A172671

$$a_n := \#\{A \in \mathbb{Z}_{\geqslant 0}^{3n\times 6} \mid \text{row sums} = 2, \text{ column sums} = n\}$$

Example: for $n = 2$:

0	0	1	0	0	1
2	0	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	1	0	0	0	1
0	0	0	1	1	0

Sequence: $1, 90, 202410, 747558000, 3536978063850, \ldots$

Questions:

- Recurrence? Efficient method for computing next terms?
- Nature of the generating function?

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ +\text{lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ + \text{lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \mathrm{Diag}\left(\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2} \right)$$

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ \text{+lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \operatorname{Diag}\Bigg(\underbrace{\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2}}_{=f(x_1,\ldots,x_6)} \Bigg)$$

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ + \text{lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \mathrm{Diag}\Bigg(\underbrace{\frac{1}{1 - x_1 x_2 - x_1 x_3 - \cdots - x_5 x_6 - x_1^2 - \cdots - x_6^2}}_{= f(x_1, \ldots, x_6)} \Bigg)$$

$$= \oint_{x_2} \cdots \oint_{x_6} f\left(\frac{x}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_5}{x_6}, x_6\right) \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_6}{x_2 \cdots x_6}$$

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ \text{+lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \mathrm{Diag}\Bigg(\underbrace{\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2}}_{=f(x_1,\ldots,x_6)} \Bigg)$$

$$= \oint_{x_2} \cdots \oint_{x_6} f\left(\frac{x}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_5}{x_6}, x_6\right) \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_6}{x_2 \cdots x_6}$$

$\longrightarrow$ The a_n satisfy a linear recurrence and their g.f. is D-finite!

Nature of the Sequence

Answer 1: Denote by c_i $(1 \leqslant i \leqslant 21)$ the number of rows of type i:

$$a_n = \sum_{\substack{0 \leqslant c_1, \ldots, c_{21} \leqslant n \\ +\text{lin. constraints}}} \binom{3n}{c_1, c_2, \ldots, c_{21}}$$

Answer 2: Interpret arrays as walks in $\mathbb{Z}^6_{\geqslant 0}$ ending at $(n, \ldots, n)$:

$$\sum_{n=0}^{\infty} a_n x^n = \operatorname{Diag}\Bigg(\underbrace{\frac{1}{1 - x_1x_2 - x_1x_3 - \cdots - x_5x_6 - x_1^2 - \cdots - x_6^2}}_{=f(x_1,\ldots,x_6)} \Bigg)$$

$$= \oint_{x_2} \cdots \oint_{x_6} f\left(\frac{x}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_5}{x_6}, x_6\right) \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_6}{x_2 \cdots x_6}$$

$\longrightarrow$ The a_n satisfy a linear recurrence and their g.f. is D-finite!
How to construct this recurrence / ODE?

Guessing

Guessing

Ansatz: $c_0 a_n + c_1 a_{n+1} + \cdots + c_r a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

Guessing

Ansatz: $\big(c_{00} + c_{01} n\big) a_n + \cdots + \big(c_{r0} + c_{r1} n\big) a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \left(\begin{array}{cccc|ccc} a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r \\ a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \end{array}\right)$$

Guessing

Ansatz: $\left(c_{00}+\cdots+c_{0d}n^d\right)a_n+\cdots+\left(c_{r0}+\cdots+c_{rd}n^d\right)a_{n+r}=0$
leads to a linear system $M\cdot x=0$ with

$$M=\left(\begin{array}{cccc|ccc|c|ccc}
a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r & \cdots & 0^d a_0 & \cdots & 0^d a_r\\
a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} & \cdots & 1^d a_1 & \cdots & 1^d a_{r+1}\\
a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} & \cdots & 2^d a_2 & \cdots & 2^d a_{r+2}\\
a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} & \cdots & 3^d a_3 & \cdots & 3^d a_{r+3}\\
a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} & \cdots & 4^d a_4 & \cdots & 4^d a_{r+4}\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots
\end{array}\right)$$

Guessing

Ansatz: $(c_{00} + \cdots + c_{0d} n^d) a_n + \cdots + (c_{r0} + \cdots + c_{rd} n^d) a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \left(\begin{array}{cccc|ccc|c|ccc} a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r & \cdots & 0^d a_0 & \cdots & 0^d a_r \\ a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} & \cdots & 1^d a_1 & \cdots & 1^d a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} & \cdots & 2^d a_2 & \cdots & 2^d a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} & \cdots & 3^d a_3 & \cdots & 3^d a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} & \cdots & 4^d a_4 & \cdots & 4^d a_{r+4} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right)$$

- There are $(r+1)(d+1)$ unknowns in the ansatz.

Guessing

Ansatz: $(c_{00} + \cdots + c_{0d}n^d)a_n + \cdots + (c_{r0} + \cdots + c_{rd}n^d)a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \left(\begin{array}{cccc|ccc|c|ccc} a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r & \cdots & 0^d a_0 & \cdots & 0^d a_r \\ a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} & \cdots & 1^d a_1 & \cdots & 1^d a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} & \cdots & 2^d a_2 & \cdots & 2^d a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} & \cdots & 3^d a_3 & \cdots & 3^d a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} & \cdots & 4^d a_4 & \cdots & 4^d a_{r+4} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right)$$

- There are $(r+1)(d+1)$ unknowns in the ansatz.
- If $a_0, \ldots, a_N$ are given, then we get $N - r + 1$ equations.

Guessing

Ansatz: $(c_{00} + \cdots + c_{0d} n^d) a_n + \cdots + (c_{r0} + \cdots + c_{rd} n^d) a_{n+r} = 0$ leads to a linear system $M \cdot x = 0$ with

$$M = \left(\begin{array}{cccc|ccc|c|ccc} a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r & \cdots & 0^d a_0 & \cdots & 0^d a_r \\ a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} & \cdots & 1^d a_1 & \cdots & 1^d a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} & \cdots & 2^d a_2 & \cdots & 2^d a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} & \cdots & 3^d a_3 & \cdots & 3^d a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} & \cdots & 4^d a_4 & \cdots & 4^d a_{r+4} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right)$$

- There are $(r+1)(d+1)$ unknowns in the ansatz.
- If $a_0, \ldots, a_N$ are given, then we get $N - r + 1$ equations.
- To trust the result, we need $N - r + 1 \geqslant (r+1)(d+1)$.

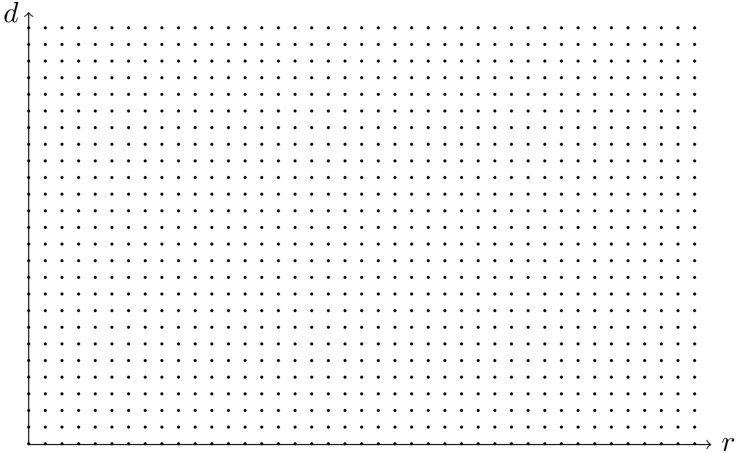
Guessing

Ansatz: $(c_{00} + \cdots + c_{0d}n^d)a_n + \cdots + (c_{r0} + \cdots + c_{rd}n^d)a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

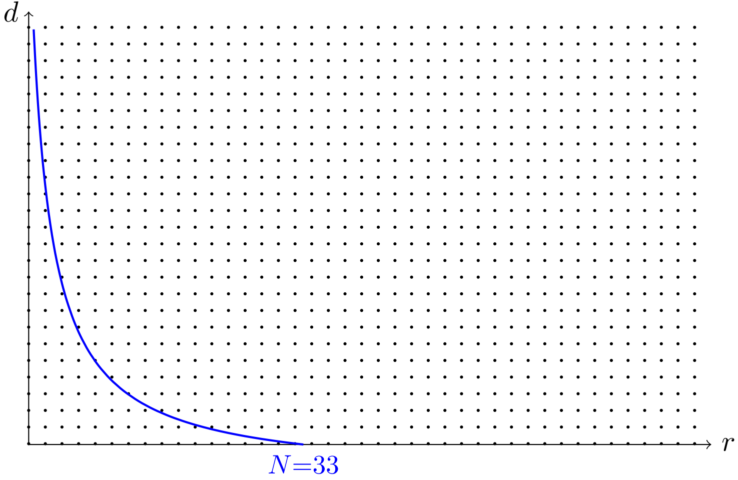
$$M = \left(\begin{array}{cccc|ccc|c|ccc} a_0 & a_1 & \cdots & a_r & 0a_0 & \cdots & 0a_r & \cdots & 0^d a_0 & \cdots & 0^d a_r \\ a_1 & a_2 & \cdots & a_{r+1} & 1a_1 & \cdots & 1a_{r+1} & \cdots & 1^d a_1 & \cdots & 1^d a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} & 2a_2 & \cdots & 2a_{r+2} & \cdots & 2^d a_2 & \cdots & 2^d a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} & 3a_3 & \cdots & 3a_{r+3} & \cdots & 3^d a_3 & \cdots & 3^d a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} & 4a_4 & \cdots & 4a_{r+4} & \cdots & 4^d a_4 & \cdots & 4^d a_{r+4} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right)$$

- There are $(r+1)(d+1)$ unknowns in the ansatz.
- If $a_0, \ldots, a_N$ are given, then we get $N - r + 1$ equations.
- To trust the result, we need $N - r + 1 \geqslant (r+1)(d+1)$.
- For sequence A172671 we know only 33 terms, i.e. we can try $(r, d) = (1, 15), (2, 9), (3, 6), (4, 5), (6, 3), (7, 2), (10, 1), (16, 0)$.

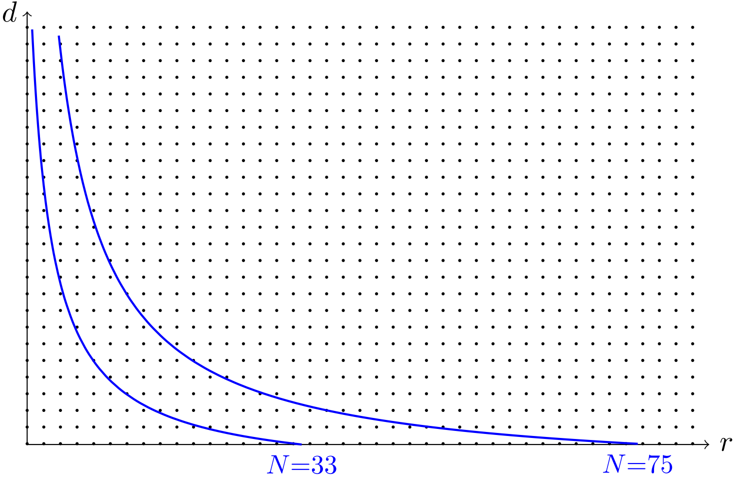
Trading Order vs. Degree



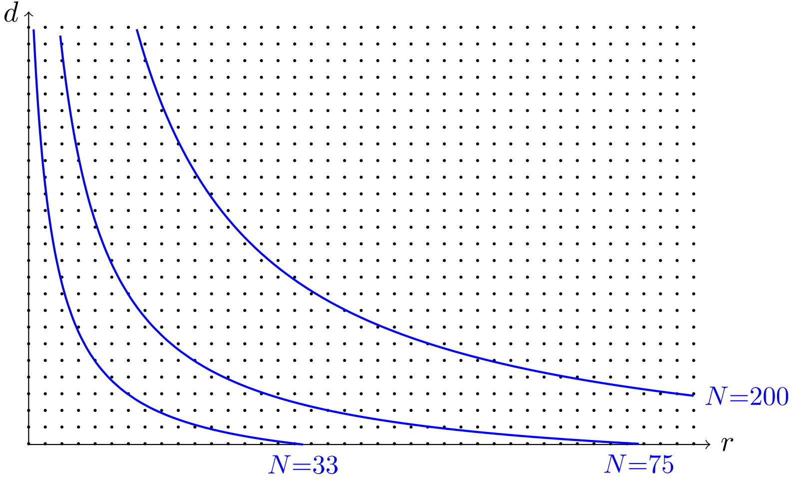
Trading Order vs. Degree



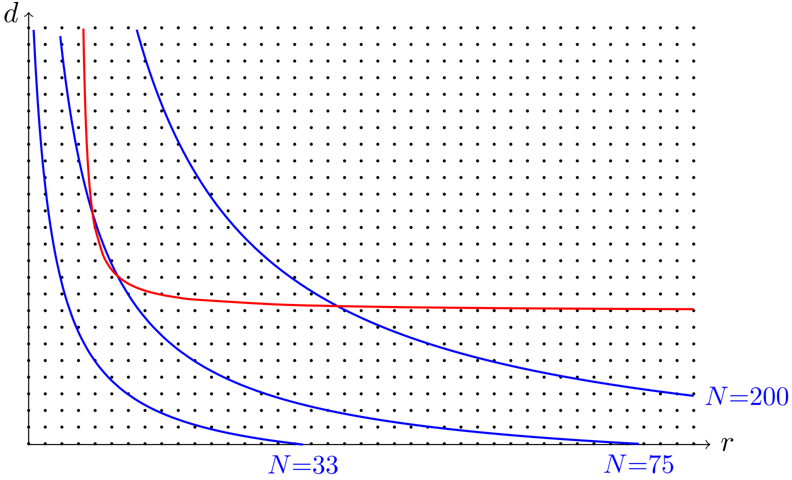
Trading Order vs. Degree



Trading Order vs. Degree



Trading Order vs. Degree



Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41.$

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r + 1)(d + 1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.
- The ODE of the g.f. has only regular singularities.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.
- The ODE of the g.f. has only regular singularities.
- The p-curvature is nilpotent.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.
- The ODE of the g.f. has only regular singularities.
- The p-curvature is nilpotent.

$\longrightarrow$ These criteria lead to nonlinear (diophantine) equations ☹

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.
- The ODE of the g.f. has only regular singularities.
- The p-curvature is nilpotent.

$\longrightarrow$ These criteria lead to nonlinear (diophantine) equations ☹

- The coefficients of the recurrence involve "small" integers.

Guessing with Little Data

If we have too little data, i.e., $N - r + 1 < (r+1)(d+1) - 1$, then $\dim(\ker M) > 1$.

Example: For A172671, need $r = 4, d = 13 \implies \dim(\ker M) = 41$.

Problem: Identify the true recurrence in the vector space $\ker M$.
$\longrightarrow$ is like looking for a needle in a haystack!

Filtering criteria:

- Unrolling the recurrence produces only integers.
- The ODE of the g.f. has only regular singularities.
- The p-curvature is nilpotent.

$\longrightarrow$ These criteria lead to nonlinear (diophantine) equations ☹

- The coefficients of the recurrence involve "small" integers.

$\longrightarrow$ Employ a lattice reduction algorithm (LLL, BKZ, ...).

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

A lattice reduction algorithm computes a basis $w_1, \ldots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

Lattice Basis

Let $v_1, \dots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \dots, c_\ell \in \mathbb{Z}\}.$$

A lattice reduction algorithm computes a basis $w_1, \dots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

Caveat: Before we had computed a $\mathbb{Q}$-vector space basis of $\ker M$.

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

A lattice reduction algorithm computes a basis $w_1, \ldots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

Caveat: Before we had computed a $\mathbb{Q}$-vector space basis of $\ker M$.

- Clearing denominators is not enough.

Lattice Basis

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

A lattice reduction algorithm computes a basis $w_1, \ldots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

Caveat: Before we had computed a $\mathbb{Q}$-vector space basis of $\ker M$.

- Clearing denominators is not enough.
- We need a basis of the $\mathbb{Z}$-module $\ker_{\mathbb{Z}} M$.

Lattice Basis

Let $v_1, \dots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \dots + c_\ell v_\ell \mid c_1, \dots, c_\ell \in \mathbb{Z}\}.$$

A lattice reduction algorithm computes a basis $w_1, \dots, w_\ell$ of $\mathcal{L}$ that consists of short vectors.

Caveat: Before we had computed a $\mathbb{Q}$-vector space basis of $\ker M$.

- Clearing denominators is not enough.
- We need a basis of the $\mathbb{Z}$-module $\ker_{\mathbb{Z}} M$.
- It can be computed, e.g. using the Hermite normal form.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$
Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

1 Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$
Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.
if $\ell = 0$, then return FAIL.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

```
Compute ℤ-module basis v_1, ..., v_ℓ ∈ ℤ^{(r+1)(d+1)} of ker_ℤ M.
if ℓ = 0, then return FAIL.
Apply LLL to v_1, ..., v_ℓ, and call the result w_1, ..., w_ℓ.
```

Algorithm

Input: $a_0, \dots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

1. Compute $\mathbb{Z}$-module basis $v_1, \dots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.
2. if $\ell = 0$, then return FAIL.
3. Apply LLL to $v_1, \dots, v_\ell$, and call the result $w_1, \dots, w_\ell$.
4. Return the recurrence corresponding to the vector w_1.

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$
Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

1 Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.
2 if $\ell = 0$, then return FAIL.
3 Apply LLL to $v_1, \ldots, v_\ell$, and call the result $w_1, \ldots, w_\ell$.
4 Return the recurrence corresponding to the vector w_1.

Improvements and Variations:

- Use different basis than standard monomials (binomials, ...)

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.
if $\ell = 0$, then return FAIL.
Apply LLL to $v_1, \ldots, v_\ell$, and call the result $w_1, \ldots, w_\ell$.
Return the recurrence corresponding to the vector w_1.

Improvements and Variations:

- Use different basis than standard monomials (binomials, ...)
- Incorporate homomorphic images and Chinese remaindering

Algorithm

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

Output: A linear recurrence of order r and degree d which matches the given data and involves short integers, or FAIL.

1 Compute $\mathbb{Z}$-module basis $v_1, \ldots, v_\ell \in \mathbb{Z}^{(r+1)(d+1)}$ of $\ker_{\mathbb{Z}} M$.

2 if $\ell = 0$, then return FAIL.

3 Apply LLL to $v_1, \ldots, v_\ell$, and call the result $w_1, \ldots, w_\ell$.

4 Return the recurrence corresponding to the vector w_1.

Improvements and Variations:

- Use different basis than standard monomials (binomials, ...)
- Incorporate homomorphic images and Chinese remaindering
- Recycle LLL-output when trying a range of degrees d

The Generic Case

Theorem [Bombieri–Vaaler, 1983] Let $M \in \mathbb{Z}^{k \times m}$ with $k < m$, and let g be the $\gcd$ of all $k \times k$ minors of M.
Then $\ker_{\mathbb{Z}} M$ contains a nonzero element $x \in \mathbb{Z}^m$ with

$$||x||_\infty \leqslant \left(\frac{1}{g} \sqrt{\det\left(MM^\top\right)} \right)^{1/(m-k)}$$

The Generic Case

Theorem [Bombieri–Vaaler, 1983] Let $M \in \mathbb{Z}^{k\times m}$ with $k < m$, and let g be the $\gcd$ of all $k \times k$ minors of M.
Then $\ker_{\mathbb{Z}} M$ contains a nonzero element $x \in \mathbb{Z}^m$ with

$$||x||_\infty \leqslant \left(\frac{1}{g}\sqrt{\det\left(MM^\top\right)}\right)^{1/(m-k)}$$

Implications:

- If the integer coefficients are larger than this bound, it is very unlikely that the recurrence can be found.

The Generic Case

Theorem [Bombieri–Vaaler, 1983] Let $M \in \mathbb{Z}^{k\times m}$ with $k < m$, and let g be the $\gcd$ of all $k \times k$ minors of M.
Then $\ker_{\mathbb{Z}} M$ contains a nonzero element $x \in \mathbb{Z}^m$ with

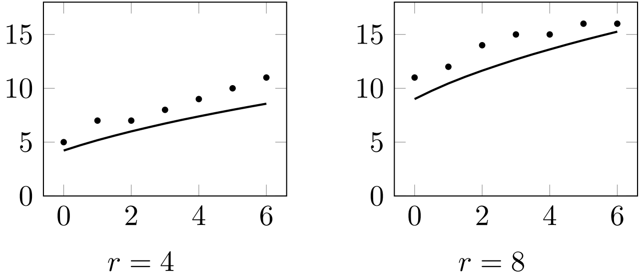
$$||x||_\infty \leqslant \left(\frac{1}{g}\sqrt{\det\left(MM^\top\right)}\right)^{1/(m-k)}$$

Implications:

- If the integer coefficients are larger than this bound, it is very unlikely that the recurrence can be found.
- Use this bound to estimate how little data we need at least: "generically", we cannot expect to succeed if

$$N \lesssim \frac{1}{2}\left(\sqrt{8(r+1)(d+1)+49}-7\right)+r-1.$$

The Generic Case



Implications:

- If the integer coefficients are larger than this bound, it is very unlikely that the recurrence can be found.
- Use this bound to estimate how little data we need at least: "generically", we cannot expect to succeed if

$$N \lesssim \frac{1}{2}\left(\sqrt{8(r+1)(d+1)+49}-7\right)+r-1.$$

Artifact Recurrences

There are other obstacles that prevent us to use too few values.

Artifact Recurrences

There are other obstacles that prevent us to use too few values.

Example: Assume we are given $a_0, \ldots, a_8$.

Artifact Recurrences

There are other obstacles that prevent us to use too few values.

Example: Assume we are given $a_0, \ldots, a_8$.

- If we seek for an order-4 and degree-5 recurrence we obtain

$$0 \cdot a_n + 0 \cdot a_{n+1} + 0 \cdot a_{n+2} + 0 \cdot a_{n+3} +$$
$$n(n-1)(n-2)(n-3)(n-4) \cdot a_{n+4} = 0,$$

  which is true for any input data (but has small coefficients).

Artifact Recurrences

There are other obstacles that prevent us to use too few values.

Example: Assume we are given $a_0, \ldots, a_8$.

- If we seek for an order-4 and degree-5 recurrence we obtain

$$0 \cdot a_n + 0 \cdot a_{n+1} + 0 \cdot a_{n+2} + 0 \cdot a_{n+3} + \\ n(n-1)(n-2)(n-3)(n-4) \cdot a_{n+4} = 0,$$

  which is true for any input data (but has small coefficients).

- Same with order-4 and degree-3, we could get something like

$$(n-4)(2234n^2 + 719n - 6083) \cdot a_n + \\ (n-4)(5672n^2 - 4375n + 6727) \cdot a_{n+1} - \\ (n-4)(9959n^2 - 4399n - 7710) \cdot a_{n+2} + \\ (n-4)(n-3)(1322n + 1174) \cdot a_{n+3} + \\ (n-4)(n-3)(n-2)1909 \cdot a_{n+4} = 0.$$

The Non-Generic Case

- There are more than 350,000 sequences in the OEIS.

The Non-Generic Case

- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,

The Non-Generic Case

- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),

The Non-Generic Case

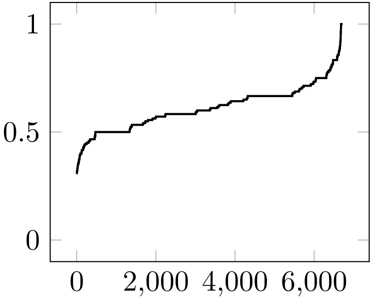
- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),
- which are not too simple (LA needs more than 10 terms),

The Non-Generic Case

- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),
- which are not too simple (LA needs more than 10 terms),
- which are not too large (LA needs no more than 250 terms).

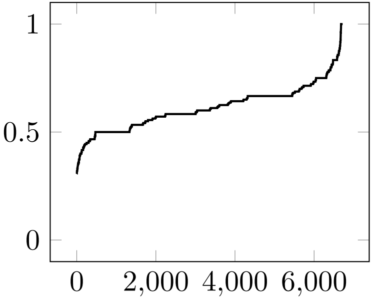
The Non-Generic Case

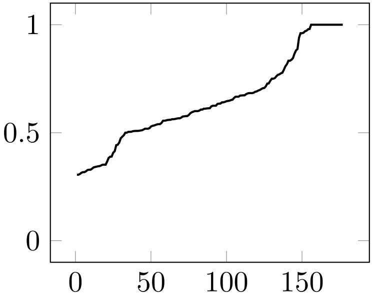
- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),
- which are not too simple (LA needs more than 10 terms),
- which are not too large (LA needs no more than 250 terms).



The Non-Generic Case

- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),
- which are not too simple (LA needs more than 10 terms),
- which are not too large (LA needs no more than 250 terms).





Result for A172671

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

Result for A172671

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^9 + 4485660756765878340n^8 + 18521224670025594405n^7\\
&+56639217843614362320n^6 + 128197997261515989990n^5 + 211964073373172447460n^4\\
&+248660072114197834440n^3 + 195845152107619591920n^2\\
&+92743576895010081600n + 19927056990544704000)a_n\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^9 + 118570454113296582n^8 + 533897028046714761n^7\\
&+1794118103056008945n^6 + 4499490897537212457n^5 + 8317813242144219813n^4\\
&+11017108466619178896n^3 + 9901273828612752684n^2\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10}\\
&-990927551678562n^9 - 6445641158908164n^8 - 30971993224981077n^7\\
&-111314492026841106n^6 - 299240095376493090n^5 - 594271149013691226n^4\\
&-847459848696773373n^3 - 821800045816910820n^2\\
&-485718284438018172n - 132150596906568240)a_{n+2}\\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10}\\
&-5264405804862n^9 - 36048494147578n^8 - 182315015737541n^7\\
&-689472630263907n^6 - 1949560872565283n^5 - 4070539427181535n^4\\
&-6099491170412670n^3 - 6211013227585736n^2\\
&-3851899366258336n - 1098712786184832)a_{n+3}\\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
&+702635490n^9 + 5025358332n^8 + 26510256652n^7\\
&+104430770292n^6 + 307166340054n^5\\
&+666220125600n^4 + 1035598237875n^3\\
&+1092435142500n^2 + 700889050000n\\
&+206542200000)a_{n+4}
\end{aligned}
$$

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

Result for A172671

Trustworthy?

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7}\\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}\\
&+248660072114197834440n^{3} + 195845152107619591920n^{2}\\
&+92743576895010081600n + 19927056990544704000)a_{n}\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7}\\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4}\\
&+11017108466619178896n^{3} + 9901273828612752684n^{2}\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10}\\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7}\\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4}\\
&-847459848696773373n^{3} - 821800045816910820n^{2}\\
&-485718284438018172n - 132150596906568240)a_{n+2}\\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10}\\
&-5264405804862n^{9} - 36048494147578n^{8} - 182315015737541n^{7}\\
&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4}\\
&-6099491170412670n^{3} - 6211013227585736n^{2}\\
&-3851899366258336n - 1098712786184832)a_{n+3}\\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7}\\
&+104430770292n^{6} + 307166340054n^{5}\\
&+666220125600n^{4} + 1035598237875n^{3}\\
&+1092435142500n^{2} + 700889050000n\\
&+206542200000)a_{n+4}
\end{aligned}
$$

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

Result for A172671

Trustworthy?

$$(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}$$
$$+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7}$$
$$+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}$$
$$+248660072114197834440n^{3} + 195845152107619591920n^{2}$$
$$+92743576895010081600n + 19927056990544704000)a_n$$

$\ldots 092056n^{10}$ $\ldots 6714761n^{7}$ $\ldots 4219813n^{4}$ $\ldots 2752684n^{2}$ $\ldots 3520)a_{n+1}$ $\ldots 208164n^{10}$ $\ldots 4981077n^{7}$ $\ldots 3691226n^{4}$ $\ldots 6910820n^{2}$ $\ldots 8240)a_{n+2}$ $\ldots 207552n^{10}$ $\ldots 5737541n^{7}$ $\ldots 7181535n^{4}$ $\ldots 7585736n^{2}$ $\ldots 4832)a_{n+3}$ $\ldots 437960n^{10}$ $\ldots 0256652n^{7}$ $\ldots 6340054n^{5}$ $\ldots 8237875n^{3}$ $\ldots 89050000n$ $\ldots 0000)a_{n+4}$

> Neil Sloane (05.03.2022, about A189281): In the text of the paper you say the coefficients are small! Au contraire. In fact the amount of data in the g.f. is comparable with the data in the original 35-term b-file for the sequence.
>
> If you print the g.f. and then print the data, the number of digits in the two printouts look about the same. When this happens, surely you should be worried. I am very worried, and I think the g.f. needs more justification.
>
> In fact the g.f. looks wrong. I use gfun all the time, and when the g.f. looks like this, like something you would find in the dumpster behind a restaurant, then I would not even consider it :D

90
202410
747558000
3536978063…
1929211769…
1154281859…
7370055389…
4937928427…
3433503127…
2458852579…
1803606070…
1349547237…
1026866094…
7926069369…
6194061046…
4893088622…
3902291784…
3138480844…
2543261429…
2074925587…
1703218238…
1405884888…
1166349337…
97212368765632828873597857210432906828323036261620913199779764525314490735250548751871000

Result for A172671

Trustworthy?

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7}\\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4}\\
&+248660072114197834440n^{3} + 195845152107619591920n^{2}\\
&+92743576895010081600n + 19927056990544704000)a_n\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7}\\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4}\\
&+11017108466619178896n^{3} + 9901273828612752684n^{2}\\
&+5411908796200065936n + 1358800904704763520)a_{n+1}\\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10}\\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7}\\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4}\\
&-847459848696773373n^{3} - 821800045816910820n^{2}\\
&-485718284438018172n - 132150596906568240)a_{n+2}\\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10}\\
&-5264405804862n^{9} - 36048494147578n^{8} - 182315015737541n^{7}\\
&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4}\\
&-6099491170412670n^{3} - 6211013227585736n^{2}\\
&-3851899366258336n - 1098712786184832)a_{n+3}\\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10}\\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7}\\
&+104430770292n^{6} + 307166340054n^{5}\\
&+666220125600n^{4} + 1035598237875n^{3}\\
&+1092435142500n^{2} + 700889050000n\\
&+206542200000)a_{n+4}
\end{aligned}
$$

Result for A172671

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10} \\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7} \\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4} \\
&+248660072114197834440n^{3} + 195845152107619591920n^{2} \\
&+92743576895010081600n + 19927056990544704000)a_n \\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10} \\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7} \\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4} \\
&+11017108466619178896n^{3} + 9901273828612752684n^{2} \\
&+5411908796200065936n + 1358800904704763520)a_{n+1} \\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10} \\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7} \\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4} \\
&-847459848696773373n^{3} - 821800045816910820n^{2} \\
&-485718284438018172n - 132150596906568240)a_{n+2} \\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10} \\
&-5264405804862n^{9} - 36048494147578n^{8} - 182315015737541n^{7} \\
&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4} \\
&-6099491170412670n^{3} - 6211013227585736n^{2} \\
&-3851899366258336n - 1098712786184832)a_{n+3} \\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10} \\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7} \\
&+104430770292n^{6} + 307166340054n^{5} \\
&+666220125600n^{4} + 1035598237875n^{3} \\
&+1092435142500n^{2} + 700889050000n \\
&+206542200000)a_{n+4}
\end{aligned}
$$

Trustworthy?

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000
8137021686675518646293987429238146291939698206862646669804019655299405410612322521011793199840
6837802728712759610153893305259995444879386272730048497289313037408331493614063937026579190230160
576696135477018756656097310539308206595297137655178128559217331447163987622287690653154248117571110400
48802599521992920089218265263126093488252491477883748511441945654829069021314936881677902667388025048400000
4142879219648880148628212753941786837923961100732936038421511856853363244953154556854132023543994137557662400
352722570320243675250582868120044596892546971115353276044155030127971499536247490129735774654399339137472372179200

Result for A172671

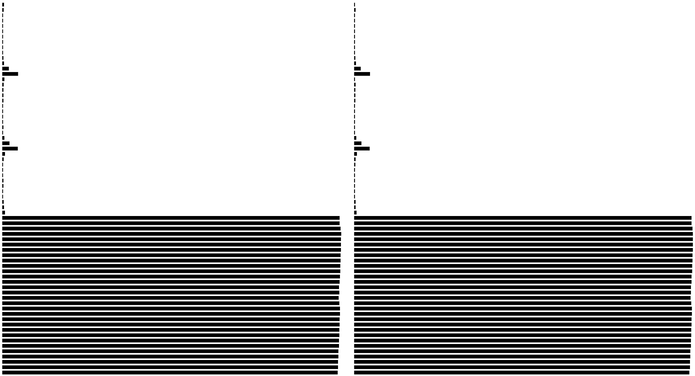
Trustworthy !

$$
\begin{aligned}
&416745(n+2)(3n+4)(3n+5)(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\times(3784n^4+62436n^3+384549n^2+1047914n+1066254)a_n\\
&+9(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\times(29681696n^7+712360704n^6+7253307424n^5\\
&+40621828312n^4+135172900470n^3+267337368752n^2\\
&+291083104767n+134667010044)a_{n+1}\\
&-9(n+3)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\times(10844944n^8+309080904n^7+3833838118n^6\\
&+27035659722n^5+118560795930n^4+331121212914n^3\\
&+575194973415n^2+568260550317n+244478848756)a_{n+2}\\
&-(n+3)(n+4)^3(3n+13)(3n+14)\\
&\times(3799136n^7+98777536n^6+1092573240n^5\\
&+6662600832n^4+24184813590n^3+52244190090n^2\\
&+62174897623n+31442101253)a_{n+3}\\
&+(n+3)(n+4)^3(n+5)^5\\
&\times(3784n^4+47300n^3+219945n^2\\
&+450988n+344237)a_{n+4}
\end{aligned}
$$

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000
8137021686675518646293987429238146291939698206862646669804019655299405410612322521011793199840
683780272871275961015389330525999544487938627273004849728931303740833149361406393702657919023016 00
57669613547701875665609731053930820659529713765517812855921733144716398762228769065315424811757111 0400
48802599521992920089218265263126093488252491477883748511441945654829069021314936881677902667388025 04840000
41428792196488801486282127539417868379239611007329360384215118568533632449531545568541320235439941 375576624000
352722570320243675250582868120044596892546971115353276044155030127971499536247490129735774654399339137472372179200

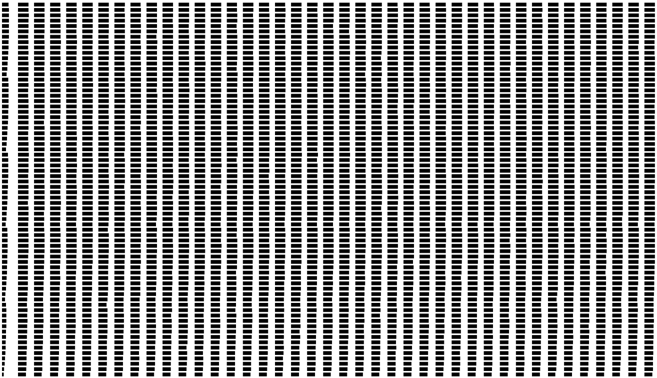
Result for A172671

First two vectors of $\ker_{\mathbb{Z}} M$:



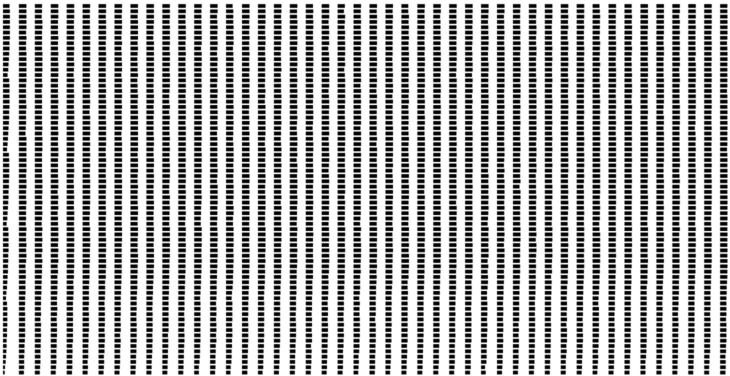
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:



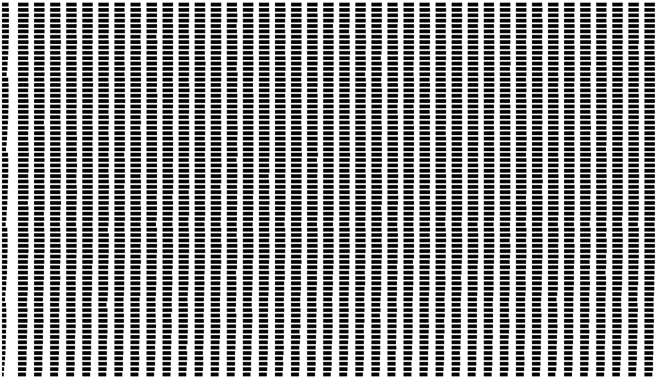
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 28$:



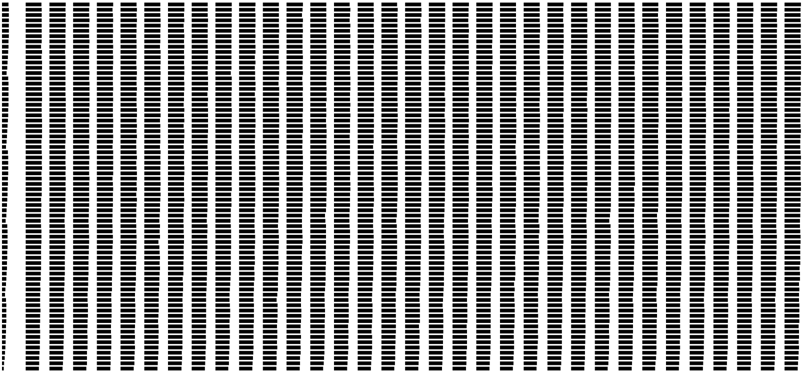
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:



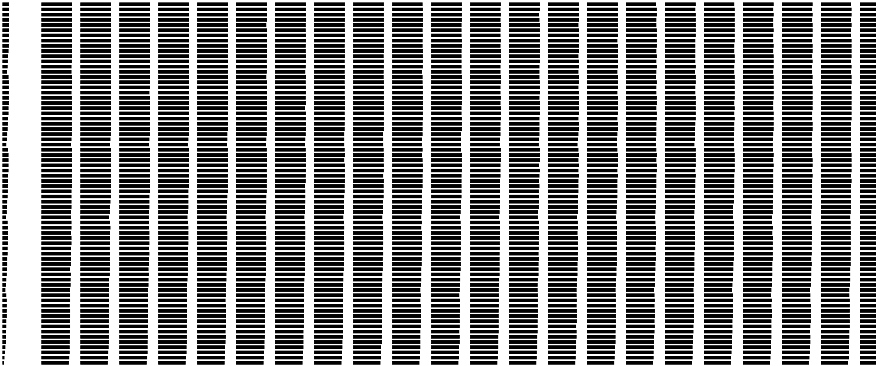
Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 40$:



Result for A172671

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 50$:



Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea.

Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea. As for A172671, if I am not mistaken for some reason the elements all have binomial(3*n,n)*binomial(2*n,n) as an integer factor. I wonder if the sequence divided by the 2 binomials has a simpler recurrence.

Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea. As for A172671, if I am not mistaken for some reason the elements all have binomial(3*n,n)*binomial(2*n,n) as an integer factor. I wonder if the sequence divided by the 2 binomials has a simpler recurrence.

Indeed, let
$$b_n := \frac{a_n}{\binom{3n}{n}\binom{2n}{n}}.$$

Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea. As for A172671, if I am not mistaken for some reason the elements all have binomial(3*n,n)*binomial(2*n,n) as an integer factor. I wonder if the sequence divided by the 2 binomials has a simpler recurrence.

Indeed, let
$$b_n := \frac{a_n}{\binom{3n}{n}\binom{2n}{n}}.$$

- Recurrence for b_n has the same order (namely, 4), but the coefficient degree drops from 13 to 8.

Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea. As for A172671, if I am not mistaken for some reason the elements all have binomial(3*n,n)*binomial(2*n,n) as an integer factor. I wonder if the sequence divided by the 2 binomials has a simpler recurrence.

Indeed, let
$$b_n := \frac{a_n}{\binom{3n}{n}\binom{2n}{n}}.$$

- Recurrence for b_n has the same order (namely, 4), but the coefficient degree drops from 13 to 8.
- With the classical linear algebra guessing, we can find this simplified recurrence using only 48 terms, instead of 75 terms.

Human Insight

Erich Kaltofen (16.07.2022): I have viewed the video of your ISSAC talk. A very interesting and clever idea. As for A172671, if I am not mistaken for some reason the elements all have binomial(3*n,n)*binomial(2*n,n) as an integer factor. I wonder if the sequence divided by the 2 binomials has a simpler recurrence.

Indeed, let
$$b_n := \frac{a_n}{\binom{3n}{n}\binom{2n}{n}}.$$

- Recurrence for b_n has the same order (namely, 4), but the coefficient degree drops from 13 to 8.
- With the classical linear algebra guessing, we can find this simplified recurrence using only 48 terms, instead of 75 terms.
- But the LLL-based guessing required only 24 terms for a_n, and it requires only 17 terms to find the simpler recurrence for b_n.

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

✗

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} ✗ \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ✗ \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ✓$$

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

- The OEIS has only 32 terms of this sequence.

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{✓}$$

- The OEIS has only 32 terms of this sequence.
- LLL-based guesser finds a recurrence of order 5 and degree 10.

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{✓}$$

- The OEIS has only 32 terms of this sequence.
- LLL-based guesser finds a recurrence of order 5 and degree 10.
- LA-guessing with order-degree trading requires 55 terms.

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{✗} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

- The OEIS has only 32 terms of this sequence.
- LLL-based guesser finds a recurrence of order 5 and degree 10.
- LA-guessing with order-degree trading requires 55 terms.

Questions:

- Can we compute more terms?

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ✓$$

- The OEIS has only 32 terms of this sequence.
- LLL-based guesser finds a recurrence of order 5 and degree 10.
- LA-guessing with order-degree trading requires 55 terms.

Questions:

- Can we compute more terms?
- Can we show that A188818 is D-finite?

A188818

Number of $n \times n$ binary arrays without the pattern 01 diagonally or antidiagonally.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} ✗ \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ✗ \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ✓$$

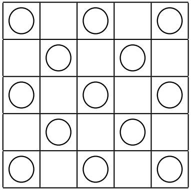
- The OEIS has only 32 terms of this sequence.
- LLL-based guesser finds a recurrence of order 5 and degree 10.
- LA-guessing with order-degree trading requires 55 terms.

Questions:

- Can we compute more terms?
- Can we show that A188818 is D-finite?
- Can we prove that the guessed recurrence is correct?

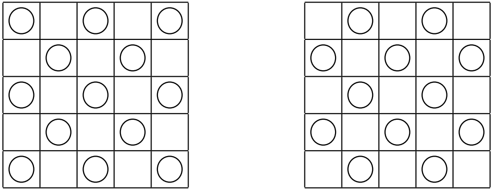
A188818

Chessboard decomposition:



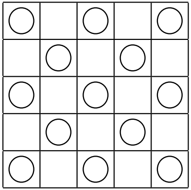
A188818

Chessboard decomposition:

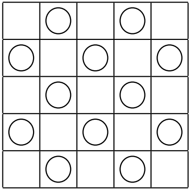


A188818

Chessboard decomposition:



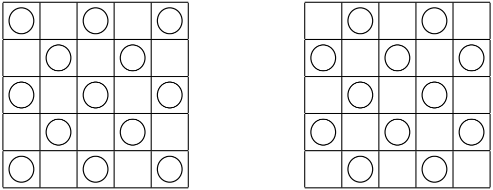
b_n



w_n

A188818

Chessboard decomposition:

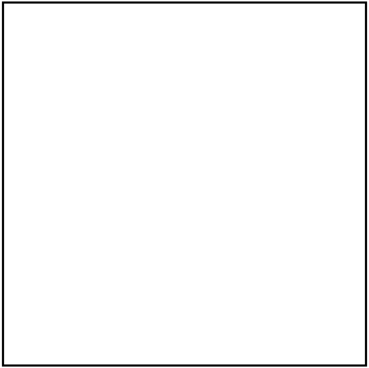


b_n w_n

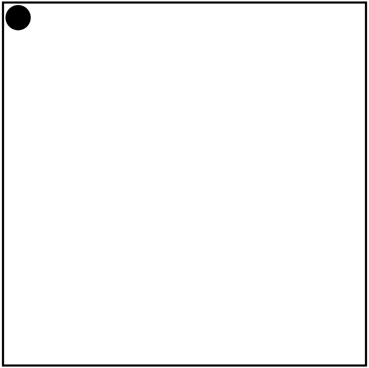
Hence, we obtain:

$$a_n = b_n \cdot w_n$$

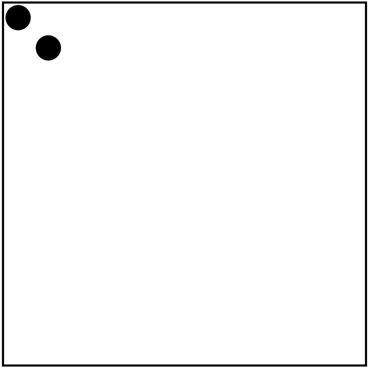
A188818



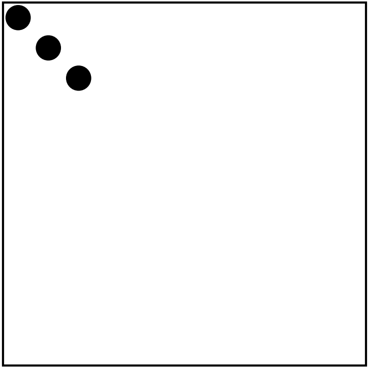
A188818



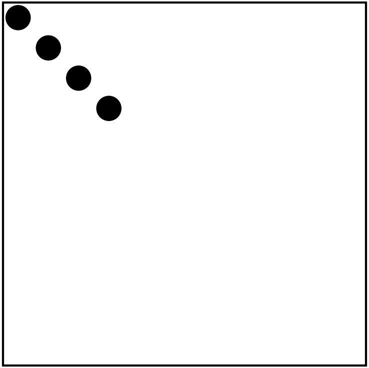
A188818



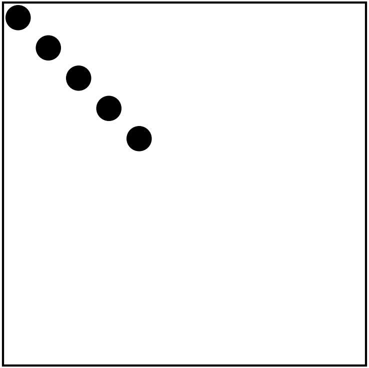
A188818



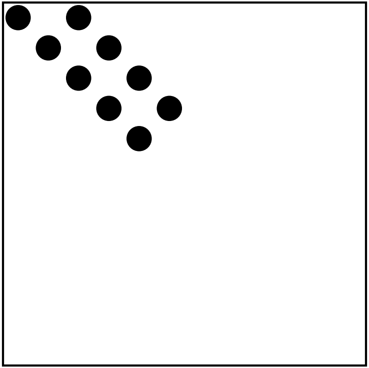
A188818



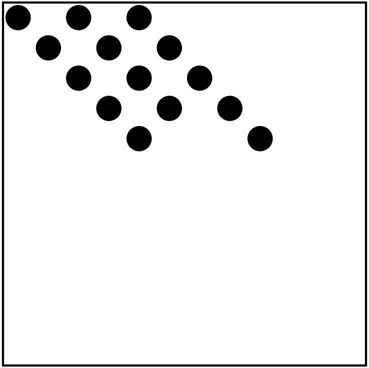
A188818



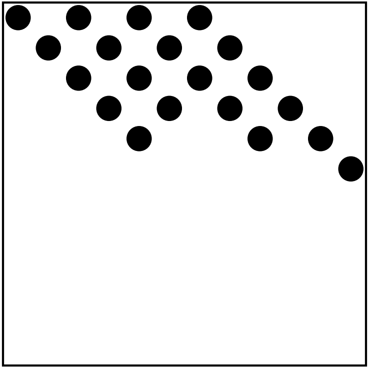
A188818



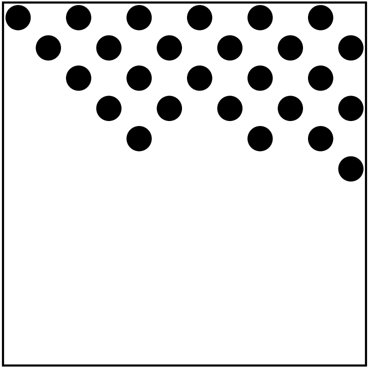
A188818



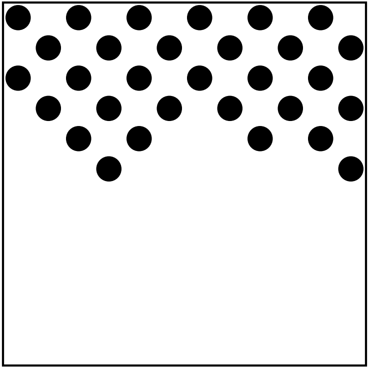
A188818



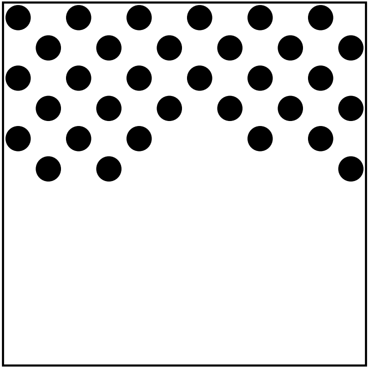
A188818



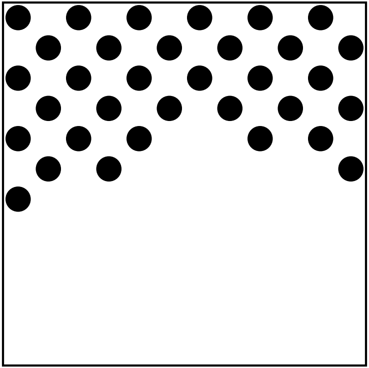
A188818



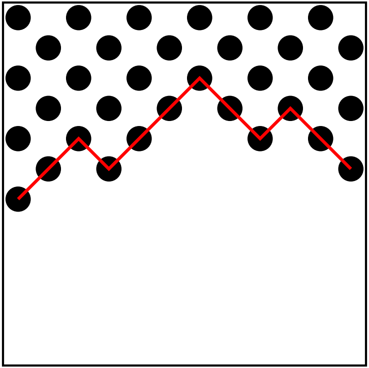
A188818



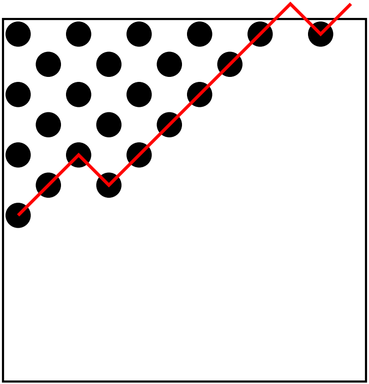
A188818



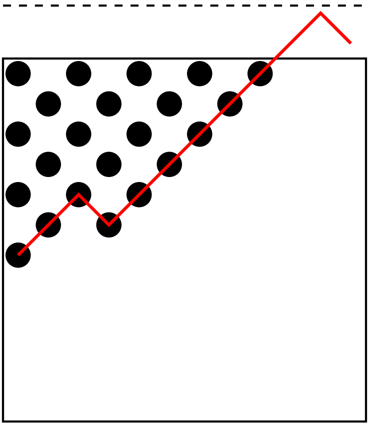
A188818



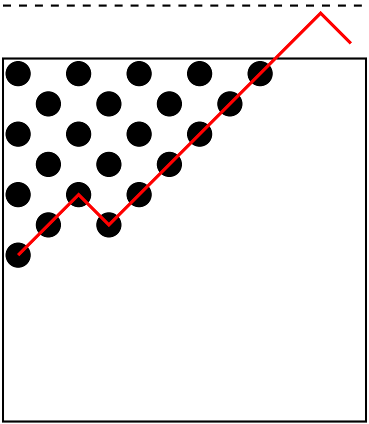
A188818



A188818



A188818



- Catalan (Dyck) paths with arbitrary start and end points:

$$\left|L\big((a,b)\to(c,d)\mid x\geqslant y\big)\right| = \binom{c+d-a-b}{c-a} - \binom{c+d-a-b}{c-b+1}.$$

A188818

Explicit representation of the sequence terms:

$a_n = b_n \cdot w_n$

A188818

Explicit representation of the sequence terms:

$$a_n = b_n \cdot w_n$$

$$b_n = \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} P(2k+1, 2\ell+1 \mid 1 \leqslant y \leqslant n+2) \qquad \text{(for } n \text{ odd)}$$

A188818

Explicit representation of the sequence terms:

$$a_n = b_n \cdot w_n$$

$$b_n = \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} P(2k+1, 2\ell+1 \mid 1 \leqslant y \leqslant n+2) \qquad \text{(for } n \text{ odd)}$$

$$= \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} \left\{ \begin{matrix} P(2k+1, 2\ell+1 \mid y \geqslant 1), & \text{if } k+\ell \leqslant \frac{n-1}{2} \\ P(n+2-2k, n+2-2\ell \mid y \geqslant 1), & \text{otherwise} \end{matrix} \right\}$$

A188818

Explicit representation of the sequence terms:

$$a_n = b_n \cdot w_n$$

$$b_n = \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} P(2k+1, 2\ell+1 \mid 1 \leqslant y \leqslant n+2) \qquad \text{(for } n \text{ odd)}$$

$$= \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} \begin{Bmatrix} P(2k+1, 2\ell+1 \mid y \geqslant 1), & \text{if } k+\ell \leqslant \frac{n-1}{2} \\ P(n+2-2k, n+2-2\ell \mid y \geqslant 1), & \text{otherwise} \end{Bmatrix}$$

$$= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left(2 \cdot \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor - k} \left(\binom{n-1}{\lfloor \frac{n}{2} \rfloor - k + \ell} - \binom{n-1}{\lfloor \frac{n+2}{2} \rfloor + k + \ell} \right) + \binom{n-1}{n-2k} \right)$$

A188818

Explicit representation of the sequence terms:

$$a_n = b_n \cdot w_n$$

$$b_n = \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} P(2k+1, 2\ell+1 \mid 1 \leqslant y \leqslant n+2) \qquad \text{(for } n \text{ odd)}$$

$$= \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} \left\{ \begin{matrix} P(2k+1, 2\ell+1 \mid y \geqslant 1), & \text{if } k+\ell \leqslant \frac{n-1}{2} \\ P(n+2-2k, n+2-2\ell \mid y \geqslant 1), & \text{otherwise} \end{matrix} \right\}$$

$$= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left(2 \cdot \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor - k} \left(\binom{n-1}{\lfloor \frac{n}{2} \rfloor - k + \ell} - \binom{n-1}{\lfloor \frac{n+2}{2} \rfloor + k + \ell} \right) + \binom{n-1}{n-2k} \right)$$

$$w_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(2 \cdot \sum_{\ell=0}^{\lfloor \frac{n-2}{2} \rfloor - k} \left(\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor - k + \ell} - \binom{n-1}{\lfloor \frac{n+3}{2} \rfloor + k + \ell} \right) + \binom{n-1}{n-2k-1} \right)$$

A188818

Explicit representation of the sequence terms:

$$a_n = b_n \cdot w_n$$

$$b_n = \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} P(2k+1, 2\ell+1 \mid 1 \leqslant y \leqslant n+2) \qquad \text{(for } n \text{ odd)}$$

$$= \sum_{k=0}^{\frac{n+1}{2}} \sum_{\ell=0}^{\frac{n+1}{2}} \begin{Bmatrix} P(2k+1, 2\ell+1 \mid y \geqslant 1), & \text{if } k+\ell \leqslant \frac{n-1}{2} \\ P(n+2-2k, n+2-2\ell \mid y \geqslant 1), & \text{otherwise} \end{Bmatrix}$$

$$= \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} \left(2 \cdot \sum_{\ell=0}^{\lfloor\frac{n-1}{2}\rfloor-k} \left(\binom{n-1}{\lfloor\frac{n}{2}\rfloor-k+\ell} - \binom{n-1}{\lfloor\frac{n+2}{2}\rfloor+k+\ell} \right) + \binom{n-1}{n-2k} \right)$$

$$w_n = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \left(2 \cdot \sum_{\ell=0}^{\lfloor\frac{n-2}{2}\rfloor-k} \left(\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor-k+\ell} - \binom{n-1}{\lfloor\frac{n+3}{2}\rfloor+k+\ell} \right) + \binom{n-1}{n-2k-1} \right)$$

D-finiteness is clear, recurrence can be derived by creative telescoping.

A189281

$a_n := \#\big\{\pi \in S_n \;\big|\; \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\big\}.$

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

Example: $a_3 = 5$ because only one $\pi \in S_3$ is excluded.

$$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}$$

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

Example: $a_3 = 5$ because only one $\pi \in S_3$ is excluded.

$$\cancel{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

A189281

$$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}.$$

Only $a_0, \ldots, a_{35}$ are known:

1	222822578	257590656485400508526570
1	2847624899	6409633590481106885238443
2	39282739034	165928838963556686281573922
5	581701775369	4462073606461933066205164757
18	9202313110506	124470791290376112779747519538
75	154873904848803	3597058248632667485834774744787
410	2762800622799362	107559658152025736992729145688602
2729	52071171437696453	3324154021716716493547315823808809
20906	1033855049655584786	106067493846954075776733869818571690
181499	21567640717569135515	3490771207487802026912252686947947027
1763490	471630531427793184474	118383998479651470880820236769742970626
18943701	10787660036599729160073	4133478159186775319059453592629838113797

A189281

$$a_n := \#\big\{\pi \in S_n \;\big|\; \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\big\}.$$

Only $a_0, \ldots, a_{35}$ are known:

1	222822578	257590656485400508526570
1	2847624899	6409633590481106885238443
2	39282739034	165928838963556686281573922
5	581701775369	4462073606461933066205164757
18	9202313110506	124470791290376112779747519538
75	154873904848803	3597058248632667485834774744787
410	2762800622799362	107559658152025736992729145688602
2729	52071171437696453	3324154021716716493547315823808809
20906	1033855049655584786	106067493846954075776733869818571690
181499	21567640717569135515	3490771207487802026912252686947947027
1763490	471630531427793184474	118383998479651470880820236769742970626
18943701	10787660036599729160073	4133478159186775319059453592629838113797

A189281

$a_n := \#\big\{\pi \in S_n \;\big|\; \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\big\}.$

We found a trustworthy recurrence of order 10 and degree 6.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.
- The guessed operator has a right factor of order 8 and degree 11.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}.$

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.
- The guessed operator has a right factor of order 8 and degree 11.
- We computed $a_{36}, a_{37}, a_{38}, a_{39}$ (!) and found that these terms were correctly predicted by the guessed recurrence.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}.$

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.
- The guessed operator has a right factor of order 8 and degree 11.
- We computed $a_{36}, a_{37}, a_{38}, a_{39}$ (!) and found that these terms were correctly predicted by the guessed recurrence.
- George Spahn and Doron Zeilberger designed a new algorithm for enumerating this sequence, and obtained $a_{40}, \ldots, a_{70}$.

A189281

$a_n := \#\{\pi \in S_n \mid \pi(k+2) - \pi(k) \neq 2 \text{ for all } k\}$.

We found a trustworthy recurrence of order 10 and degree 6.

- The first vector in the LLL-basis is much shorter than the other vectors.
- Even with just 29 of the 36 terms the recurrence can be detected.
- The guessed operator has a right factor of order 8 and degree 11.
- We computed $a_{36}, a_{37}, a_{38}, a_{39}$ (!) and found that these terms were correctly predicted by the guessed recurrence.
- George Spahn and Doron Zeilberger designed a new algorithm for enumerating this sequence, and obtained $a_{40}, \ldots, a_{70}$.
- Rintaro Matsuo even got 300 terms by dynamic programming.

A189282

$a_n := \#\{\pi \in S_n \mid \pi(k+3) - \pi(k) \neq 3 \text{ for all } k\}.$

- Only the terms $a_0, \ldots, a_{30}$ were given in the OEIS (by Vaclav Kotesovec).

A189282

$a_n := \#\{\pi \in S_n \mid \pi(k+3) - \pi(k) \neq 3 \text{ for all } k\}.$

- Only the terms $a_0, \ldots, a_{30}$ were given in the OEIS (by Vaclav Kotesovec).
- We computed $a_{31}, \ldots, a_{36}$, taking some weeks of computation time, but could not find a recurrence.

A189282

$a_n := \#\{\pi \in S_n \mid \pi(k+3) - \pi(k) \neq 3 \text{ for all } k\}.$

- Only the terms $a_0, \ldots, a_{30}$ were given in the OEIS (by Vaclav Kotesovec).
- We computed $a_{31}, \ldots, a_{36}$, taking some weeks of computation time, but could not find a recurrence.
- George Spahn and Doron Zeilberger designed a new algorithm for enumerating this sequence, and obtained $a_{37}, \ldots, a_{48}$.

A189282

$a_n := \#\{\pi \in S_n \mid \pi(k+3) - \pi(k) \neq 3 \text{ for all } k\}.$

- Only the terms $a_0, \ldots, a_{30}$ were given in the OEIS (by Vaclav Kotesovec).
- We computed $a_{31}, \ldots, a_{36}$, taking some weeks of computation time, but could not find a recurrence.
- George Spahn and Doron Zeilberger designed a new algorithm for enumerating this sequence, and obtained $a_{37}, \ldots, a_{48}$.
- Still: too little data to guess a recurrence with our method!

A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \ldots$$

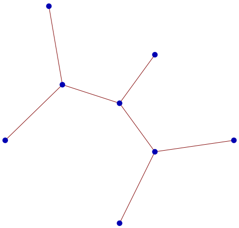
A339987

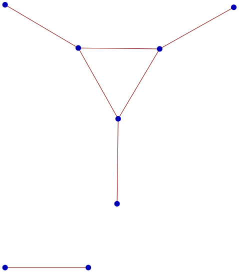
Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \ldots$$

Example: For $n = 4$ there are two unlabeled graphs which yield

$$8 \cdot \binom{7}{2} \cdot 5 \cdot \binom{4}{2} + \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 = 5040 + 3360 = 8400.$$





A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \ldots$$

Example: For $n = 4$ there are two unlabeled graphs which yield

$$8 \cdot \binom{7}{2} \cdot 5 \cdot \binom{4}{2} + \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 = 5040 + 3360 = 8400.$$

Results:

- In the OEIS 40 terms are known.

A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \dots$$

Example: For $n = 4$ there are two unlabeled graphs which yield

$$8 \cdot \binom{7}{2} \cdot 5 \cdot \binom{4}{2} + \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 = 5040 + 3360 = 8400.$$

Results:

- In the OEIS 40 terms are known.
- LLL-guesser finds a recurrence of order 5 and degree 10.

A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \ldots$$

Example: For $n = 4$ there are two unlabeled graphs which yield

$$8 \cdot \binom{7}{2} \cdot 5 \cdot \binom{4}{2} + \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 = 5040 + 3360 = 8400.$$

Results:

- In the OEIS 40 terms are known.
- LLL-guesser finds a recurrence of order 5 and degree 10.
- LA-guessing needs at least 70 terms.

A339987

Number of labeled graphs on $2n$ vertices s.t. $n-1$ vertices have degree 3 and the remaining $n+1$ vertices have degree 1:

$$1, 4, 90, 8400, 1426950, 366153480, 134292027870, \ldots$$

Example: For $n = 4$ there are two unlabeled graphs which yield

$$8 \cdot \binom{7}{2} \cdot 5 \cdot \binom{4}{2} + \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 = 5040 + 3360 = 8400.$$

Results:

- In the OEIS 40 terms are known.
- LLL-guesser finds a recurrence of order 5 and degree 10.
- LA-guessing needs at least 70 terms.
- How difficult would it be to compute more terms?

Minimal Number of Terms for Guessing

Guessing from a Single Term

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\implies p(10000000) = 583000086700003459999802.$$

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\implies p(10000000) = 583000086700003459999802.$$

Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

$$(n+2)D_{n+2} - (6n+9)D_{n+1} + (n+1)D_n = 0. \qquad (*)$$

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\Longrightarrow\ p(10000000) = 583000086700003459999802.$$

Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

$$(n+2)D_{n+2} - (6n+9)D_{n+1} + (n+1)D_n = 0. \qquad (*)$$

- Search space: all recurrences of order 2 and degree 1 with coefficients and initial values $\leqslant 9$ (in absolute value).

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\implies p(10000000) = 583000086700003459999802.$$

Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

$$(n+2)D_{n+2} - (6n+9)D_{n+1} + (n+1)D_n = 0. \qquad (*)$$

- Search space: all recurrences of order 2 and degree 1 with coefficients and initial values $\leqslant 9$ (in absolute value).
- Select those combinations which produce integral terms $a_2, \ldots, a_{20}$, and compare with the D_n sequence:

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\implies p(10000000) = 583000867000345999802.$$

Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

$$(n+2)D_{n+2} - (6n+9)D_{n+1} + (n+1)D_n = 0. \qquad (*)$$

- Search space: all recurrences of order 2 and degree 1 with coefficients and initial values $\leqslant 9$ (in absolute value).
- Select those combinations which produce integral terms $a_2, \ldots, a_{20}$, and compare with the D_n sequence:

n	2	3	4	5	6	7	8	9	$\cdots$	20
#	85994	48240	3056	853	25	8	1	1	$\cdots$	1

Guessing from a Single Term

Fact: A polynomial can be interpolated from a single value, if a bound on the coefficients is known, e.g.:

$$p(x) = 583x^3 + 867x^2 + 346x - 198$$
$$\implies p(10000000) = 583000086700003459999802.$$

Toy example: The central Delannoy numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy a recurrence of order 2 and degree 1:

$$(n+2)D_{n+2} - (6n+9)D_{n+1} + (n+1)D_n = 0. \qquad (*)$$

- Search space: all recurrences of order 2 and degree 1 with coefficients and initial values $\leqslant 9$ (in absolute value).
- Select those combinations which produce integral terms $a_2, \ldots, a_{20}$, and compare with the D_n sequence:

n	2	3	4	5	6	7	8	9	$\cdots$	20
#	85994	48240	3056	853	25	8	1	1	$\cdots$	1

- Thus, $(*)$ can be recovered from the single term $D_8 = 265729$.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.
- In practice, it often succeeds to reduce the required number of sequence terms to 50% and less.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.
- In practice, it often succeeds to reduce the required number of sequence terms to 50% and less.
- For example, the recurrence for A172671 could be guessed using 24 terms, while LA requires at least 75 terms.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.
- In practice, it often succeeds to reduce the required number of sequence terms to 50% and less.
- For example, the recurrence for A172671 could be guessed using 24 terms, while LA requires at least 75 terms.
- We identified trustworthy, yet-unknown recurrences for several entries in the OEIS, which could not be found otherwise.

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.
- In practice, it often succeeds to reduce the required number of sequence terms to 50% and less.
- For example, the recurrence for A172671 could be guessed using 24 terms, while LA requires at least 75 terms.
- We identified trustworthy, yet-unknown recurrences for several entries in the OEIS, which could not be found otherwise.
- The output of the algorithm is a guess and thus non-rigorous. Independent verification is necessary to obtain a theorem!

Conclusion

- We designed a method for guessing recurrence equations, using fewer terms than classical LA methods.
- Our algorithm is applicable to medium-sized examples where computing more terms is prohibitively expensive.
- In practice, it often succeeds to reduce the required number of sequence terms to 50% and less.
- For example, the recurrence for A172671 could be guessed using 24 terms, while LA requires at least 75 terms.
- We identified trustworthy, yet-unknown recurrences for several entries in the OEIS, which could not be found otherwise.
- The output of the algorithm is a guess and thus non-rigorous. Independent verification is necessary to obtain a theorem!
- We were not able to find a recurrence for the notorious Av(1324) sequence. . .