Picard-Fuchs equations for Feynman integrals

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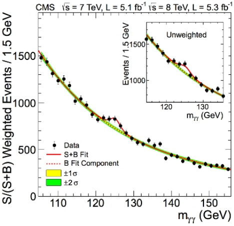
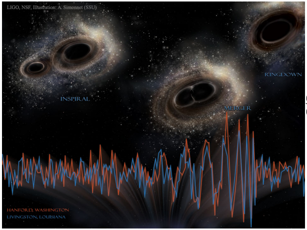
& &

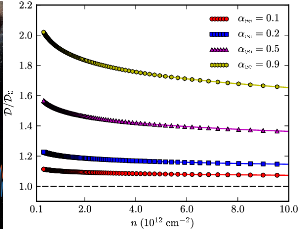
Joint PolSys–SpecFun Seminar

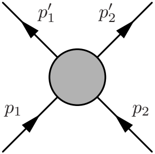
based on work in progress with

Charles Doran, Andrey Novoseltsev









Scattering amplitudes are the fundamental tools for making contact between quantum field theory description of nature and experiments

- Comparing particule physics model against datas from accelators
- Post-Minkowskian expansion for Gravitational wave physics
- Various condensed matter and statistical physics systems

Feynman Integrals: parametric representation

Feynman integral are given by projective space integrals

$$I_\Gamma(\underline{\nu}, D; \underline{s}, \underline{m}) = \int_{\Delta_n} \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega-\frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\omega}} \prod_{i=1}^{n} x_i^{\nu_i-1}\,\Omega_0 \qquad \omega = \sum_{i=1}^{n} \nu_i - \frac{D}{2}$$

with the volume form on $\mathbb{P}^{n-1}$

$$\Omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0, \dots, x_n \geqslant 0 | [x_1, \dots, x_n] \in \mathbb{P}^{n-1}\}$$

Feynman Integrals: parametric representation

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times X(\underline{m}^2; \underline{x}) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

- Homogeneous polynomial of degree L with $u_{a_1,\dots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L\\ 0\leqslant a_i\leqslant 1}} u_{a_1,\dots,a_n} \prod_{i=1}^n x_i^{a_i}$$

- the hyperplane

$$X(\underline{m}^2; \underline{x}) := \sum_{n=1}^n m_i^2 x_i$$

- Homogeneous polynomial of degree $L+1$

$$\mathcal{V}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L+1\\ 0\leqslant a_i\leqslant 1}} S_{a_i,\cdots,a_n} \prod_{i=1}^n x_i^{a_i}$$

Feynman Integrals: parametric representation

The integrand is an algebraic differential form in $H^{n-1}(\mathbb{P}^{n-1}\backslash\mathbb{X}_\Gamma)$ on the complement of the graph hypersurface

$$\mathbb{X}_\Gamma := \{\mathcal{U}_\Gamma(\underline{x}) \times \mathcal{F}_\Gamma(\underline{x}) = 0, \underline{x} \in \mathbb{P}^{n-1}\}$$

- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities

Feynman integral and periods

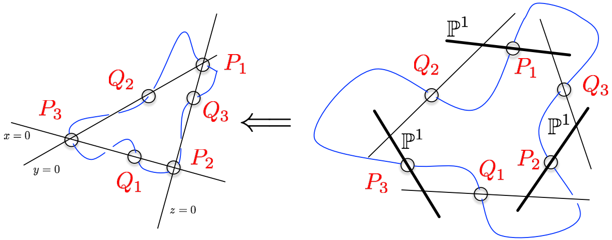
The domain of integration Δ_n is not an homology cycle because

$$\partial\Delta_n \cap \mathbb{X}_\Gamma = \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$$

we have to look at the relative cohomology

$$H^\bullet(\mathbb{P}^{n-1}\backslash X_\Gamma; \Pi_n \backslash \Pi_n \cap \mathbb{X}_\Gamma)$$

The normal crossings divisor $\Pi_n := \{x_1\cdots x_n = 0\}$ and $\mathbb{X}_\Gamma$ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



Differential equation

The Feynman integral *are* period integrals of the relative cohomology after performing the appropriate blow-ups

$$\mathfrak{M}(\underline{s},\underline{m}^2) := H^\bullet(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{X_F};\widetilde{\Pi_n}\backslash\widetilde{\Pi_n}\cap\widetilde{X_\Gamma})$$

Since the integrand varies with the physical variables $\{S_{\underline{a}^i}, m_1^2, \dots, m_n^2\}$ one needs to study a variation of (mixed) Hodge structure

One can show that the Feynman integral are **holonomic D-finite functions** [Bitoun et al.;Smirnov et al.]

A Feynman integrals satisfies inhomogenous differential equations with respect to any set of variables $\underline{z} \in \{S_{\underline{a}^i}, m_1^2, \dots, m_n^2\}$

$$\mathcal{L}_{PF}\, I_\Gamma = \mathcal{S}_\Gamma$$

Generically there is an inhomogeneous term $\mathcal{S}_\Gamma \neq 0$ due to the boundary components $\partial\Delta_n$

Feynman integral D-module

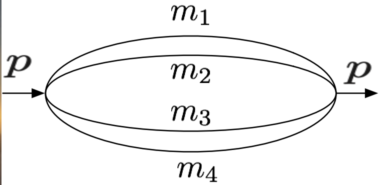
We want to address the questions of how to derive

- How can we derive efficiently the complete system of differential equations (i.e. the minimal order PF)

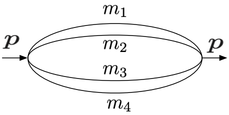
$$\mathcal{L}_{\rm PF}\, I_\Gamma = \mathcal{S}_\Gamma$$

- Of which geometry the Feynman integral are period integrals?
- Understand the algebraic geometry that determines the motive $\mathfrak{M}(\underline{s}, \underline{m}^2)$ leading to the above (D-module) of system of differential equations ?

The sunset graphs family



The sunset family of graph



The graph polynomial for the $n-1$-loop sunset with $\omega = D/2 = 1$

$$\mathcal{F}_n^{\ominus}(\underline{x}) = x_1 \cdots x_n \left(\phi_n^{\ominus}(\underline{x}) - p^2\right);\ \phi_n^{\ominus}(\underline{x}) = \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\left(m_1^2 x_1 + \cdots + m_n^2 x_n\right)$$

The Feynman integral in $D = 2$ is convergent

$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_{x_1 \geqslant 0, \ldots, x_n \geqslant 0} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

The sunset integrals and L-function values

For the special value $p^2 = m_1^2 = \cdots = m_n^2 = 1$ the sunset Feynman integral becomes a pure period integral [Bloch, Kerr, Vanhove]

$$I_n^{\ominus}(1,\ldots,1) = \int_{x_i \geqslant 0} \frac{\prod_{i=1}^{n-1} d\log x_i}{1 - \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)(x_1 + \cdots + x_n)}$$

- Using impressive numeric experimentations [Broadhust] found that $I_n^{\ominus}(1,\ldots,1)$ is given by L-function values in the critical band.
- For large n the L-function are from moments Kloosterman sums over finite fields

The sunset integrals and L-function values

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These special values realise explicitly Deligne's conjecture relating period integrals to L-values in the critical band

$n = 3$: elliptic curve case : $I_3^{\ominus}(1,\ldots,1) = \frac{1}{2}\zeta(2)$
$n = 4$: $K3$ Picard rank 19 : $I_4^{\ominus}(1,\ldots,1) = \frac{12\pi}{\sqrt{15}} L(f_{K3}, 2)$ [Bloch, Kerr, Vanhove]

- $L(f_{K_3}, s)$ is the L-function of $H^2(K3, \mathbb{Q}_\ell)$ [Peters, Top, v. der Vlugt]
- Functional equation $L(f_{K3}, s) \propto L(f_{K3}, 3 - s)$
- $f_{K3} = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n} q^{m^2+4n^2+mn}$

The classical sunset period integrals

We can consider the period integral by changing the domain of integration to the torus $\mathbb{T}_n = \{|x_1| = \cdots = |x_n| = 1\}$

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}_n} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

is given by the series in terms of generalized Apéry numbers near $p^2 = \infty$

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \sum_{m \geqslant 0} (p^2)^{-1-m} \sum_{r_1+\cdots+r_n=m} \left(\frac{(r_1 + \cdots + r_n)!}{r_1! \cdots r_n!} \right)^2 \prod_{i=1}^{n} (m_i^2)^{r_i}$$

The series has been studied in the past by [Verrill].

The classical sunset period integrals

The Feynman integral for $0 \leqslant p^2 \leqslant (m_1 + \cdots + m_n)^2$

$$I_n^{\ominus}(p^2, \underline{m}^2) = 2^{n-1} \int_0^{\infty} u I_0(\sqrt{p^2} u) \prod_{i=1}^{n} K_0(m_i u) du$$

The classical period for $p^2 \geqslant (m_1 + \cdots + m_n)^2$

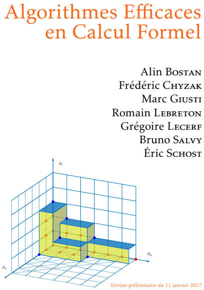
$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2} \int_0^{\infty} u K_0(\sqrt{p^2} u) \prod_{i=1}^{n} I_0(m_i u) du$$

where we have the modified Bessel function of the first kind

$$I_0(z) = \frac{1}{2i\pi} \int_{|t|=1} e^{-\frac{z}{2}(t+\frac{1}{t})} d\log t; \qquad K_0(z) = \int_0^{+\infty} e^{-\frac{z}{2}(t+\frac{1}{t})} d\log t$$

There are exponential period integrals in the sense of the non classical exponential motives of [Fresán, Jossen]. Fascinating quadratic relations satisfied by the Bessel moments generalizing Riemann identity [Broadhurst, Roberts; Zhou; Fresán, Jossen, Sabbah, Yu]

Creative Telescoping



A Fast Approach to Creative Telescoping

Christoph Koutschan

Abstract. In this note we reinvestigate the task of computing creative telescoping relations in differential-difference operator algebras. Our approach is based on an ansatz that explicitly includes the denominators of the delta parts. We contribute several ideas of how to make an implementation of this approach reasonably fast and provide such an implementation. A selection of examples shows that it can be superior to existing methods by a large factor.

Mathematics Subject Classification (2010). Primary 68W30; Secondary 33F10.

Keywords. holonomic functions, special functions, symbolic integration, symbolic summation, creative telescoping, Ore algebra, WZ theory.

1. Introduction

The method of creative telescoping nowadays is one of the central tools in computer algebra for attacking definite integration and summation problems. Zeilberger with his celebrated holonomic systems approach [17] was the first to recognize its potential for making these tasks algorithmic for a large class of functions. In the realm of holonomic functions, several algorithms for computing creative telescoping relations have been developed in the past. The methodology described here is not an algorithm in the strict sense because it involves some heuristics. But since it works pretty well on nontrivial examples we found it worth to be written down. Additionally we believe that it is the method of choice for really big examples. Our implementation is contained in the Mathematica package `HolonomicFunctions` as the command `FindCreativeTelescoping`. The package can be downloaded from the RISC combinatorics software webpage:

http://www.risc.uni-linz.ac.at/research/combinat/software/

Throughout this paper we will work in the following setting. We assume that a function f to be integrated or summed satisfies some linear difference-differential relations which we represent in a suitable operator algebra (Ore algebra). We use the symbol D_x to denote the derivation operator w.r.t. x and S_n for the shift operator w.r.t. n. Such an algebra can be viewed as a polynomial ring in the respective operators, with coefficients being rational functions in the corresponding variables, subject to the commutation rules $D_x x = x D_x + 1$ and $S_n n = n S_n + S_n$. Ideally, all the relations for f generate a ∂-finite left ideal, i.e., a zero-dimensional left ideal in the operator algebra. If additionally f is holonomic (a notion that can be made formal by D-module theory), then the existence of creative telescoping relations is guaranteed by theory (i.e., by the elimination property of holonomic modules). Chyzak, Kauers, and Salvy [6] have shown that creative telescoping is also possible for higher-dimensional ideals under certain conditions. We tacitly assume that any input to a creative

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We want to derive the differential equation

$$\mathcal{L}_{\rm PF} \int_\Gamma f_\Gamma(\underline{S}, \underline{m}^2; \underline{x})\Omega_0 = \mathcal{S}_\Gamma$$

For a given subset of the physical parameters $\underline{z} := (z_1, \ldots, z_r) \subset \{\underline{S}, \underline{m}^2\}$ we want to construct a differential operator $T_{\underline{z}}$ such that

$$T_{\underline{z}}\Omega_\Gamma = 0$$

such that

$$T_{\underline{z}} = \mathcal{L}_{\rm PF}(\underline{S}, \underline{m}^2, \underline{\partial_z}) + \sum_{i=1}^{n} \partial_{x_i} Q_i(\underline{S}, \underline{m}^2 \underline{\partial_z}; \underline{x}, \underline{\partial_x})$$

where the finite order differential operator

$$\mathcal{L}_{\rm PF}(\underline{S},\underline{m}^2,\partial_{\underline{z}}) = \sum_{\substack{0\leqslant a_i\leqslant o_i\\ 1\leqslant i\leqslant r}} p_{a_1,\dots,a_r}(\underline{S},\underline{m}^2)\prod_{i=1}^{r}\left(\frac{d}{dz_i}\right)^{a_i}$$

$$Q_i(\underline{S},\underline{m}^2,\partial_{\underline{z}}) = \sum_{\substack{0\leqslant a_i\leqslant o'_i\\ 1\leqslant i\leqslant r}}\sum_{\substack{0\leqslant b_i\leqslant \tilde o_i\\ 1\leqslant i\leqslant n}} q^{(i)}_{a_1,\dots,a_r}(\underline{S},\underline{m}^2,\underline{x})\prod_{i=1}^{r}\left(\frac{d}{dz_i}\right)^{a_i}\prod_{i=1}^{n}\left(\frac{d}{dx_i}\right)^{b_i}$$

- The orders o_i, o'_i, $\tilde{o}_i$ are positive integers
- $p_{a_1,\dots,a_r}(\underline{S},\underline{m}^2)$ polynomials in the kinematic variables
- $q^{(i)}_{a_1,\dots,a_r}(\underline{S},\underline{m}^2,\underline{x})$ rational functions in the kinematic variable and the projective variables $\underline{x}$.

Integrating over a cycle γ gives

$$0 = \oint_\gamma T_{\underline{z}} f_\Gamma \Omega_0 = \mathcal{L}_{\rm PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma f_\Gamma \Omega_0 + \oint_\gamma d\beta_\Gamma$$

For a cycle $\oint_\gamma d\beta_\Gamma = 0$ we get

$$\mathcal{L}_{\rm PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma f_\Gamma \Omega_0 = 0$$

For the Feynman integral I_Γ we have

$$0 = \int_{\Delta_n} T_{\underline{z}} f_\Gamma \Omega_0 = \mathcal{L}_{\rm PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma + \int_{\Delta_n} d\beta_\Gamma$$

since $\partial\Delta_n \neq \emptyset$

$$\mathcal{L}_{\rm PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma = \mathcal{S}_\Gamma$$

This can done using the creative telescoping method introduced by Doron Zeilberger (1990) and the algorithm by F. Chyzak because the Feynman integral are D-finite [Bitoun, Bogner, Klausen, Panzer]

This works in all case even when the graph hypersurface does not have isolated singularities (which is the generic case)

This algorithm gives the D-module of annihilator and with the inhomogeneous term

We can use the Creative Telescoping algorithm for exploring the properties of the Feynman integral. This gives some very useful insight in the underlying algebraic geometry (order of the PF operators, their singularities, etc.)

Application: the multiloop sunset integral in $D = 2$

In the case of the sunset integral in two dimensions the Bessel representation is a one-dimensional integral $p^2 < (m_1 + \cdots + m_n)^2$

$$I_n(p^2, \underline{m}^2) = 2^{n-1} \int_0^\infty x I_0(\sqrt{p^2} x) \prod_{i=1}^{n} K_0(m_i x)\, dx\,,$$

and the classical period integral for $p^2 > (m_1 + \cdots + m_n)^2$

$$\pi_n(p^2, \underline{m}^2) = 2^{n-1} \int_0^\infty x K_0(\sqrt{p^2} x) \prod_{i=1}^{n} I_0(m_i x)\, dx\,,$$

The Bessel functions I_0 and K_0 have the same annihilator.
In this case the telescoper reads with
$\underline{z} = \{z_1, \ldots, z_r\} \subset \{p^2, m_1^2, \ldots, m_n^2\}$

$$T_z = \mathcal{L}_{\rm PF}\left(p^2, \underline{m}, \frac{d}{d\underline{z}}\right) + \frac{d}{dx} Q\left(p^2, \underline{m}^2, x, \frac{d}{dx}, \frac{d}{d\underline{z}}\right)$$

The motivic geometry

Sunset graphs toric variety $X_{p^2}(A_n)$ [Verrill]

The sunset graph polynomial

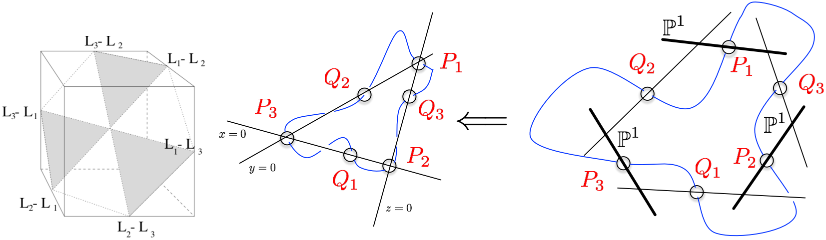
$$\mathcal{F}_n^{\ominus} = x_1 \cdots x_n \left(\left(\sum_{i=1}^{n} m_i^2 x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - p^2 \right)$$

is a character of the adjoint representation of A_{n-1} with support on the polytope generated by the A_{n-1} root lattice

- The Newton polytope Δ_n for $\mathcal{F}_n^{\ominus}$ is reflexive with only the origin as interior point
- The toric variety $X(A_{n-1})$ is the graph of the Cremona transformations $X_i \to 1/X_i$ of $\mathbb{P}^{n-1}$
  $X(A_{n-1})$ is obtained by blowing up the strict transform of the points, lines, planes etc. spanned by the subset of points $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, ..., $(0, \ldots, 0, 1)$ in $\mathbb{P}^{n-1}$

Two-loop Sunset toric variety $X(A_2)$

$$(m_1^2x_1 + m_2^2x_2 + m_3^2x_3)(x_1x_2 + x_1x_3 + x_2x_3) = p^2x_1x_2x_3$$



- The toric variety is $X(A_2) = Bl_3(\mathbb{P}^2) = dP_6$ blown up at 3 points
- The subfamily of anticanonical hyperspace is non generic
  The combinatorial structure of the **NEF** partition describes precisely the mass deformations
- True for all n

Sunset graphs pencils of variety $\mathcal{X}_{p^2}(A_n)$ [Verrill]

For $p^2 \in \mathbb{P}^1$ we define the pencil in the ambient toric variety $X(A_{n-1})$

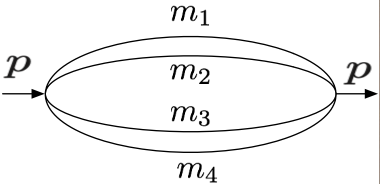
$$\mathcal{X}_{p^2}(A_{n-1}) = \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_{n-1}) | x_1 \cdots x_n \left(\sum_{i=1}^{n} m_i^2 x_i\right) \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 x_1 \cdots x_n = 0\}$$

The fiber at $p^2 = \infty$ is $\Pi_n = \{x_1 \cdots x_n = 0\}$

Since Π_n is linearly equivalent to the anti-canonical divisor of $X(A_{n-1})$ the family has trivial canonical divisor: We have a family of (singular) Calabi-Yau $n-2$-fold

This is specific to this family of associated with root lattice of A_n

The Iterative fibration



The Iterative fibration

The sunset family $\left(\sum_{i=1}^{n} m_i^2 x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 = 0$ is birational to a complete intersection variety in $\mathbb{P}^n$

$$\frac{1}{x_0} + \sum_{i=1}^{n} \frac{1}{x_i} = 0; \qquad p^2 x_0 + \sum_{i=1}^{n} m_i^2 x_i = 0$$

Obviously $X(A_{n-1})$ is obtained from $X(A_{n-2})$ with the substitutions

$$\frac{1}{x_{n-1}} \to \frac{1}{x_{n-1}} + \frac{1}{x_n}; \qquad m_{n-1}^2 x_{n-1} \to m_{n-1}^2 x_{n-1} + m_n^2 x_n$$

$X(A_{n-1})$ is fibrered over $X(A_1) = \mathbb{P}^1$ with generic fibers $X(A_{n-2})$

$$X(A_{n-2}) \to X(A_{n-1}) \to X(A_1) = \mathbb{P}^1$$

The Iterative fibration

The geometric phenomenon at work that the n-loop sunset corresponds to a family of Calabi-Yau $(n-1)$-folds each of which is a double cover of the (rational) total space of a family of $(n-1)$-loop sunset Calabi-Yau $(n-2)$-folds.

At the level of the integrals this

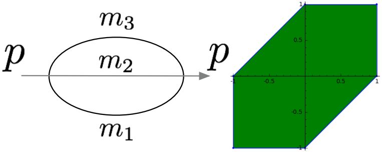
$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_0^{+\infty} I_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

and for the classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2i\pi}\int_{|t|=1} \pi_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

This construction allows to understand the geometry and build the PF operator for all loop orders [Doran, Novoseltsev, Vanhove]

The two-loop sunset graph [Bloch, Kerr, Vanhove]



The pencil of sunset elliptic curve

$$\mathcal{X}_{p^2}(A_2) = \{(p^2, \underline{x}) \in \mathbb{P}^2 \times X(A_2) | (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) = p^2 x_1 x_2 x_3\}$$

The j-invariant is

$$j_{\ominus}(p^2, \underline{m}^2) = \frac{\left(\prod_{i=1}^4 (p^2 - \mu_i^2) + 16 p^2 \prod_{i=1}^3 m_i^2\right)^3}{(p^2)^2 \prod_{i=1}^3 m_i^4 \prod_{i=1}^4 (p^2 - \mu_i^2)}$$

with $\mu_i^2 = (\pm m_1 \pm m_2 \pm m_3)^2$

The two-loop sunset graph [Bloch, Kerr, Vanhove]

The j-invariant is

$$j_\ominus(p^2,\underline{m}^2) = \frac{\left(\prod_{i=1}^4(p^2-\mu_i^2)+16p^2\prod_{i=1}^3 m_i^2\right)^3}{(p^2)^2\prod_{i=1}^3 m_i^4\prod_{i=1}^4(p^2-\mu_i^2)}$$

The fibers types are

- Generic case $m_1\neq m_2\neq m_3$

$$I_2(0)+I_6(\infty)+4I_1(\mu_i^2); \qquad \mu_i^2=(\pm m_1\pm m_2\pm m_3)^2$$

- single mass $m_1=m_2=m_3\neq 0$: modular curve $X_1(6)$

$$I_2(0)+I_6(\infty)+I_3(m^2)+I_1(9m^2)$$

The Feynman integral is an elliptic dilogarithm [Bloch, Kerr,Vanhove]

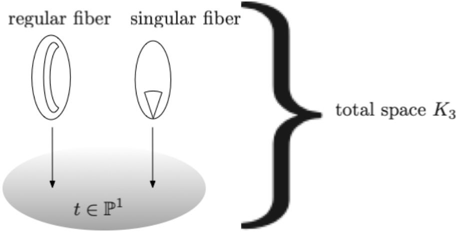
$$H^2(\mathbb{P}^2\backslash\{x_1x_2x_3=0\},\mathbb{X}_\ominus,\mathbb{Q}(2))$$

The 3-loop case : pencil of $K3$

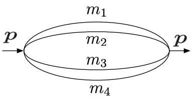
$$\mathcal{X}_{p^2}(A_3) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_3) | \left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4\right) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_4}\right) = p^2\}$$

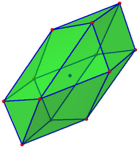
The graph hypersurface defines a $K3$ hypersurface

By the iteration we know that this $K3$ is elliptically fibered with for fibers given by the sunset elliptic curve



The 3-loop case : pencil of $K3$ [Doran, Novoseltsev, Vanhove]





$$\mathcal{X}_{p^2}(A_3) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_3) | \left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4\right)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_4}\right) = p^2\}$$

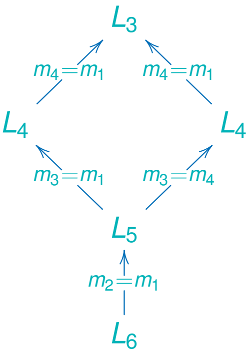
Generic anticanonical $K3$ hypersurface in the toric threefold X_{Δ° has Picard rank 11

The physical locus for the sunset has at least Picard rank 16

masses	fibers	Mordell-Weil	Picard rank
(m_4, m_1, m_2, m_3)	$8I_1 + 2I_2 + 2I_6$	2	16
$(m_4 = m_1, m_2, m_3)$	$8I_1 + I_4 + 2I_6$	2	17
$(m_4, m_1, m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	1	17
$(m_4 = m_1, m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	1	18
$(m_4 = m_1 = m_2, m_3)$	$8I_1 + I_4 + 2I_6$	3	18
$(m_4, m_1 = m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	2	18
$(m_4 = m_1 = m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	2	19

$|Pic| = 19$ motive of an elliptic 3-log $H^3(\mathbb{P}^3 \backslash \Delta_4, \mathbb{X}_4, \mathbb{Q}(3))$ [Bloch, Kerr, Vanhove]

The Picard-Fuchs operator



$$L_r = (\alpha \frac{d}{dp^2} + \beta) \circ L_{r-1}$$

The Picard-Fuchs operators for the Feynman integral for general parameters $m_4 \neq m_1 \neq m_2 \neq m_3$

$$L_6 = \sum_{r=0}^{6} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 6 and degree 25

$$q_6(p^2) = \tilde{q}_6(p^2) \times \prod_{\epsilon_i = \pm 1} (p^2 - (\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3 + \epsilon_4 m_4)^2)$$

with $\tilde{q}_6(p^2)$ degree 17 with apparent singularities

The 4-loop case : pencil of CY 3-fold

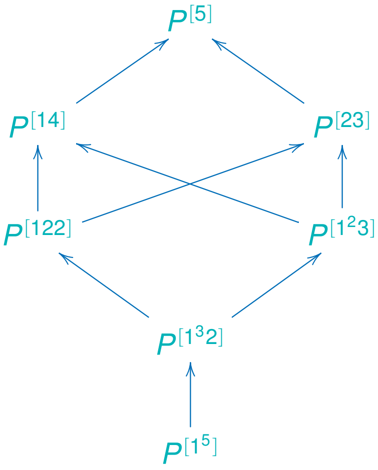
$$\mathcal{X}_{p^2}(A_4) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_4) | \left(m_1^2 x_1 + \cdots + m_5^2 x_5\right) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_5}\right) = p^2\}$$

This gives a pencil of nodal Calabi-Yau 3-fold

For a (small or big) resolution $\hat{W}$ is

- $h^{12}(\hat{W}) = 5$ for the 5 masses case : 30 nodes
- $h^{12}(\hat{W}) = 1$ for the 1 mass case $m_1 = \cdots = m_5$: 35 nodes
- $h^{12}(\hat{W}) = 0$ for $p^2 = m_1 = \cdots = m_5 = 1$: rigid case birational to the Barth-Nieto quintic
  - $I_5^{\ominus}(1, \ldots, 1) = 48\zeta(2)L(f, 2)$ [Broadhurst]
  - f weight 4 and level 6 modular form $f = (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$
  - This L-series is precisely the one for $H^3(X(A_4), \mathbb{Q}_\ell)$ [Verrill]
  - Functional equation $L(f, s) \propto L(f, 4 - s)$
  - Again we have a manifestation of Deligne's conjecture

The 4-loop case : pencil of CY 3-fold



The Picard-Fuchs operators for the Feynman integral for general parameters $m_1 \neq \cdots \neq m_5$

$$L_{12} = \sum_{r=0}^{12} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 12 and degree 121

The one identifies two masses the order of the differential operator decreases by 2

$$L_{12} \to L_{10} \to L_8 \to L_6 \to L_4$$

$$L_r = \left(\alpha \left(\frac{d}{dp^2}\right)^2 + \beta \frac{d}{dp^2} + \gamma\right) \circ L_{r-2}$$

Conclusion

- We have put forward a new approach for deriving the differential equation for Feynman integrals
- We have explained that the sunset graph have a natural nested Calabi-Yau structure allowing to understand they geometry easily

Generic Feynman graphs is more intricate

- For Feynman graph with $\deg(\mathcal{F})_\Gamma = L$ in $\mathbb{P}^n$ with $n > L + 1$ we do not have a Calabi-Yau
  - Two-loop motivic elliptic curve for the Hodge structure [Bloch, Doran, Kerr, Vanhove (work in progress)]: natural classification of the master integral topologies and algebraic geometry of del Pezzo surfaces