Stieltjes Moment Problem: Constructive approach to unique and non-unique solutions

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Plan

1. Definition of Stieltjes moment problem
2. Unique and non-unique solutions
3. Analysis of Polynomial Killers
4. Construction of non-unique solutions
   a) $\Gamma(an+b)$; b) e^{n^2}
5. Stieltjes Classes, open problems

MOMENT PROBLEM

Given set of $\varrho(n)$, $n = 0, 1, 2 \ldots \infty$
find $W(x) \geq 0$, such that

- $$\int_0^R x^n W(x)\,dx = \varrho(n)$$

$$R = \lim_{n\to\infty} [\varrho(n)]^{1/n}$$

$R < \infty \rightarrow$ Hausdorff problem
$R = \infty \rightarrow$ Stieltjes problem

Mellin Transform of $W(x)$

$$\mathcal{M}[W(x); s] = \int_0^\infty x^{s-1} W(x)\,dx$$

Solution of Stieltjes Moment problem:

- $$W(x) = \mathcal{M}^{-1}[\varrho(s-1); x]$$

$$W(x) > 0 \Longleftrightarrow \left\{ \begin{array}{l} \det(\varrho(i+j-2))_{1\leq i,j\leq n} \\ \det(\varrho(i+j-1))_{1\leq i,j\leq n} \end{array} \right\} > 0$$

for all n

Hankel dets

(... hopeless)

Polynomial Killers – $\omega(x)$, $x \geq 0$
– Functional characterisation

Functional Space

- $S = S(R_+) = \{\omega(x) : \omega \in C^\infty(R_+),$
$\lim_{x \to 0,\infty} x^l \omega^{(m)}(x) = 0;\ l, m = 0, 1, 2 \ldots\}$
- $\Phi = \Phi(R_+) = \{\omega(x) \in S,\ \int_0^\infty x^k \omega(x)\,dx = 0,$
$k = 0, 1, 2 \ldots\}$

Φ = Lizorkin Space (Russian Literature)

All of them: **BOUNDED functions**

Example: $\boxed{\omega(x) = \exp\left(-\frac{\ln^2(x)}{4}\right) \sin\left(\frac{\pi}{2} \ln(x)\right)}$

Mellin transform of $\omega(x)$ vanishes at integers $s = 1, 2, 3 \ldots$:

$$\int_0^\infty x^{s-1} \omega(x)\,dx = 2\sqrt{\pi} \exp\left(s^2 - \frac{\pi^2}{4}\right) \sin(\pi s)$$

(... Stieltjes, Heyde)

Attention: a particularity —

$\omega(x)$ has all, positive and negative, moments equal to zero!

$$\int_0^\infty x^k \omega(x)\,dx = 0, \quad k = -\infty \dots \infty$$

Is this an exceptional phenomenon? Can we generate other polynomial killers?

The space of polynomial Killers (PK) is very rich: (theorem)

a) take a PK $\omega(x)$ from Φ;

b) and $\psi(x) \in S$

c) convolve $\omega(x)$ with $\psi(x)$

$$\tilde{\omega}(x) \equiv \int_0^\infty \underset{\Phi}{\omega\left(\frac{x}{t}\right)} \underset{S}{\psi(t)} \frac{1}{t}\,dt$$ is again PK.

How to adapt given PK to a given moment problem?

Reminder:

Direct Mellin inversion

$$f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x)\,dx$$

$$f(\underline{x}) = \mathcal{M}^{-1}[f^*(s); x] = \underbrace{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f^*(s)\,\underline{x}^{-s}\,ds}$$

Done explicitly for:

I) $f_I^*(s) = \frac{\Gamma(as+b-a)}{\Gamma(b)} \sin(\pi K s)$; $a>0$, $K=\pm 1 \ldots$, $b \in \mathbb{R}$

(J.-M. Sixdeniers, KAP, A.I. Solomon, J. Phys. A 32, 7543 (1999))

II) $f_{II}^*(s) = q^{-a(s-1)^2 - b(s-1)} \sin(\pi K s)$, $0<q\leq 1$, $a>0$, $b\in\mathbb{R}$

(parametrized lognormal)

(J. M. Sixdeniers, Thesis PhD, Univ. P6, 2001)

$\Longrightarrow$ For distributions with moments $\Gamma(an+b)$ and q^{-an^2-bn} the polynomial Killers have been obtained.

Attempting to construct Polynomial Killers $\boxed{\omega(x)}$

Idea: a) write down moments $\tilde{\varrho}(s-1)$ which vanish for integer values of s: "∞" of possibilities!

b) try to make a link with "initial" moments $\varrho(s-1)$

c) try to Mellin invert the moments $\tilde{\varrho}(s-1)$

$$\Longrightarrow \int_0^\infty x^n W_0(x)\,dx = \varrho(n) \quad \text{– "initial" moments}$$

$$\int_0^\infty x^{s-1}\omega(x)\,dx = \varrho(s-1)\cdot\sin(\pi k s) = \tilde{\varrho}(s-1)$$

Polynomial Killer (not positive)

$$\Longrightarrow \omega(x) = \mathcal{M}^{-1}[\varrho(s-1)\sin(\pi k s); x]$$

? convolution ?

No! since $\sin(\pi k s)$ is not a Mellin trans. of any function.

Consult: I. N. Sneddon, The use of Integral Transforms (1972)

Analytical Forms of Polynomial Killers

I: originating from moments:

$$q(n) = q_{a,b}(n) = \Gamma(an+b)/\Gamma(b)$$

$a > 2, \; b > 0$

$\omega_I(a,b,k,x) \sim$

$$\sim x^{\frac{b-a}{a}} \exp(-x^{\frac{1}{a}}) \cdot \sin\left(\pi k \left(\frac{a-b}{a}\right) + x^{\frac{1}{a}} \tan\left(\frac{\pi k}{a}\right)\right)$$

$a > 2|k|$

$k = \pm 1, \pm 2, \ldots$

Comments:

Numerical Prefactor does not matter

Non-uniciity relevent for $a \geq 3$

Principal solution:

$$W_0 = \frac{x^{(b-a)/a} \exp(-x^{\frac{1}{a}})}{\alpha \Gamma(\beta)} = W_0(x)$$

It factors out in the formula for ω_I.

$$\int_0^\infty x^n \omega_I(a,b,k,x)\, dx \equiv 0, \quad n = 0,1,2\ldots$$

Some concrete examples:

$$x^{1/3}\exp(-x^{1/3})\cos\left(\frac{\pi}{6}+\sqrt{3}\,x^{1/3}\right)$$

$$\exp(-x^{1/4})\sin(x^{1/4})$$

$$\exp(-x^{\frac{1}{5}})\sin\left(\frac{\pi}{5}+\tan\left(\frac{\pi}{5}\right)x^{1/5}\right)\frac{1}{x^{1/5}}$$

$$\exp(-x^{1/6})\sin\left(\frac{\pi}{3}+\frac{1}{\sqrt{3}}x^{1/6}\right)\frac{1}{x^{1/3}}$$

— * —

$$x^{2/3}\exp(-x^{1/3})\sin\left(\frac{\pi}{3}+\sqrt{3}\,x^{1/3}\right)$$

$$x^{1/4}\exp(-x^{1/4})\cos\left(\frac{\pi}{4}+x^{1/4}\right)$$

$$\exp(-x^{1/5})\sin\left(x^{1/5}\cdot\tan\left(\frac{\pi}{5}\right)\right)$$

$$\vdots$$

Analytical Forms of Polynomial Killers

II: originating from moments

$$\varrho_{a,b,q}(n) = q^{-an^2-bn}, \quad 0<q\leq 1, \quad a>0, \quad b\in\mathbb{R}$$

(reparametrized lognormal distribution)

$$\omega_{\mathrm{II}}(a,b,q,x) =$$

$$\sim \exp\left(-\frac{\ln^2(x)}{4a\ln(\frac{1}{q})}\right)\cdot\frac{\sin\left(\frac{\pi}{2a}(2a-b)+\frac{\pi\ln(x)}{2a\ln(1/q)}\right)}{x^{(2a-b)/2a}}$$

Principal solution:

$$\sim \exp\left(-\frac{\ln^2(x)}{4a\ln(\frac{1}{q})}\right) x^{\frac{b-2a}{2a}} \text{ factors out!}$$

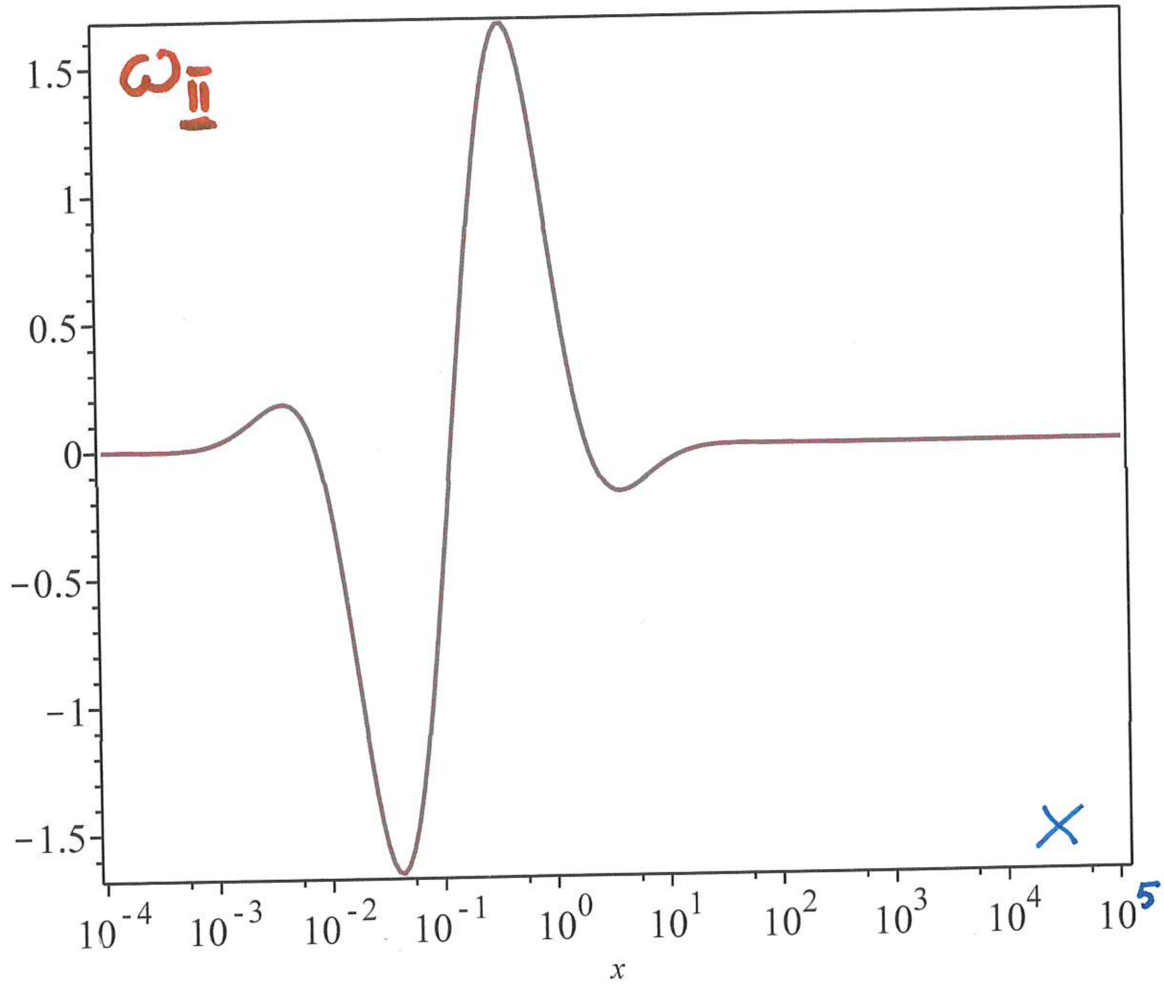
$$\int_0^\infty x^n \omega_{\mathrm{II}}(a,b,q,x)\,dx \equiv 0, \quad n=0,1,\ldots$$

Example:

$$\omega_{\mathrm{II}}(2,1,\tfrac{1}{2},x) = \frac{\exp\left(-\frac{\ln^2(x)}{8\ln(2)}\right)}{x^{3/4}}\cos\left(\frac{\pi}{4}+\frac{\pi\ln(x)}{4\ln(2)}\right)$$

Polynomial Killer
related to extended lognormal

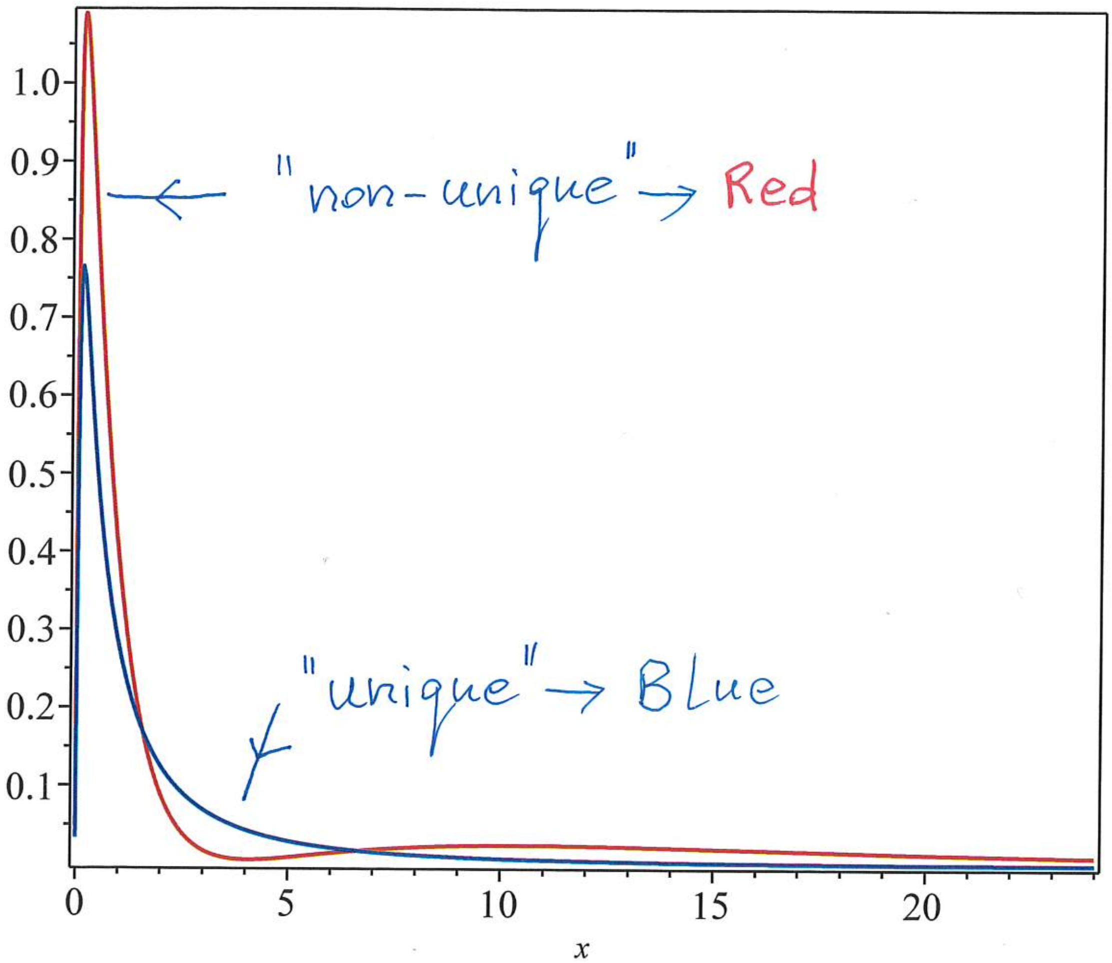
$$\omega_{\underline{II}}(2,1,1/2,x)=\frac{\exp\left(-\frac{\ln^2(x)}{8\ln(2)}\right)}{x^{3/4}}\cos\left(\frac{\pi}{4}+\frac{\pi\ln(x)}{4\ln(2)}\right)$$



Attention:
semilogplot

Lognormal Distribution

```
> plot([exp(-ln(x)^2/4)/(sqrt(2*Pi)*x)-exp(-(ln(x))^2/4)*sin(Pi*ln(x)
/2)/9,exp(-ln(x)^2/4)/(2*sqrt(Pi)*x)],x=0..24,axes=boxed,color=
[red,blue]);
```



Blue curve : $$\frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\exp\left(-\frac{\ln^2(x)}{4}\right)$$

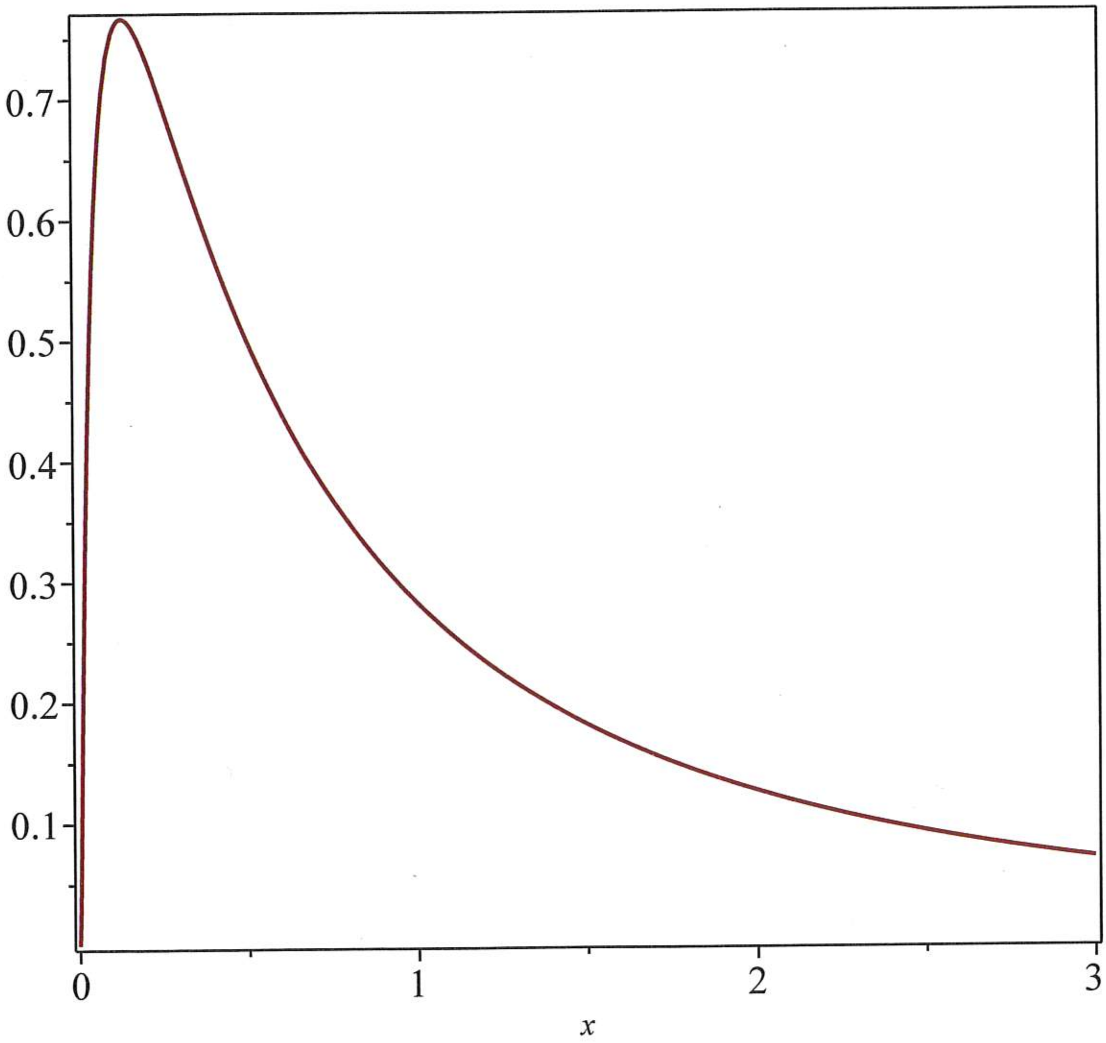
Red curve: $$-\frac{1}{9}\exp\left(-\frac{\ln^2(x)}{4}\right)\sin\left(\frac{\pi\ln(x)}{2}\right)+\frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\exp\left(-\frac{\ln^2(x)}{4}\right)$$

Red curve : $$\frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\underbrace{\exp\left(-\frac{\ln^2(x)}{4}\right)\cdot\left[1-\frac{\sqrt{2\pi}\,x}{9}\sin\left(\frac{\pi\ln(x)}{2}\right)\right]}_{\text{Stieltjes Class}}$$

```
> #      LOGNORMAL DISTRIBUTION
```

$\exp(-\ln^2(x)/4)/(2\sqrt{\pi}\cdot x)$

```
>
> #        exp(-ln(x)^2/4)/(2*sqrt(Pi)*x)
```



```
> #   Calculation of k-th moment of lognormal
distribution
> int(x^k*exp(-ln(x)^2/4)/(2*sqrt(Pi)*x),x=0..
infinity);
```

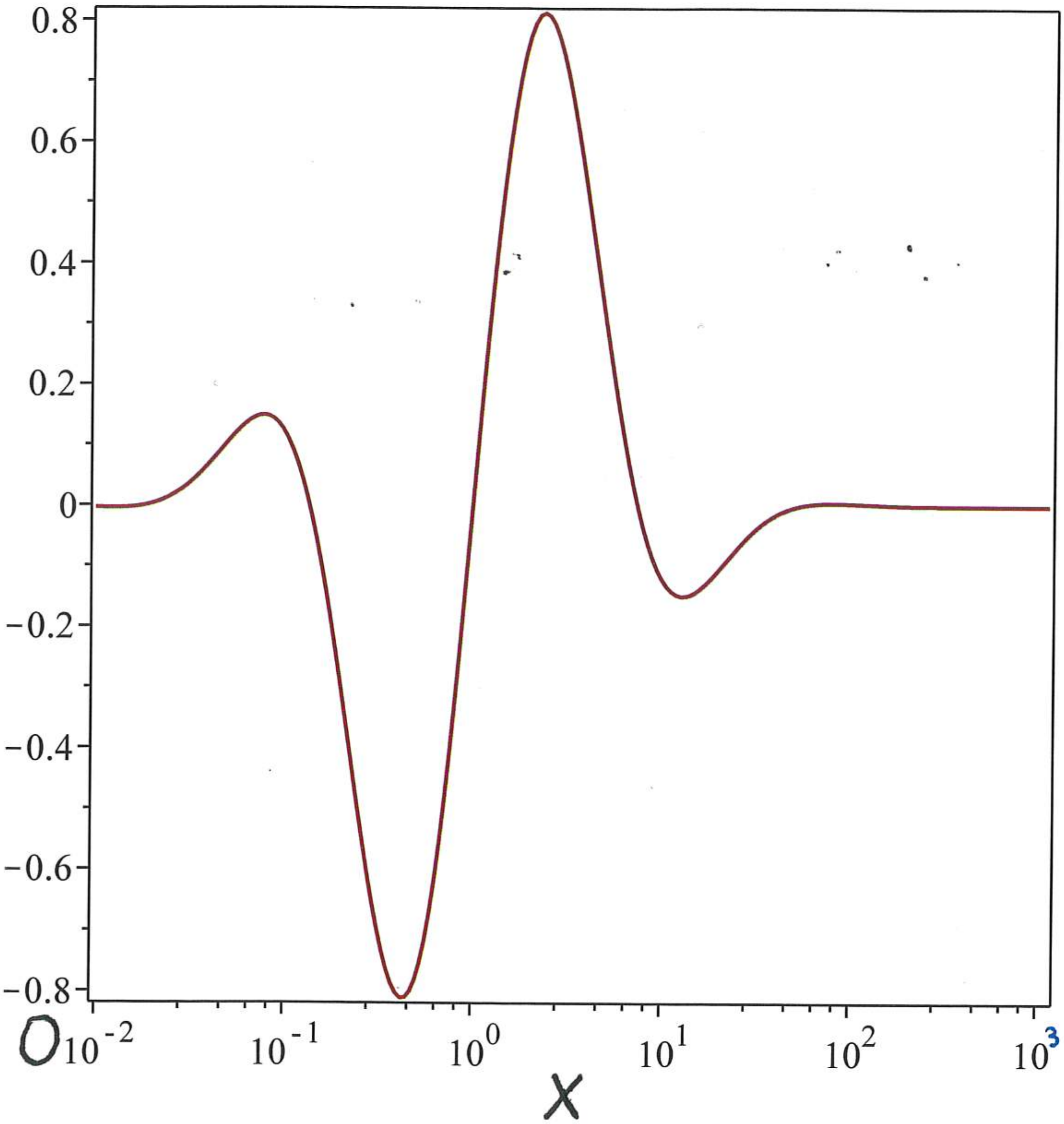
$$e^{k^2} \rightarrow \exp(k^2) \qquad (1)$$

Rapidly growing k-th moment $\rightarrow \exp(k^2)$

"Classical" polynomial killer $\omega(x)$

(Stieltjes)

```
> exp(-(ln(x))^2/4)*sin(Pi*ln(x)/2)
```



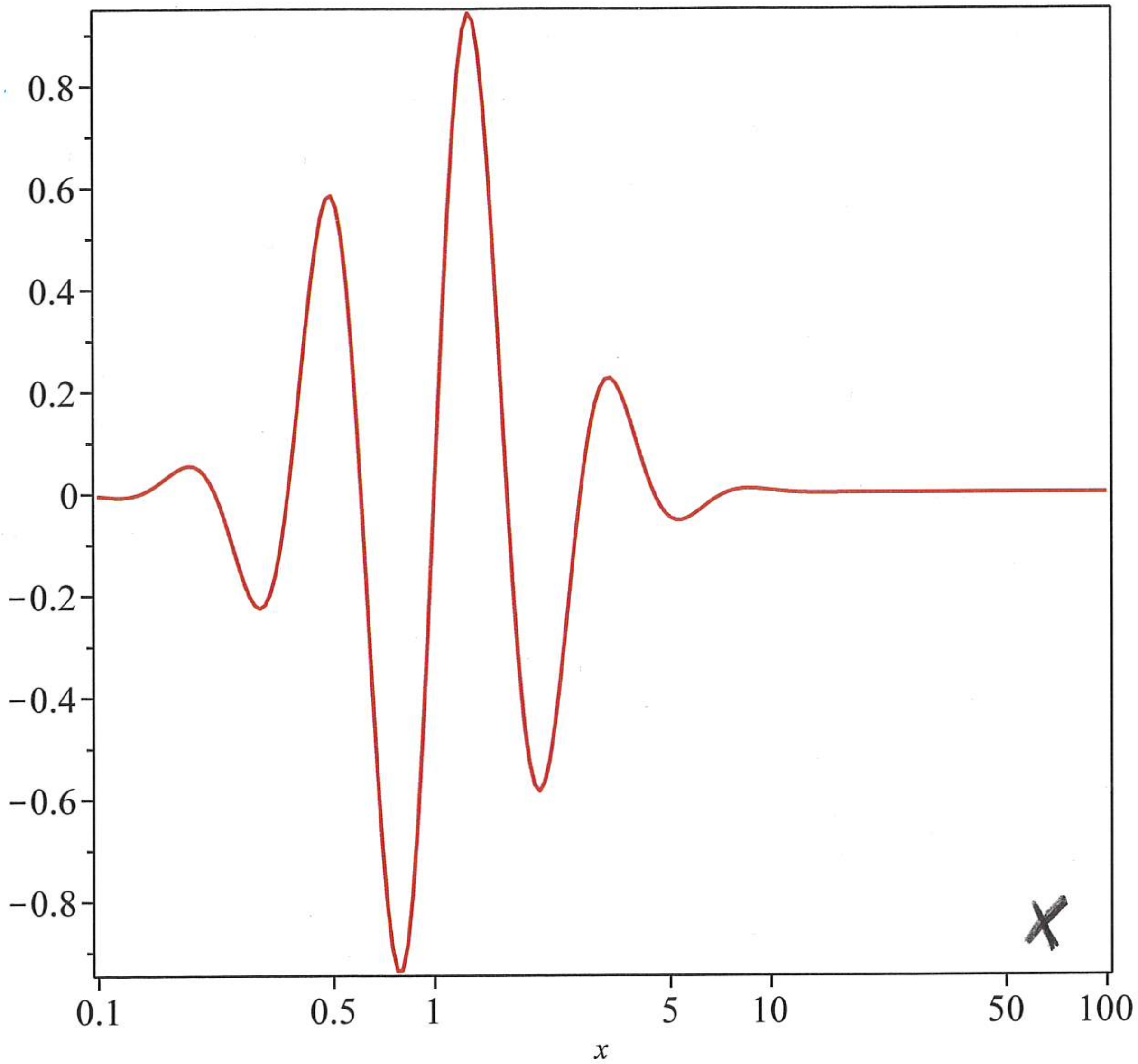
Semilog plot

A) $\lim_{x \to 0} \omega(x) = \lim_{x \to \infty} \omega(x) = 0$;

B) $|\omega(x)| \lesssim 0.8$; bounded function

Stieltjes Polynomial Killer

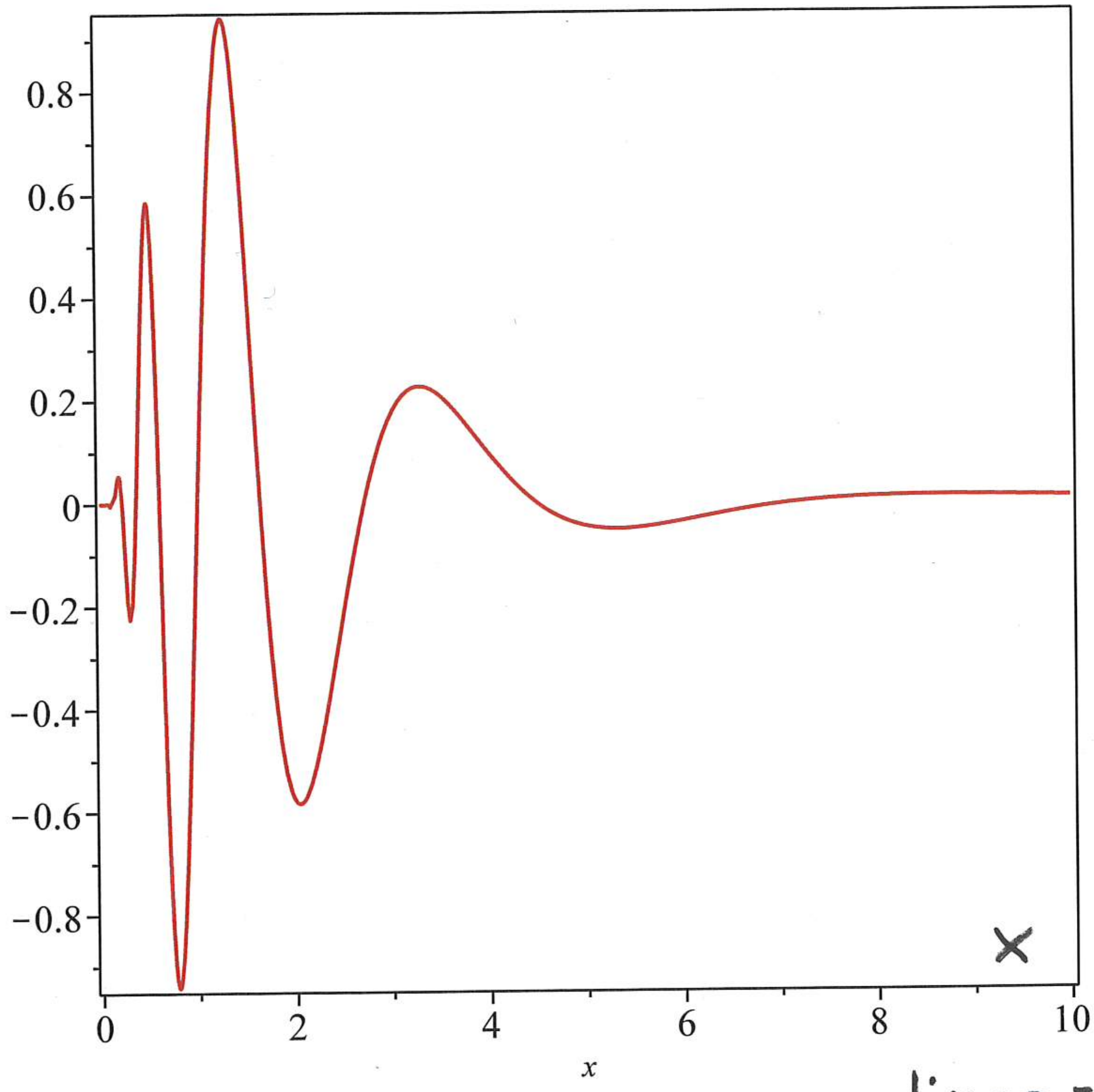
$$\exp(-\ln^2(x))\sin(2\pi\ln(x))$$



semilog plot

Stieltjes Polynomial Killer

$\exp(-\ln^2(x))\sin(2\pi\ln(x))$



linear scale

Mellin Convolution of $f(x)$ and $g(x)$

$$\mathcal{M}[f(x);s] = f(s) \text{ and } \mathcal{M}[g(x);s] = g(s)$$

$$\Downarrow$$

$$\mathcal{M}\left[\int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{1}{t} dt; s\right] \vdots f(s) \cdot g(s)$$

Parent formules (a choice)

$$\mathcal{M}\left[x^\alpha \int_0^\infty f(x \cdot t) g(t) dt; s\right] = f(s+\alpha) g(1-s-\alpha)$$

$$\mathcal{M}\left[\int_0^\infty f\left(\frac{t}{x}\right) g(t) dt; s\right] = f(-s) g(s+1)$$

$$\mathcal{M}\left[\int_0^\infty f(x^\alpha t^\beta) g(t^\gamma) dt; s\right] = \frac{1}{|\alpha|} f\left(\frac{s}{\alpha}\right) \frac{1}{|\gamma|} g\left(\frac{\alpha-\beta s}{\alpha\gamma}\right)$$

Mellin convolution conserves positivity

$$\mathcal{M}[e^{-x}; s] = \int_0^\infty x^{s-1} e^{-x} dx \equiv \Gamma(s)$$

$$\Rightarrow \mathcal{M}^{-1}[\Gamma^2(s); x] = \int_0^\infty \exp\left(\frac{-x}{t}\right) \exp(-t) \frac{1}{t} dt =$$

$$= \int_0^\infty \exp\left(-\frac{x}{t} - t\right) \frac{1}{t} dt = 2K_0(2\sqrt{x})$$

Bessel

Meijer G-function

$\Rightarrow$ inverse Mellin transform of ratios of products of Gamma functions.

two lists only....

• $\boxed{\text{Meijer } G} = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}\alpha_1\dots\alpha_n, & \alpha_{n+1}\dots\alpha_p\\ \beta_1\dots\beta_m, & \beta_{m+1}\dots\beta_q\end{matrix}\right) =$

$\text{MeijerG}\left([[\cancel{\alpha_1\dots\alpha_n}],[\alpha_{n+1}\dots\alpha_p]],[[\beta_1\dots\beta_m],[\cancel{\beta_{m+1}\dots\beta_q}]],z\right) =$

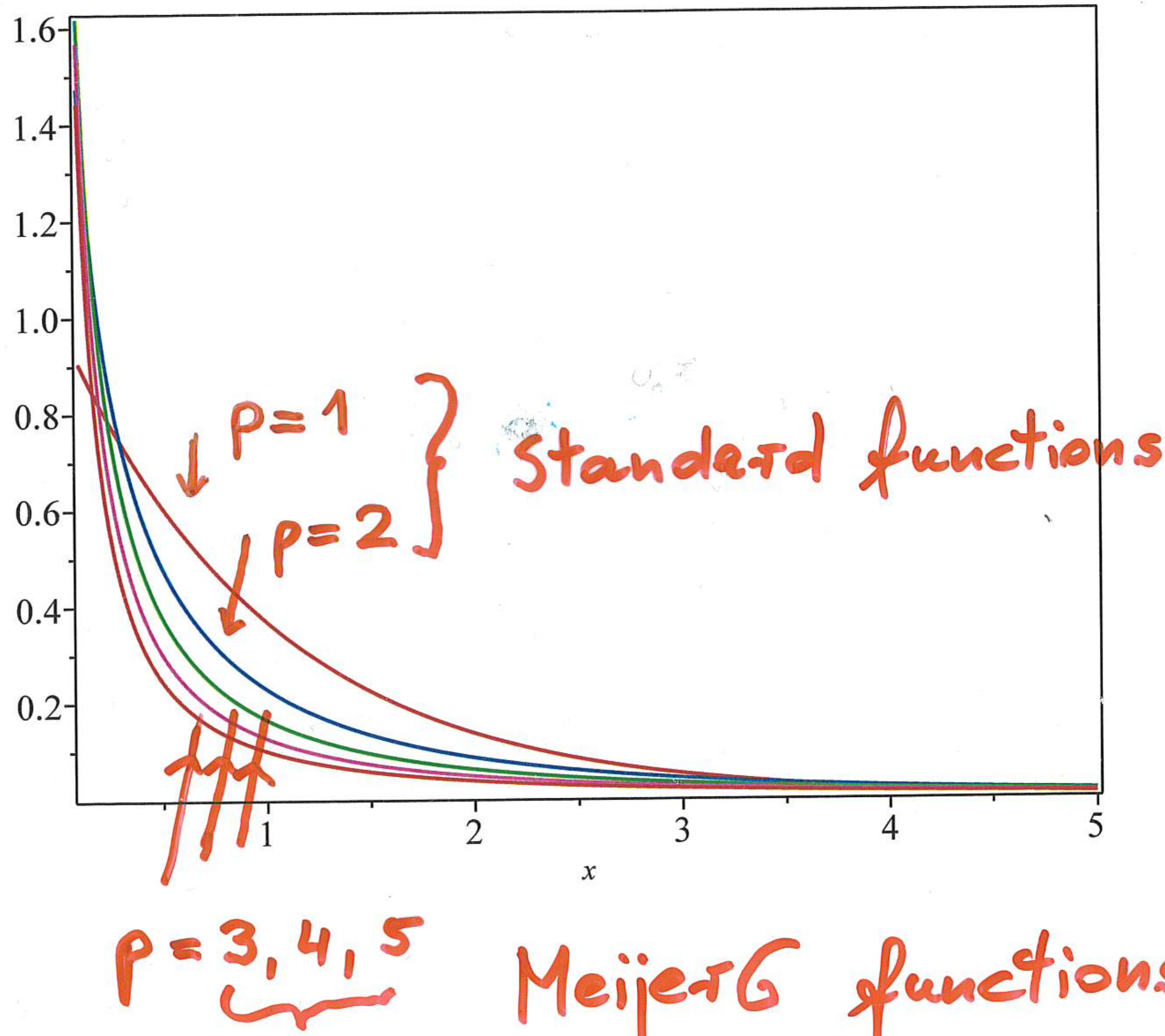
$$\mathcal{M}^{-1}\left[\frac{\prod_{j=1}^{m}\Gamma(\beta_j+s)\prod_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-s)\prod_{j=n+1}^{p}\Gamma(\alpha_j+s)};z\right]$$

In combinatorial applications mostly in the form:

$$\mathcal{M}^{-1}\left[\frac{\prod_j\Gamma(\beta_j+s)}{\prod_j\Gamma(\alpha_j+s)};x\right]$$

Mellin Convolution at work

I : moments $(n!)^p$



... Stieltjes moments

Mellin positivity "bricks"
(a choice)

$$\Gamma(s+\alpha) \longrightarrow x^{\alpha} e^{-x}$$

$$[\Gamma(s)]^2 \longrightarrow K_0(2\sqrt{x})$$

$$\frac{\Gamma(s)}{\Gamma(s+a)} \longrightarrow (1-x)^{a-1}, \quad x<1$$

$\vdots$

etc (infinity of possibilities)

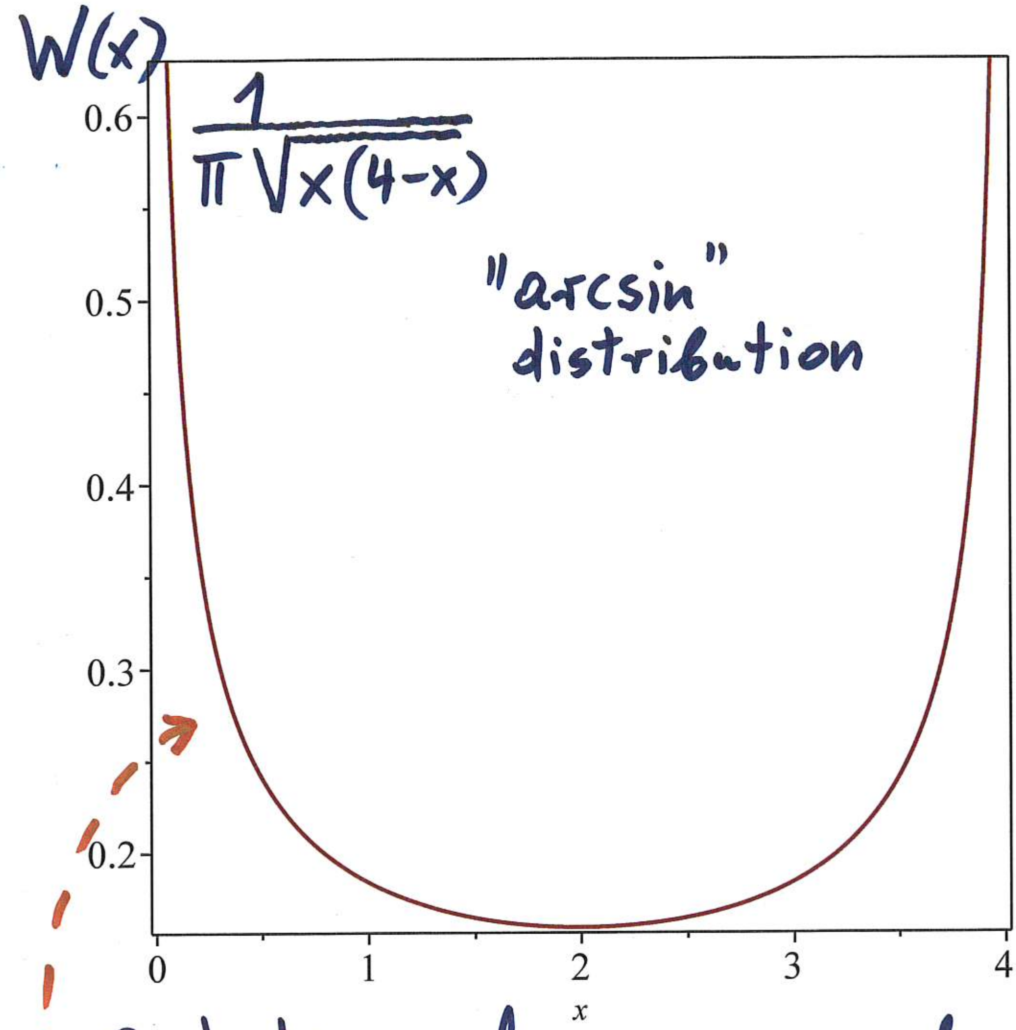
Lesson:

Examine $\dfrac{\Gamma\ldots\ \Gamma\ldots\ \Gamma\ldots\ \Gamma\ldots\ldots\ldots}{\Gamma\ldots\ \Gamma\ldots\ \Gamma\ldots\ \Gamma\ldots\ \Gamma\ldots}$ for the presence of such "bricks". This conceived as convolution will prove the positivity.

"Allowed" forms in Meijer G-functions

$$\Gamma(a \pm s)$$

Mellin convolution at work II



Solution for moments $\binom{2n}{n}$

$$W(x) = \mathrm{MeijerG}\left([[\,], [0]], [[-1/2], [\,]], \tfrac{x}{4}\right)/(4\sqrt{\pi})$$

What will be solution for $\binom{2n}{n}^2$?

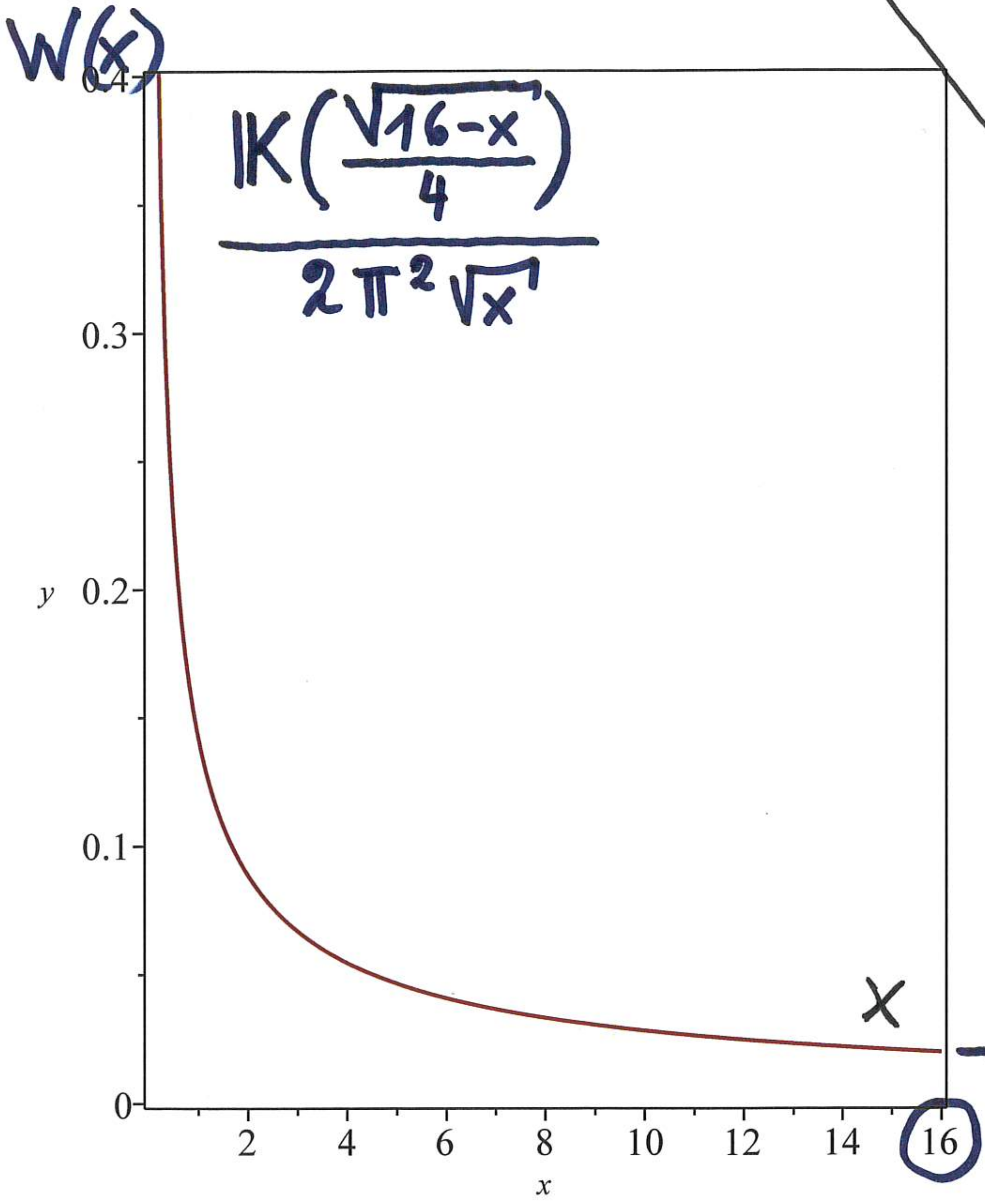
Double the parameters and support.

... Hausdorff moments

Moments $\binom{2n}{n}^2$

$$W(x) = \text{MeijerG}\left([\,], \underbrace{[0,0]}, \underbrace{[-\tfrac{1}{2}, -\tfrac{1}{2}]}, [\,], \frac{x}{16}\right)$$

Doubling of Parameters



$$\frac{\mathbb{K}\left(\frac{\sqrt{16-x}}{4}\right)}{2\pi^2\sqrt{x}}$$

$\frac{1}{16\pi}$

$\mathbb{K}(\ldots)$ – Elliptic function $\mathbb{K}$

List of Collaborators on this and related problems (in random order)

K. Życzkowski, V. Minh, J.-M. Sixdeniers, Ch. Tollu, P. Blasiak, G. Koshevoy, Ph. Flajolet, G.H.E. Duchamp, K. Górska, A. Solomon, S. Smith, W. Młotkowski, A. Bostan, N. Behr, M. Nowak, A. Horzela