Reconstruction algorithms for sums of affine powers

Specfun seminar - Paris 05-2018

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Contents

Motivation: algebraic complexity

Introduction

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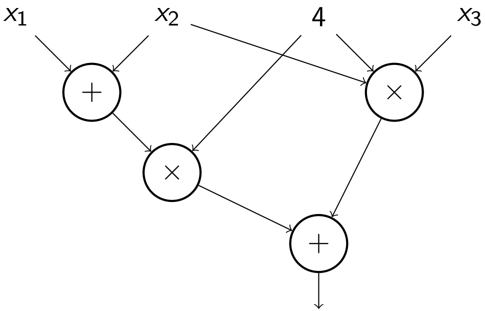

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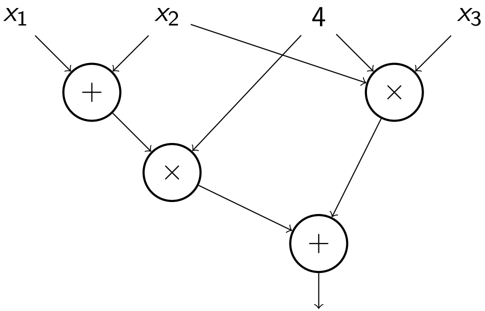

Figure: Circuit computing the polynomial $4(x_1 + x_2) + 4x_2x_3$.

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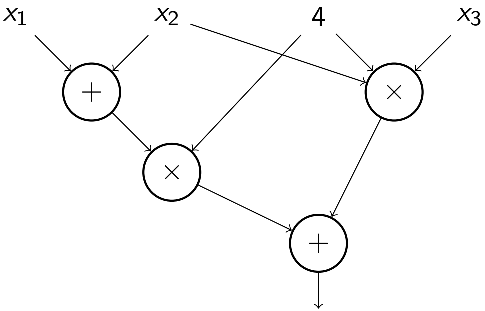

Figure: Circuit computing the polynomial $4(x_1 + x_2) + 4x_2x_3$.

- Hardness in the case of circuits: *depth* and *size*.

Complexity classes

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Definition (VP)

The class VP consists of all families of polynomials $\{f_n\}$ such that:

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Example

$$\mathrm{DET}_n(X) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)}$$

Definition (VNP)

The class VNP consists of all families of polynomials $\{f_n\}$ such that there exists a family $\{g_n\}$ in VP with:

$$f_n(x_1,\dots,x_{k(n)}) = \sum_{w\in\{0,1\}^{p(n)}} g_{p(n)}\left(x_1,\dots,x_{k(n)},\quad w_1\dots,w_{p(n)}\right)$$

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PERM is VNP-complete.

Models of interest

Let $\mathbb{F}$ be any characteristic zero field. We consider f an **univariate** polynomial with **coefficients in $\mathbb{F}$**, this is, $f \in \mathbb{F}[x]$.

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Model (Univariate $\Sigma \wedge \Sigma$)

$$\sum_{i=1}^{k} \alpha_i (x - a_i)^{e_i} \quad \text{with } \alpha_i, a_i \in \mathbb{F}$$

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A polynomial can be written in many ways in this model, for example $f = 10x^4 + 20x^2 + 2 \in \mathbb{R}[x]$ can be written as:

$$f = \; 10\,(x-0)^4 \; + \; 20\,(x-0)^2 \; + \; 2\,(x-0)^0 =$$

Sums of affine powers

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$$\begin{aligned} f &= 10\,(x-0)^4 + 20\,(x-0)^2 + 2\,(x-0)^0 = \\ &= (x+1)^5 - (x-1)^5 \end{aligned}$$

Associated measure

For $f \in \mathbb{F}[x]$,

Definition

$$\mathrm{AffPow}_{\mathbb{K}}(f) := \min\left\{ k : f(x) = \sum_{i=1}^{k} \alpha_i (x - a_i)^{e_i} \quad \text{with } \alpha_i, a_i \in \mathbb{K} \right\}$$

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In fact, $\mathrm{AffPow}_{\mathbb{R}}(f) = 2$.

Related models

Model (Sparsest shift)

$$f(x) = \sum_{i=1}^{s} \alpha_i (x-a)^{e_i}$$

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Related models

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Model (Waring decomposition)

$$f(x) = \sum_{i=1}^{s} \alpha_i (x-a_i)^d \text{ where } d = \deg(f)$$

$f = (x+1)^5 - (x-1)^5$ is not a Waring decomposition!

Goal: reconstruction algorithms

Problem

Given a polynomial $f \in \mathbb{F}[x]$, compute the exact value $s = \mathit{AffPow}_{\mathbb{F}}(f)$ and a decomposition with s terms.

$$f = \sum_{i=0}^{d} f_i x^i \longrightarrow \boxed{\textbf{Algorithm}} \longrightarrow \begin{array}{c} \mathrm{AffPow}(f) = s \\ f = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i} \end{array}$$

Structural results

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Real polynomials

Uniqueness

Theorem (Koiran,Garcia-Marco'15)

Consider a polynomial identity of the form:

$$\sum_{i=1}^{k} \alpha_i (x - a_i)^{e_i} = 0$$

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Define $n_e := \#\{e_i : e_i \leq e\}$.
If $2n_e < \lceil (e+3)/2 \rceil$ *for all* $e \in \mathbb{N}$, *then* $AffPow_{\mathbb{R}}(f) = s$ *and the optimal representation of* f *is unique.*

Real polynomials

- Let $f = (x+1)^d - (x-1)^d + i(x+i)^d - i(x-i)^d \in \mathbb{R}[X]$. $\mathrm{AffPow}_{\mathbb{C}}(f) \leq 4$ but $\mathrm{AffPow}_{\mathbb{R}}(f) = \lfloor (d+1)/4 \rfloor$.

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- Orthogonality of Waring rank and sparsest shift:
  For $f \in \mathbb{R}[X]$,

$$\mathrm{Waring}_{\mathbb{R}}(f) + \mathrm{Sparsest}_{\mathbb{R}}(f) \geq \frac{d+3}{2}$$

  except if $f = \alpha(x-a)^d$.

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- $2\mathrm{Waring}_{\mathbb{R}}(f) \geq \mathrm{Waring}_{\mathbb{R}}(f) + \mathrm{AffPow}_{\mathbb{R}}(f) \geq \frac{d+3}{2}$,
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Structural results

Characteristic zero

Uniqueness

Proposition

Consider a polynomial identity of the form:

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Let $f \in \mathbb{F}[x]$ be a nonzero polynomial of degree d, then:

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Proposition

If f is a **generic** *polynomial, then $\mathit{AffPow}_{\mathbb{F}}(f) = \left\lceil \frac{d+1}{2} \right\rceil$*

Characteristic zero

- Let $f = (x+1)^d - dx^{d-1}$. $\mathrm{AffPow}_{\mathbb{C}}(f) = 2$ but $\mathrm{Waring}_{\mathbb{C}}(f) \geq d-1$ and $\mathrm{Sparsest}_{\mathbb{C}}(f) \geq (d+1)/2$.

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- This is tight for

$$f = \sum_{j=1}^{\sqrt{d}} (x+\xi^j)^d = \sqrt{d} \sum_{\substack{0 \leq i \leq d \\ i \equiv 0 \pmod{\sqrt{d}}}} \binom{d}{i} x^{d-i}$$

  where ξ is a $\sqrt{d}$-th primitive root of unity:
  $\mathrm{Waring}_{\mathbb{C}}(f) \leq \sqrt{d}$ and $\mathrm{Sparsest}_{\mathbb{C}}(f) \leq \lceil (d+1)/\sqrt{d} \rceil$.

The tool: Shifted Differential Equations

Shifted differential equations

Definition (SDE)

A SDE(k) is an order k differential equation

$$\sum_{i=0}^{k} P_i(x)\, g^{(i)}(x) = 0$$

where $P_i \in \mathbb{F}[x]$ is a polynomial of degree $\deg P_i \leq i$.

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Remark

$$f \text{ satisfies an SDE}(k)$$
$$\Updownarrow$$
$$\left\{ x^j f^{(i)}(x) \;:\; 0 \leq i \leq k,\, 0 \leq j \leq i \right\} \text{ is } \mathbb{F}\text{-linearly dependent.}$$

An example

Let $f = x^d + x^{d-1}$. The polynomials:

$$\left.\begin{array}{rcrcr} f & = & x^d & + & x^{d-1} \\ f' & = & d\,x^{d-1} & + & (d-1)\,x^{d-2} \\ x\,f' & = & d\,x^d & + & (d-1)\,x^{d-1} \\ f'' & = & d(d-1)\,x^{d-2} & + & (d-1)(d-2)\,x^{d-3} \\ x\,f'' & = & d(d-1)\,x^{d-1} & + & (d-1)(d-2)\,x^{d-2} \\ x^2 f'' & = & d(d-1)\,x^d & + & (d-1)(d-2)\,x^{d-1} \end{array}\right\}$$

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are linearly dependent. Indeed,

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so f satisfies the following SDE(2):

$$x^2 g'' - 2(d-1)xg' + d(d-1)g = 0.$$

Small SDEs

Proposition

If $f(x) = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i}$*, then* f *satisfies an* $SDE(2s-1)$*, which is also satisfied by the* $(x - a_i)^{e_i}$*'s.*

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$$\subseteq \left\{ (x-a)^d \ : \ e-k \le d \le e \right\}$$

Therefore, $C_k(\,(x-a)^e\,) \le k+1$.

It is enough to have

$$s(k+1) < \frac{(k+1)(k+2)}{2}$$

Algorithms

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2. **Find the solutions of the SDE of the form $(x - a)^e$.**

3. **Write f as a linear combination of these solutions**
   (and hope it is the good one)

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So, there is a SDE fulfilling our wishes.

1. **Find a "small" SDE satisfied by f.**
   Hope that the powers $(x - a_i)^{e_i}$ satisfy the same equation.

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So, there is a SDE fulfilling our wishes.

Issue:

What if we do not find the 'good' SDE?

2. **Find the solutions of the SDE of the form** $(x-a)^e$.

Step 2

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Some issues:

- How large should e be ? $\Rightarrow$ **solved by the unexpected corollary**
- We may obtain some "false positives".

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We just have to solve a linear system.
If step 1 is good, we know that there is at least one solution.

Some issues:

- What if there are several solutions?
- How do we find the "shortest one"?

Distinct nodes

Large exponents

The key lemma

Lemma

Let $f \in \mathbb{F}[x]$ be written as

$$f = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i},$$

where the $a_i \in \mathbb{F}$ are **all distinct**. *Whenever f satisfies a SDE(k), if*

$$e_i \text{ is big}$$

then $(x - a_i)^{e_i}$ satisfies the same SDE.

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$$e_i \geq ks + \binom{s}{2}$$

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An algorithm for large exponents

$$f = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i} \quad \text{with distinct } a_i, \quad e_i > \frac{5s^2}{2}$$

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d) If f satisfies a SDE(k) with $k \leq 2s - 1$, then so does $(x - a_i)^{e_i}$,

e) We have $e_i \leq \deg(f) + (s^2/2)$.

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Lemma

We have $|B| \leq r$ and B is $\mathbb{F}$-linearly independent.

Distinct nodes

Large and small exponents

Weaken the hypothesis

Theorem

Let $f(x) = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i}$ *with*

- $a_i \in \mathbb{F}$ *all distinct*
- $n_k \leq (3k/4)^{1/3} - 1$

Then $\mathit{AffPow}(f) = s$ *and there is an polynomial time algorithm for the reconstruction problem.*

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- Find a SDE(t) satisfied by f.
- Compute the set of large exponents solutions

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- Compute the set of large exponents solutions
- Reconstruct coefficients of large exponents using the right derivative.
- Substract them and go on until 0 is found.

The algorithm

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Step 2. Consider $B := \{(x - b_i)^{d_i} \mid 1 \leq i \leq l\}$, the set of all the solutions of the SDE of the form $(x - b)^e$, assume that $d_1 \geq d_2 \geq \cdots \geq d_l$.

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Step 4. We set $j := d_{r+1} + 1$ and write $f^{(j)}$ as $f^{(j)} = \sum_{i=1}^{r} \beta_i \cdot \frac{d_i!}{(d_i-j)!}(x - b_i)^{d_i-j}$ with $\beta_1, \ldots, \beta_r \in \mathbb{F}$.

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Step 5. We set $\widetilde{f} := \sum_{i=1}^{r} \beta_i (x-b_i)^{d_i}$ and $h := f - \widetilde{f}$.

Towards repeated nodes.

Small intervals

Let $\delta \in \mathbb{N}$. We aim now at reconstructing expressions of the form

$$f = \sum_{i=1}^{s} \alpha_i (x - a_i)^{e_i}$$

such that whenever $a_i = a_j$, then $|e_i - e_j| \leq \delta$.

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such that whenever $a_i = a_j$, then $|e_i - e_j| \leq \delta$.

We **rewrite** f as

$$f = \sum_{i=1}^{t} Q_i(x)(x - a_i)^{e_i},$$

where

- Q_i is a polynomial of degree $\leq \delta$, and
- $a_i \neq a_j$ when $i \neq j$.

Repeated nodes: the key lemma

Lemma

Let $\delta \in \mathbb{N}$ and let $f \in \mathbb{F}[x]$ be written as

$$f = \sum_{i=1}^{t} Q_i(x)\,(x - a_i)^{e_i},$$

with distinct $a_i \in \mathbb{F}$ and $\deg(Q_i) \leq \delta$.

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If f satisfies a $SDE(k)$ and

$$e_i \geq t(k + \delta) + \binom{t}{2},$$

then $Q_i(x)\,(x - a_i)^{e_i}$ satisfies the same SDE.

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Let

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$$g(x) = R(x)(x - c)^e,$$

with $e < d + \frac{d^2}{8}$, where $\deg(R) \leq \delta$.

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Step 3. Write $f = \sum_{i=1}^{p} \lambda_i g_i$ with $\lambda_i \in \mathbb{F}$ and output the expression.

Multivariate reconstruction

Multivariate model

Model (Multivariate AffPow)

$$\sum_{i=1}^{k} \alpha_i \ell_i^{e_i} \quad \text{with } \alpha_i \in \mathbb{F}, \deg(\ell_i) = 1$$

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We will design algorithms in the "black box" setting: they have access to the input polynomial only through an oracle so that for any point $a \in \mathbb{F}^n$, we can obtain $f(a)$ in a single step by querying this oracle. We will use:

- Change of basis

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- Homogeneous components

Essential variables

$$f(x_1, x_2, x_3) = x_1^3 + x_1^2 x_2 - 2x_1^2 x_3 - 2x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

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Proposition (Carlini)

For a polynomial $f \in \mathbb{F}[X]$, we have

$$EssVar(f) = \dim_{\mathbb{F}} \left\langle \frac{\partial f}{\partial x_i} \,\middle|\, 1 \leq i \leq n \right\rangle$$

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\mathrm{EssVar}(f) \leq \mathrm{AffPow}(f)
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From reconstruction to polynomial equivalence

Take f such that $\mathrm{EssVar}(f) = \mathrm{AffPow}(f)$, i.e. $f = \sum_{i=1}^{n} \ell_i^{e_i}$.

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The Hessian matrix

$$H_f(X) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

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In particular,

$$\det(H_f(X)) = \det(A)^2 \det(H_g(A \cdot X + b)).$$

Algorithm overview

When $g = \sum_{i=1}^{n} x_i^{e_i}$, we have

$$\frac{\partial^2 g}{\partial x_i \cdot \partial x_j} = \begin{cases} 0 & \text{if } i \neq j, \\ e_i(e_i - 1)x_i^{e_i - 2} & \text{if } i = j \end{cases}$$

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Lemma

Let f be a regular polynomial such that $f(X) = \sum_{i=1}^{n} \ell_i(X)^{e_i}$ where $\ell_1(X), \ldots, \ell_n(X)$ are affine forms and $e_i \geq 2$. Then we have

$$\det(H_f(X)) = c \cdot \prod_{i=1}^{n} \ell_i(X)^{e_i - 2}$$

where $c \in \mathbb{F}$ is a nonzero constant.

Quadratic polynomials

Proposition (Folklore)

Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2 and let $f, g \in \mathbb{F}[X]$ be homogeneous quadratic polynomials. Then,

$$f \sim g \iff EssVar(f) = EssVar(g).$$

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Theorem

Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2 and let $f \in \mathbb{F}[X]$ be a polynomial of degree at most 2. Then, there exists a unique $r \in [\![0, n]\!]$ such that

i) $f \equiv \sum_{i=1}^{r} x_i^2$,

ii) $f \equiv \sum_{i=1}^{r} x_i^2 + c$ *with* $c \in \mathbb{F} \setminus \{0\}$, *or*

iii) $f \equiv \sum_{i=1}^{r-1} x_i^2 + x_r$.

Moreover, only one of these three scenarios can hold and $r = \mathit{EssVar}(f)$.

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$$\det([H_f(X)]_{k,k}) = c \cdot \prod_{i=1}^{n-1} \ell_i(X)^{e_i - 2}$$

Wrapping up

Theorem

There exists a polynomial-time randomized algorithm that receives as input a blackbox access to a regular polynomial $f \in \mathbb{F}[X]$ *and finds an optimal decomposition of* f *in the Affine Powers model if* $\mathit{AffPow}(f) = n$*, or rejects otherwise.*

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- Set $h(X) = g(X + X_0)$, and write $h = \sum_{i=1}^{t} \alpha_i [\ell_i]^{m_i+2} + [h]_{\leq 2}$.

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- If $D = 0$, repeat previous procedure with $\det([H_f(X)]_{k,k})$ for all k.

Uniqueness

For $s \in \mathbb{N}^*$, denote by $E_n := \{\underline{e} = (e_1, \ldots, e_n) \in (\mathbb{N}^*)^n \mid e_1 \geq \cdots \geq e_n\}$.

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with ℓ_i, t_i linear forms and $\underline{e} = (e_1, \dots, e_n), \underline{d} = (d_1, \dots, d_n) \in E_n$, then, $e_i = d_i$ for all i, and there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $\alpha_i \ell_i^{e_i} = \beta_{\sigma(i)} t_{\sigma(i)}^{d_{\sigma(i)}}$ if $e_i \geq 3$.

Repeated affine forms.

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Test if $f \equiv g$ with $g = \sum_{i=1}^{n} \sum_{j=1}^{t_i} \alpha_{i,j}\, x_i^{e_{i,j}}$.

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Given $f \in \mathbb{F}[X]$, is $f \equiv g$ with $g = \sum_{i=1}^{n} g_i(x_i)$?

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Given an n-variate polynomial $f(X) \in \mathbb{F}[X]$, there exists an algorithm that finds a decomposition of f as

$$f(A \cdot X) = p(x_1, \ldots, x_t) + q(x_{t+1}, \ldots, x_n),$$

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If f has a univariate decomposition, does taking an optimal decomposition for each g_i yield an optimal decomposition of f ?

The algorithm idea

Proposition

Let $f \in \mathbb{F}[X]$*, and let the* g_i*'s be univariate polynomials sorted by decreasing degree. Let* $d_i := \deg(g_i)$ *and* $k := \max\{i : d_i \geq 3\}$*. Let* $\ell_1, \ldots, \ell_n$ *be linear forms such that* $f = \sum_{i=1}^{n} g_i(\ell_i)$*. Then,*

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$$\det(H_f(X)) = c \cdot \prod_{i=1}^{k} \prod_{j=1}^{d_i - 2} (\ell_i - \alpha_{i,j}),$$

where $c \in \mathbb{F}$*, and* $\alpha_{i,j}$ *are the roots of* $g_i''(x)$ *for* $1 \leq i \leq k$*.*

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Moreover, if $\ell_1, \ldots, \ell_n$ are linearly independent, for any solution $X_0 \in \mathbb{F}^n$ to the system $B \cdot X_0 = (\alpha_{1,1}, \ldots, \alpha_{k,1})^T$, where B is the $k \times n$ matrix whose rows are the coefficients of the $\ell_1, \ldots, \ell_k$, we have that

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(a) $[f(X + X_0)]_{\geq 3} = \sum_{i=1}^{k} h_i(\ell_i)$ for some unique $h_i \in \mathbb{F}[x]$, and

(b) $\mathrm{EssVar}([f(X + X_0)]_2) = |\{i \,|\, \deg(g_i) = 2\}|$.

The bivariate case

If $f = f_1(x_1) + f_2(x_2)$, set $s_i := \mathrm{AffPow}(f_i)$ and write

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If $e_{1,1} \leq 1$ and $e_{2,1} \leq 1$, define $\mathrm{UnivAffPow}(f) := s_1 + s_2 - 1$, and otherwise $\mathrm{UnivAffPow}(f) := s_1 + s_2$.

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Lemma

Let $s, d \in \mathbb{Z}^+$ and $b_1, \ldots, b_s$ different nonzero elements of $\mathbb{F}$. If

$$\lambda_1 x_1^d + \lambda_2 x_2^d = \sum_{i=1}^{s} \gamma_i (x_1 + b_i x_2)^d,$$

with $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\gamma_i \in \mathbb{F}$ not all zero, then $s \geq d$.

Allowing more affine forms.

Previous algorithm fails

Base case: $f \equiv g$ with $g = \sum_{i=1}^{n} x_i^{e_i} + \ell^e = h + \ell^e$.

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with $P(X) = \sum_{i=1}^{n} \beta_i^2 \left(\prod_{j \neq i} e_j^{\underline{2}}\, \ell_j(X)^{e_i-2} \right) \in \mathbb{F}[X]$.

Higher order Hessian

Definition (Symmetric 4-th order Hessian)

$$\forall a \leq b, i \leq j, \quad (\overline{H}_f)_{(a,b),(i,j)} = \frac{\partial^4 f}{\partial x_a \partial x_b \partial x_i \partial x_j}$$

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Proposition

Let $n \in \mathbb{N}^*$, $m := \binom{n+1}{2}$ and $f = \sum_{i=1}^{m} \ell_i^{e_i}$, with $\ell_i = \sum_{j=1}^{n} b_{i,j} x_j + b_{i,0}$ affine forms and $e_i \geq 4$ for all i. Let U be the square $m \times m$ matrix with entries $U_{(i,j),k} := b_{k,i}\, b_{k,j}$ for all $1 \leq k \leq m$, $1 \leq i \leq j \leq n$. If $\det(U) \neq 0$*, there exists $c \neq 0$ such that*

$$\det(\overline{H}_f(X)) = c \cdot \prod_{i=1}^{m} \ell_i^{e_i - 4},$$

Proof

By linearity of the symmetric 4-th order Hessian, we have

$$\overline{H}_f(X) = \sum_{k=1}^{m} \overline{H}_{\ell_k}(X) = \sum_{k=1}^{m} e_k^4 \, \ell_k^{e_k - 4}(u_k \cdot u_k^T) = U \cdot D \cdot U^T,$$

where $D = \mathrm{Diag}(e_1^4 \, \ell_1^{e_1 - 4}, \ldots, e_m^4 \, \ell_m^{e_m - 4})$, and u_k is the column vector whose (i, j)-th entry is $b_{k,i} b_{k,j}$ with $1 \leq i \leq j \leq n$. Thus,

$$\det(\overline{H}_f(X)) = \det(U)^2 \prod_{k=1}^{m} e_k^4 \, \ell_k^{e_k - 4}.$$

Lemma

Let $n \in \mathbb{N}^*$ *and* $m := \binom{n+1}{2}$*, and consider the set of variables* $\mathcal{V} := \{y_{(k,l),i} \mid 1 \le k \le l \le n,\ 1 \le i \le n\}$*. Let* U *be the* $m \times m$ *square matrix with entries* $U_{(i,j),(k,l)} := y_{(k,l),i}\, y_{(k,l),j}$*, where* $1 \le i \le j \le n$*,* $1 \le k \le l \le n$*. Then,* $\det(U) \in \mathbb{Z}[\mathcal{V}]$ *is a nonzero polynomial of degree* $2m$*.*

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Consider $\tilde{U}$ given by: $y_{(k,l),i} \mapsto 1$ if $i \in \{k, l\}$; or $y_{(k,l),i} \mapsto 0$ otherwise.

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Theorem

Let $n \ge 2$ and $m := \binom{n+1}{2}$. Let $\ell_i = \sum_{j=1}^{n} b_{i,j} x_j + b_{i,0}$: $1 \le i \le m$ whose coefficients $b_{i,j}$ are taken uniformly at random from a finite set S and take $f := \sum_{i=1}^{m} \ell_i^{e_i} \in \mathbb{F}[X]$ with $e_i \ge 4$ for all i. Then, $\det(\overline{H}_f(X)) \neq 0$ with probability at least $1 - \frac{2m}{|S|}$.

Conclusion

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- Can we bound the bit size of an optimal decomposition by a polynomial function of the size of f?
  - Does Algorithm *Distinct Nodes* run in polynomial time?

Open questions II

A **generic** polynomial f of degree d has $\mathrm{AffPow}(f) = \left\lceil \frac{d+1}{2} \right\rceil$.

- For each $d \in \mathbb{N}$, can you provide a polynomial f_d of degree d and $\mathrm{AffPow}(f_d) = \left\lceil \frac{d+1}{2} \right\rceil$?

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Best answers known:

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For every $k \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{F}$, the polynomial $f = [(x - a_1)(x - a_2)]^k$ of degree $d = 2k$ satisfies that

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Theorem

When $\mathbb{F} = \mathbb{R}$, we provide polynomials f of degree d such that $AffPow(f) \geq d/4$.

Open questions III

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- We proved that $\mathrm{UnivAffPow}(f) = \mathrm{AffPow}(f)$ for bivariate polynomials. What about the general case?

Thank you for your attention!