Universality classes for weighted lattice paths

Where probability and ACSV meet

Marni Mishna

with: Julien Courtiel (Paris 13), Stephen Melczer (Waterloo/Lyon) and Kilian Raschel (CNRS; Tours)

Department of Mathematics
Simon Fraser University

Inria – SpecFun
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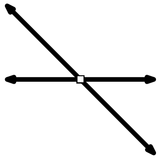
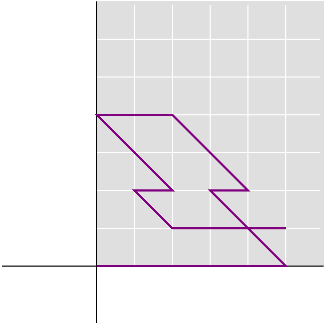
The Gouyou-Beauchamps model

The story of a single lattice path model

Let $\mathcal{W}$ be the set of walks in the first quadrant with steps:





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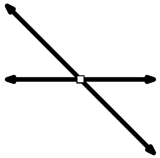
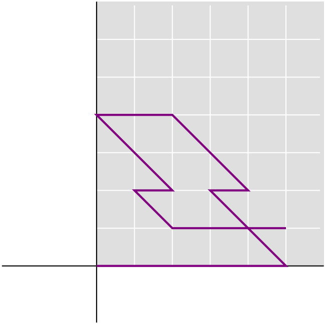




THEOREM

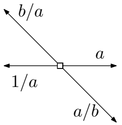
If w_n is the number of walks in $\mathcal{W}$ of length n, then

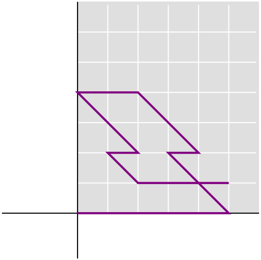
$$w_n \sim \frac{8}{\pi} 4^n n^{-2}.$$

Proof: Direct formula; Bostan Kauers 09; Melczer Wilson 16

The story of a single lattice path model

Let $\mathcal{W}_{a,b}$ be the set of weighted walks in the first quadrant with steps:





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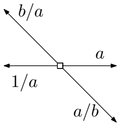
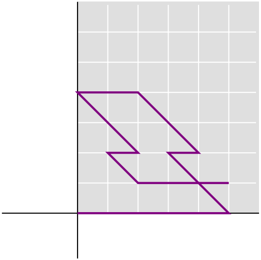




$wt(\omega) = a^5 b^1$

NEW THEOREM Courtiel, Melczer, M., Raschel 16+

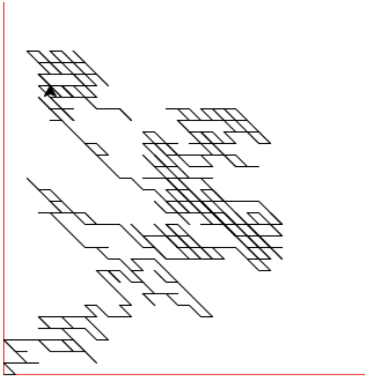
Let $w_n(a,b)$ be the number of walks in $\mathcal{W}_{a,b}$ of length n. Then

$$w_n(a,b) \sim ...$$

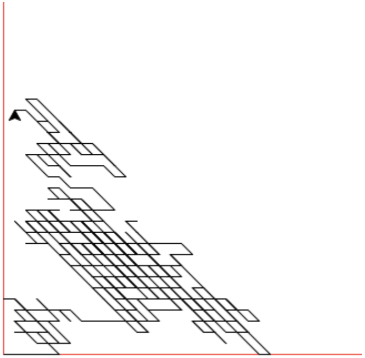
Proof: Kernel method + Analytic Combinatorics on Several Variables (ACSV)

GB Walks with 800 steps

Unweighted



Weighted, biased out of the first quadrant



Probability version: Exit times

Unweighted model generating function

$$W(t) = 1 + t + 3t^2 + 6t^3 + 20t^4 + 50t^5 + 175t^6 + \dots$$

Probability of staying in the quadrant after 6 steps:

$$\frac{w_6}{4^6} = \frac{175}{4^6} \sim 0.04$$

Probability version: Exit times

Weighted model generating function

$$1 + at + (1 + b + a^2)t^2 + (2ab + a^3 + 3a)t^3 + \ldots$$

Probability of staying in the quadrant after 3 steps:

$$\frac{w_3(a,b)}{S(1,1)^3} = \frac{2ab + a^3 + 3a}{(a + a^{-1} + ab^{-1} + b^{-1}a)^3}$$

Inventory: $S(x,y) = ax + \frac{1}{ax} + \frac{ax}{by} + \frac{by}{ax}$

Probability version: Exit times

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The weightings must be central: The probability of a given walk depends only on its length and its endpoint. We give explicit conditions for this in our work.

Natural Questions

$$w_n(a, b) \sim C\rho^{-n} n^{\alpha}$$

1. How do the weights intervene?
2. What is the range of possible asymptotic behaviour?
3. What affects the exponential growth ρ ? the critical exponent α?
4. How do parameters like the choice of cone, starting point, and drift affect the formula?
5. What is the best way to study this?

Our contribution

Use weighted models to understand the source and nature of combinatorial factors.

Asymptotic enumeration formula

THEOREM Courtiel Melczer M. Raschel 16+

As $n \to \infty$, the number $w_n(a,b)$ of weighted GB walks of length n, and ending anywhere while staying in $\mathbb{R}_+^2$, satisfies, as $n \to \infty$,

$$w_n(a,b) = \kappa \cdot \qquad \rho^{-n} \cdot n^{-\alpha} \cdot (1 + o(1)).$$

Condition	ρ^{-1}	α
$a = b = 1$	4	2
$\sqrt{b} < a < b$	$(1+b)(a^2+b)(ab)^{-1}$	0
$a < 1$ and $b < 1$	4	5
$b > 1$ and $\sqrt{b} > a$	$2(b+1)\sqrt{b}^{-1}$	$3/2$
$a > 1$ and $a > b$	$(1+a)^2 a^{-1}$	$3/2$
$b = a^2 > 1$	$2(b+1)\sqrt{b}^{-1}$	$1/2$
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$a = 1, b < 1$ or $b = 1, a < 1$	4	3

Asymptotic enumeration formula deluxe

THEOREM Courtiel Melczer M. Raschel 16+

As $n \to \infty$, the number $w_n(a,b)$ of weighted GB walks of length n, starting from (i,j) and ending anywhere while staying in $\mathbb{R}_+^2$, satisfies, as $n \to \infty$,

$$w_n(a,b) = \kappa \cdot V^{[n]}(i,j) \cdot \rho^{-n} \cdot n^{-\alpha} \cdot (1 + o(1)).$$

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Values for κ and the harmonic function $V^{[n]}(i,j)$

$a=b=1$: $\kappa = \frac{8}{\pi}$

$$\frac{(i+1)(j+1)(i+j+2)(i+2j+3)}{6}$$

$\sqrt{b} < a < b$: $\kappa = 1$

$$a^{-(4+2i+2j)} b^{-(2+2j)} \left(\left(a^{1+j}-1\right)\left(a^{1+j}+1\right)\left(a^{2+i+j}-b^{2+i+j}\right)\left(a^{2+i+j}+b^{2+i+j}\right) b^{-i-1} - \left(a^{2+i+j}-1\right)\left(a^{2+i+j}+1\right)\left(a^{1+j}-b^{1+j}\right)\left(a^{1+j}+b^{1+j}\right)\right)$$

$a<1, b<1$: $\kappa = \frac{64}{\pi(b-1)^4}$

$$\frac{(1+j)(1+i)(3+i+2j)(2+i+j)}{a^i b^j}\left(\frac{a^2b^2+a^2b-4ab+b+1}{(a-1)^4} + (-1)^{n+i}\frac{a^2b^2+a^2b+4ab+b+1}{(a+1)^4}\right).$$

$b>1, \sqrt{b}>a$: $\kappa = \frac{\sqrt{2}}{\sqrt{\pi} b^2}$

$$\left(\frac{b^{3+i+2j}(1+i)+\left(b^{1+j}-b^{2+i+j}\right)(3+i+2j)-i-1}{a^i b^{i/2+2j}}\right)\left(\frac{1}{(\sqrt{b}-a)^2}+(-1)^{i+n}\frac{1}{(\sqrt{b}+a)^2}\right).$$

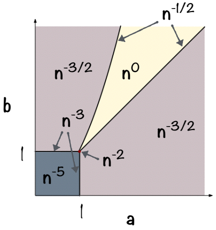
$a>1, a>b$: $\kappa = \frac{(a+1)^3\sqrt{a}}{2\sqrt{\pi}(a-b)^2}$

$$(2+i+j)\left(a^{-2-j}-a^j\right) b^{-j} a^{-1-i} + (1+j)\left(1-a^{-4-2i-2j}\right) b^{-j} a^j$$

$$\vdots$$

Visualize the asymptotic formula

We can plot the different regions of the formula.



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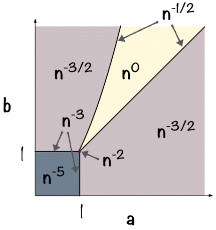

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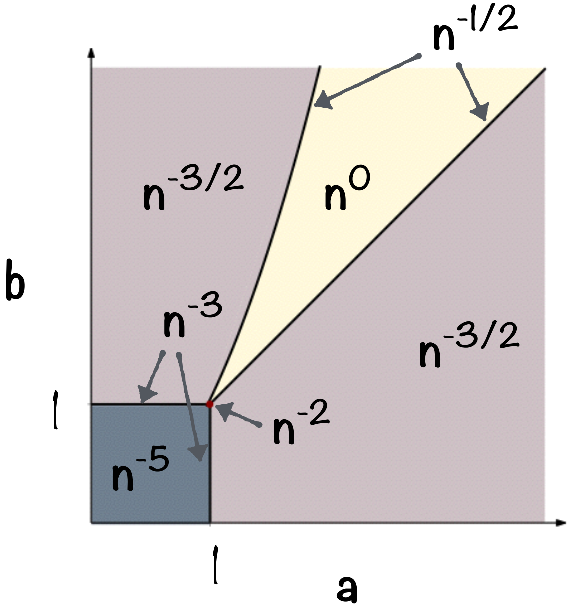
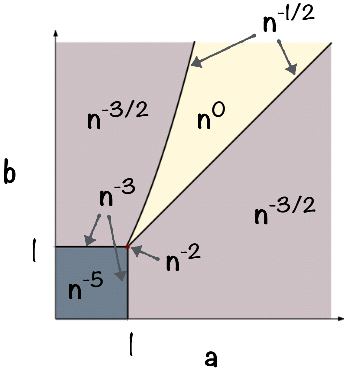
Remark that the exponential growth is continuous.

Visualize the asymptotic formula



Universality classes

A universality class is a family of objects with the same critical exponent.



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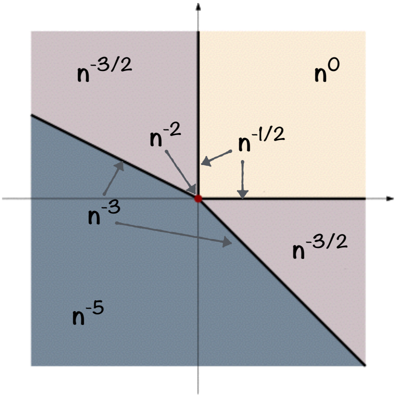
Universality classes... as a function of the drift

The drift is the vector sum of the steps: $(a - a^{-1} + \frac{a}{b} - \frac{b}{a}, \frac{b}{a} - \frac{a}{b})$



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Universality classes... as a function of the drift

The drift δ is the vector sum of the steps:

$$\delta = \left(a - \frac{1}{a} + \frac{a}{b} - \frac{b}{a}, \frac{b}{a} - \frac{a}{b} \right)$$

Universality classes... as a function of the drift

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$$\text{e.g.} a < 1, b = 1 \implies \delta = (2a - \frac{2}{a}, \frac{1}{a} - a) = (2X, -X), X < 0$$

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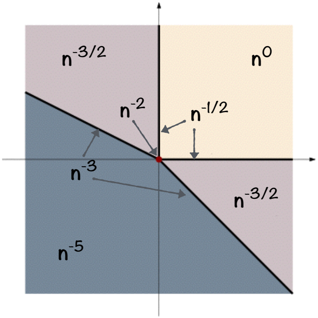

- Is there a diagram like this for any model?
- Are the regions always cones?
- What can be proved at a general level?

TECHNIQUE:
ANALYTIC COMBINATORICS IN SEVERAL VARIABLES
(ACSV)

Strategy

$$\text{GOAL:} \qquad w_n(a,b) \sim C\rho^{-n} n^{-\alpha}$$

1. $W_{a,b}(t)$ as a diagonal of a rational function

$$[t^n]W_{a,b}(t) = [x^n y^n z^n] \frac{P(x,y)}{(1 - zxyS(x^{-1}, y^{-1}))\,(x-1)(y-1)}.$$

2. Express $[t^n]W_{a,b}(t)$ as a generalized Cauchy integral.
3. Identify *contributing critical points*
   1. Rescale the integral to put critical points at origin ($\implies \rho$)
   2. Apply powerful theorems to get asymptotic estimates ($\implies \alpha$)

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Spoiler alert: The inventory $S(x,y)$ tells almost the whole story $\implies$ generality in the approach.

Diagonal Expressions

Δ: The (complete) diagonal operator

$$\Delta \sum_{n\geq 0} \left(\sum_{\mathbf{i}\in\mathbb{Z}^d} f_{\mathbf{i}}(n) z_1^{i_1} \cdots z_d^{i_d} \right) t^n := \sum_{n\geq 0} f_{n,\ldots,n}(n) t^n.$$

Bousquet-Mélou, Mishna 10; Kauers Yatchak 15, Melczer, Wilson 16

$$W(t) = [x^{\geq} y^{\geq}] \frac{(1-\overline{x})(1+\overline{x})(1-\overline{y})\left(1-\overline{x}^2 y\right)(1-x\overline{y})(1+x\overline{y})}{1-t(x+\overline{x}+x\overline{y}+\overline{x}y)}.$$

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$$R(x,y) = \frac{yz^2(y-b)(a-x)(a^2y-bx^2)(ay-bx)(ay+bx)}{(1-xyzS(x^{-1},y^{-1}))}$$

$$W_{a,b}(t) = \frac{1}{a^4b^3z^2} \cdot \Delta\left(\frac{R(x,y)}{(1-x)(1-y)}\right)$$

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For free: Excursion generating function

$$E(t) = \frac{1}{a^4 b^3 z^2} \cdot \Delta R(x,y)$$

A diagonal extraction is a contour integral computation

THEOREM: Multivariate Cauchy Integral Formula

Suppose that $F(x, y, t) \in \mathbb{Q}(x, y, t)$ is analytic at $(0, 0, 0)$ with a power series expansion $F(x, y, t) = \sum_{i_1, i_2, i_3 \geq 0} a_{i_1, i_2, i_3} x^{i_1} y^{i_2} t^{i_3}$ at the origin. Then for all $n \geq 0$,

$$a_{n,n,n} = \frac{1}{(2\pi i)^3} \int_T \frac{F(x, y, t)}{(xyt)^n} \cdot \frac{dx\, dy\, dt}{xyt},$$

where T is a poly-disk defined by $\{|x| = \epsilon_1, |y| = \epsilon_2, |z| = \epsilon_3\}$, for the ϵ_j sufficiently small.

The exponential growth

$$F(x, y, z) = \sum a_{i,j,k} x^i y^j z^k \in \mathbb{N}[x, \bar{x}, y, \bar{y}][[z]]$$

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Valid for points in the disk of convergence $\mathcal{D}$

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Absolute convergence $\implies$ for $(x, y, z) \in \mathcal{D}$, the sum converges... so does subseries $\sum a_{nnn}(|xyz|)^n$

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That is, $\Delta F = \sum a_{nnn} t^n$ converges for $t = |xyz|$ when $(x, y, z) \in \mathcal{D}$.

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Here, the bound is provably tight.

Punchline

$$\rho = \sup_{(x,y,z)\in\overline{\mathcal{D}}} |xyz|$$

The Critical Points

How to find this sup? $\mathfrak{D}$?

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Definition

The critical points of $\frac{G(x,y,z)}{H(x,y,z)}$ satisfy

$$H(x,y,z) = 0 \quad H_x(x,y,z) = H_y(x,y,z) \quad H_x(x,y,z) = H_z(x,y,z)$$

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Here,

$$H(x,y,z)=(1-xyzS(x^{-1},y^{-1}))(x-1)(y-1).$$

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$$H(x,y,z) = 0 \quad H_x(x,y,z) = H_y(x,y,z) \quad H_x(x,y,z) = H_z(x,y,z)$$

Here,
$$H(x,y,z) = (1 - xyzS(x^{-1}, y^{-1}))(x-1)(y-1).$$
The equations imply critical points look like
$$\left(x, y, \left(xyS(x^{-1}, y^{-1})\right)^{-1}\right.$$

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Punchline (matches Garbit & Raschel)

$$\rho = \sup_{(x,y,z)\in\overline{\mathcal{D}}} |xyz| = \frac{1}{S(x_s,y_s)}$$

Critical points as a function of a and b

Inventory:

$$S(x,y) = ax + \frac{1}{ax} + \frac{ax}{by} + \frac{by}{ax}$$

Global minimum of $S(x,y)$:

$$\left(\frac{1}{a}, \frac{1}{b}\right)$$

Critical point:

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$a > 1$**?**

1. $a = b = 1 \implies \quad \rho^{-1} = S(1,1) = 4$
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COROLLARY

The exponential growth changes smoothly, as the evaluation of a Laurent polynomial.

The constant and the critical exponent

THEOREM Hörmander; Pemantle, Wilson

Suppose that the functions $A(\boldsymbol{\theta})$ and $\phi(\boldsymbol{\theta})$ in d variables are smooth in a neighbourhood $\mathcal{N}$ of the origin and that ϕ has a critical point at $\boldsymbol{\theta} = \mathbf{0}$ plus some technical conditions. Then for any integer $M > 0$ there exist effective constants $C_0, \ldots, C_M$ such that

$$\int_X A(\boldsymbol{\theta})\, e^{-n\phi(\boldsymbol{\theta})} \mathrm{d}\boldsymbol{\theta} = \left(\frac{2\pi}{n}\right)^{d/2} \det(\mathcal{H})^{-1/2} \cdot \sum_{k=0}^{M} C_k n^{-k} + O\left(n^{-M-1}\right).$$

$C_0 = \phi(\mathbf{0})$; If $A(\boldsymbol{\theta})$ vanishes to order L at the origin then (at least) the constants $C_0, \ldots, C_{\lfloor \frac{L}{2} \rfloor}$ are all zero.

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$$R(x, y) = \frac{yz^2(y-b)(a-x)(a^2y-bx^2)(ay-bx)(ay+bx)}{(1-xyzS(x^{-1}, y^{-1}))}$$

A WORD OR TWO ON CENTRAL WEIGHTS

Central weights are ideal for generating functions

1. Central weights: the weight depends only on the endpoint: equiprobable
2. THM: $wt((i, j)) = a_0 a_1^i a_2^j$
3. PROP: The complete generating function of a weighted model is an algebraic substitution of the unweighted model.
4. The finiteness of the group of a model is unchanged by central weights.

Generating function connections

$$Q_a(x,y;t) = \sum_n t^n \sum_{\substack{w \text{ walk ending} \\ \text{at } (k,\ell) \text{ with } n \text{ steps}}} \left(\prod_{s\in\mathcal{S}} a_i^{n_s(w)} \right) x^k y^\ell a_0^{-n}.$$

PROPOSITION

Let $Q_a(x,y;z)$ be the generating function of walks with a central weighting $a_s = \beta \prod_{k=1}^{d} \alpha_k^{\pi_k(s)}$ and $Q(x,y;z)$ the generating function of unweighted walks with the same set of steps. Then

$$Q_a(x,y;z) = Q(a_1 x, a_2 y; a_0 z). \tag{1}$$

COR: This generates an infinite colletion of non-D-finite models.

A Wider Picture

Context: Small step 2D lattice models

Bostan, Kauers 09

OEIS Tag	Steps	Equation sizes			Asymptotics	OEIS Tag	Steps	Equation sizes			Asymptotics
A000012		1, 0	1, 1	1, 1	1	A000079		1, 0	1, 1	1, 1	2^n
A001405		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n}{\sqrt{n}}$	A000244		1, 0	1, 1	1, 1	3^n
A001006		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$	A005773		2, 1	2, 3	2, 2	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A126087		3, 1	2, 5	2, 2	$\frac{12\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^{3n/2}}{n^{3/2}}$	A151255		6, 8	4, 16	–	$\frac{24\sqrt{2}}{\pi}\frac{2^{3n/2}}{n^2}$
A151265		6, 4	4, 9	6, 8	$\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151266		7, 10	5, 16	–	$\frac{\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A151278		7, 4	4, 12	6, 8	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151281		3, 1	2, 5	2, 2	$\frac{1}{2}3^n$
A005558		2, 3	3, 5	–	$\frac{8}{\pi}\frac{4^n}{n^2}$	A005566		2, 2	3, 4	–	$\frac{4}{\pi}\frac{4^n}{n}$
A018224		2, 3	3, 5	–	$\frac{2}{\pi}\frac{4^n}{n}$	A060899		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A060900		2, 3	3, 5	8, 9	$\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}\frac{4^n}{n^{2/3}}$	A128386		3, 1	2, 5	2, 2	$\frac{6\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n 3^{n/2}}{n^{3/2}}$
A129637		3, 1	2, 5	2, 2	$\frac{1}{2}4^n$	A151261		5, 8	4, 15	–	$\frac{12\sqrt{3}}{\pi}\frac{2^n 3^{n/2}}{n^2}$
A151282		3, 1	2, 5	2, 2	$\frac{A^2B^{3/2}}{2^{3/4}\Gamma(\frac{1}{2})}\frac{B^n}{n^{3/2}}$	A151291		6, 10	5, 15	–	$\frac{4}{3\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A151275		9, 18	5, 24	–	$\frac{12\sqrt{30}}{\pi}\frac{(\sqrt{24})^n}{n^2}$	A151287		7, 11	5, 19	–	$\frac{\sqrt{8}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
A151292		3, 1	2, 5	2, 2	$\frac{\sqrt[4]{3}C^2D^{3/2}}{8\Gamma(\frac{1}{2})}\frac{D^n}{n^{3/2}}$	A151302		9, 18	5, 24	–	$\frac{\sqrt{5}}{3\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A151307		8, 15	5, 20	–	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$	A151318		2, 1	2, 3	2, 2	$\frac{\sqrt{5/2}}{\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A129400		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{6^n}{n^{3/2}}$	A151297		7, 11	5, 18	–	$\frac{\sqrt{3}C^{7/2}}{2\pi}\frac{(2C)^n}{n^2}$
A151312		4, 5	3, 8	–	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	A151323		2, 1	2, 3	4, 4	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(\frac{1}{4})}\frac{6^n}{n^{3/4}}$
A151326		7, 14	5, 18	–	$2\sqrt{3}$ 6^n	A151314		9, 18	5, 24	–	$EF^{7/2}$ $(2F)^n$

Foire Aux Questions

What is the next (easiest) way to generalize small step walks?
length of step? dimension?

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Conjecture for sub-exponential growth Garbit, Mustafa, Raschel 16+

Suppose that $\mathcal{S}$ is a non-singular step set. Let

$$(x_s, y_s) = \underset{x \geq 1, y \geq 1}{\arg\min}\, S(x, y).$$

Then the asymptotic growth of the number of walks in the first quadrant is given by the following table.

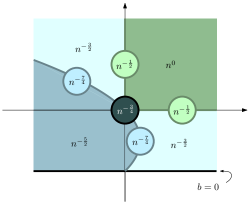
	$\nabla S(x_s, y_s) = 0$	$S_x(x_s, y_s) = 0$ or $S_y(x_s, y_s) = 0$	$S_x(x_s, y_s) > 0$ and $S_y(x_s, y_s) > 0$
$(x_s, y_s) = (1, 1)$	$S(1,1)^n\, n^{-p_1/2}$ balanced	$S(1,1)^n\, n^{-1/2}$ axial	$S(1,1)^n n^0$ free
$x^* = 1$ or $y^* = 1$	$S(x_s, y_s)^n\, n^{-p_1/2-1}$ transitional	$\min\{S(x_s, 1), S(1, y_s)\}^n\, n^{-3/2}$ directed	(not possible)
$x_s > 1$ and $y_s > 1$	$S(x_s, y_s)^n n^{-p_1-1}$ reluctant	(not possible)	(not possible)

$$c = \frac{S_{xy}(x_s,y_s)}{\sqrt{S_{xx}(x_s,y_s) S_{yy}(x_s,y_s)}} \qquad p_1 = \pi / \arccos(-c)$$

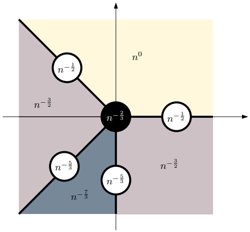
PROVABLE: Prove in case of a finite orbit sum.

Drift diagrams for other models

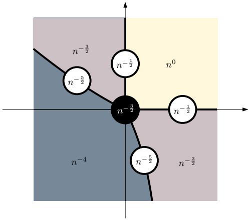
Kreweras



Gessel



Tandem



OPEN: The regions are *not* always cones! What's the story?

Conclusion

Main result

Asymptotic enumeration formula for weighted Gouyou-Beauchamps model

Implications

- Simplified context for ACSV: good entry point?
- Understanding of the mechanism of how drift drives asymptotics
- New discrete harmonic functions
- Discovery of universality classes

Could it be true?

The location of the critical point of the INVENTORY defines the universality classes of the weighted walks.
The Non-D-finite generating functions of lattice walks are diagonals of *something* of similar structure.

Questions for you

- Find explicit generating functions for weighted small step walks.
- Can we team up creative telescoping and ACSV for mutual simplification?
- How can we uncover the link between the properties of the harmonic function constant and holonomy?

Merci Beaucoup!