```
> restart : with(DEtools) :
>
> ## GenHypSols files
> read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/Hyper_gen_t\
ransf.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/recover_\
pullback.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/log_reco\
ver_pullback.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/irrat_rec\
over_pullback.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/rat_reco\
ver_pullback.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/recover_\
coeffs.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/GenHyp\
Sols.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/GenHypSols/extras.
txt`;
>
> ## utilities files
> read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/utilities/utilities_diff.txt`;
>
> ## RemovSing files
> read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/RemovSing/to_modify/Rem\
ovSing.txt`;
>
> ## AppSing files
> read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/AppSing/AppSing.txt`;
>
> ## miniISOLDE files
> read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/miniISOLDE/SplitUnivari\
ate.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/miniISOLDE/RegSing\
Univariate.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/miniISOLDE/MoserU\
nivariate.txt`;
read `/Users/suzy/Documents/Work/Packages/code_versions/version2.0/miniISOLDE/ExpUniv\
ariate.txt`;
#read`/Users/suzy/Documents/Work/Packages/code_versions/version2.0/miniISOLDE/ExpUn\
ivariate_debug.txt`;
>
>
>
```

Generalized hypergeometric systems

Example 1

>

> $diffop2de\left(subs\left(\mathrm{D}=Dx, \mathrm{D}^2+\frac{2\,\mathrm{D}}{x-1}-\frac{1}{x-1}\right), y(x), [Dx, x]\right);$

$$-\frac{y(x)}{x-1}+\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\right)}{x-1}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,y(x) \qquad (1.1.1)$$

> $dsolve(\%);$

$$y(x)=\frac{\_C1\,\mathrm{BesselI}\left(1, 2\sqrt{x-1}\right)}{\sqrt{x-1}}+\frac{\_C2\,\mathrm{BesselY}\left(1, 2\,\mathrm{I}\sqrt{x-1}\right)}{\sqrt{x-1}} \qquad (1.1.2)$$

Generating Example 1

>

> $L := generalized\_hyper\_eqn\_coeff([\ ], [2], \mathrm{D}, Dx, x, 1, 0);$

$$L := \mathrm{D}^2+\frac{2\,\mathrm{D}}{x}-\frac{1}{x} \qquad (1.2.1)$$

> $diffop2de(subs(\mathrm{D}=Dx, L), y(x), [Dx, x]); dsolve(\%);$

$$-\frac{y(x)}{x}+\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\right)}{x}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,y(x)$$

$$y(x)=\frac{\_C1\,\mathrm{BesselI}\left(1, 2\sqrt{x}\right)}{\sqrt{x}}+\frac{\_C2\,\mathrm{BesselK}\left(1, 2\sqrt{x}\right)}{\sqrt{x}} \qquad (1.2.2)$$

>

> $Lf := changeOfVars(\mathrm{L}, \mathrm{x}-1, \mathrm{D}, \mathrm{x});$

$$Lf := \mathrm{D}^2+\frac{2\,\mathrm{D}}{x-1}-\frac{1}{x-1} \qquad (1.2.3)$$

> $diffop2de(subs(\mathrm{D}=Dx, \mathrm{Lf}), y(x), [Dx, x]);$

$$-\frac{y(x)}{x-1}+\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\right)}{x-1}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,y(x) \qquad (1.2.4)$$

> $dsolve(\%);$

$$y(x)=\frac{\_C1\,\mathrm{BesselI}\left(1, 2\sqrt{x-1}\right)}{\sqrt{x-1}}+\frac{\_C2\,\mathrm{BesselY}\left(1, 2\,\mathrm{I}\sqrt{x-1}\right)}{\sqrt{x-1}} \qquad (1.2.5)$$

Matricial representation of Example 1

> $B := op2matrix\_q(L, \mathrm{D});$

$$B := \begin{bmatrix} 0 & 1 \\ \frac{1}{x} & -\frac{2}{x} \end{bmatrix} \quad \textbf{(1.3.1)}$$

> f := (x − 1); BB := changeOfVars_matrix(B, x, f);

$$f := x - 1$$

$$BB := \begin{bmatrix} 0 & 1 \\ \frac{1}{x-1} & -\frac{2}{x-1} \end{bmatrix} \quad \textbf{(1.3.2)}$$

Example 2

>

> $diffop2de\left(subs\left(D = Dx, generalized\_hyper\_eqn\_coeff\left(\left[\frac{1}{5}\right], \left[\frac{1}{3}\right], D, Dx, x, 1, 1\right)\right), y(x), [Dx, x]\right);$

$$-\frac{1}{5}\frac{y(x)}{x} + \frac{1}{15}\frac{(-15x+5)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{d^2}{dx^2}y(x) \quad \textbf{(1.4.1)}$$

> dsolve(%);

$$y(x) = \_C1\, x^{2/3}\, \mathrm{KummerM}\left(\frac{13}{15}, \frac{5}{3}, x\right) + \_C2\, x^{2/3}\, \mathrm{KummerU}\left(\frac{13}{15}, \frac{5}{3}, x\right) \quad \textbf{(1.4.2)}$$

Generating Example 2

> $L := generalized\_hyper\_eqn\_coeff\left(\left[\frac{1}{5}\right], \left[\frac{1}{3}\right], D, Dx, x, 1, 1\right);$

$$L := D^2 + \frac{1}{15}\frac{(-15x+5)\,D}{x} - \frac{1}{5x} \quad \textbf{(1.5.1)}$$

> diffop2de(subs(D = Dx, L), y(x), [Dx, x]); dsolve(%);

$$-\frac{1}{5}\frac{y(x)}{x} + \frac{1}{15}\frac{(-15x+5)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{d^2}{dx^2}y(x)$$

$$y(x) = \_C1\, x^{2/3}\, \mathrm{KummerM}\left(\frac{13}{15}, \frac{5}{3}, x\right) + \_C2\, x^{2/3}\, \mathrm{KummerU}\left(\frac{13}{15}, \frac{5}{3}, x\right) \quad \textbf{(1.5.2)}$$

> $Lf := exp\_transf\left(\frac{-15\cdot x+5}{15\cdot x}, \frac{-1}{5\cdot x}, x^2, D, x\right);$

$$Lf := D^2 - \frac{1}{3}\frac{(6x^3+3x-1)\,D}{x} + \frac{1}{15}\frac{15x^5+15x^3-35x^2-3}{x} \quad \textbf{(1.5.3)}$$

> diffop2de(subs(D = Dx, Lf), y(x), [Dx, x]);

$$\frac{1}{15}\,\frac{\left(15\,x^5+15\,x^3-35\,x^2-3\right)y(x)}{x}-\frac{1}{3}\,\frac{\left(6\,x^3+3\,x-1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\right)}{x}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,y(x) \qquad (1.5.4)$$

> *dsolve*(%);

$$y(x)=\_C1\,x^{2/3}\,\mathrm{KummerM}\left(\frac{13}{15},\frac{5}{3},x\right)\mathrm{e}^{\frac{1}{3}\,x^3}+\_C2\,x^{2/3}\,\mathrm{KummerU}\left(\frac{13}{15},\frac{5}{3},x\right)\mathrm{e}^{\frac{1}{3}\,x^3} \qquad (1.5.5)$$

Matricial representation of Example 2

> $B := op2matrix\_q(L, \mathrm{D});$

$$B := \begin{bmatrix} 0 & 1 \\ \frac{1}{5\,x}\ \ \frac{1}{3} & \frac{3\,x-1}{x} \end{bmatrix} \qquad (1.6.1)$$

> $BB := exp\_transf\_matrix(B, x^2, x);$

$$BB := \begin{bmatrix} -x^2 & 1 \\ \frac{1}{5\,x}\ \ -\frac{1}{3} & \frac{3\,x^3-3\,x+1}{x} \end{bmatrix} \qquad (1.6.2)$$

>

Transformations

Gauge transformation

> $H20 := generalized\_hyper\_sys\_coeff\left([7], \left[2, \frac{3}{2}\right], \mathrm{D}, Dx, x, 2, 1\right);$

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{7}{x^2} & \frac{x-3}{x^2} & -\frac{9}{2\,x} \end{bmatrix} \qquad (2.1.1)$$

>

> $T := Matrix\left(3, 3, \left[x, 0, 0, \frac{1}{x+1}, 0, 1, 0, 1, \frac{1}{x}\right]\right);$

$$T := \begin{bmatrix} x & 0 & 0 \\ \frac{1}{x+1} & 0 & 1 \\ 0 & 1 & \frac{1}{x} \end{bmatrix} \quad \textbf{(2.1.2)}$$

> *A1* := *gauge_transf_matrix*(*H20*, *T*, *x*);

$$A1 := \begin{bmatrix} -\frac{1}{x+1} & 0 & \frac{1}{x} \\ \frac{7x^3+15x^2+3x-3}{(x+1)^2 x^2} & -\frac{11}{2x} & \frac{1}{2} \quad \frac{2x^2-13x-13}{(x+1)x^2} \\ \frac{2}{(x+1)^2} & 1 & \frac{1}{x+1} \end{bmatrix} \quad \textbf{(2.1.3)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, t, *0*);
Generalized_exponents_Maple_format(*A1*, *x*, *t*, *0*);

$$\left[\left[x=t, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right]$$

$$\left[\left[x=t, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right] \quad \textbf{(2.1.4)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, infinity);
Generalized_exponents_Maple_format(*A1*, *x*, *t*, infinity);

$$\left[\left[\frac{1}{x}=t^2, \frac{1}{t}, 1, \left[-\frac{5}{2}\right]\right], \left[x=t^2, 0, 1, [7]\right]\right]$$

$$\left[\left[\frac{1}{x}=t^2, \frac{1}{t}, 1, \left[-\frac{3}{2}\right]\right], \left[x=t^2, 0, 1, [8]\right]\right] \quad \textbf{(2.1.5)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, -*1*);
Generalized_exponents_Maple_format(*A1*, *x*, *t*, -*1*);

$$[[x+1=t, 0, 3, 0]]$$

$$\left[\left[x+1=t, 0, 3, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right]\right] \quad \textbf{(2.1.6)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *5*);
Generalized_exponents_Maple_format(*A1*, *x*, *t*, *5*);

$$[[x-5=t,0,3,0]]$$
$$[[x-5=t,0,3,0]] \tag{2.1.7}$$

Exp-product transformation

> *H20* := *generalized_hyper_sys_coeff*$\left([7], \left[2, \frac{3}{2}\right], \mathrm{D}, Dx, x, 2, 1\right)$;

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{7}{x^2} & \frac{x-3}{x^2} & -\frac{9}{2x} \end{bmatrix} \tag{2.2.1}$$

> *A2* := *exp_transf_matrix*$\left(H20, \frac{2}{(x+1)^2} + \frac{1}{(x-5)^3} + \frac{1}{x^2}, x\right)$;

$$A2 := \left[\left[-\frac{3x^5-42x^4+198x^3-239x^2-175x-125}{x^2(x+1)^2(x-5)^3}, 1, 0\right], \left[0, -\frac{3x^5-42x^4+198x^3-239x^2-175x-125}{x^2(x+1)^2(x-5)^3}, 1\right], \left[\frac{7}{x^2}, \frac{x-3}{x^2}, -\frac{1}{2}\,\frac{9x^6-111x^5+330x^4+486x^3-2053x^2-1475x-250}{x^2(x+1)^2(x-5)^3}\right]\right] \tag{2.2.2}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *0*);
> *Generalized_exponents_Maple_format*(*A2*, *x*, *t*, *0*);

$$\left[\left[x=t, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right]$$
$$\left[\left[x=t, -\frac{1}{t}, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right] \tag{2.2.3}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, infinity);
> *Generalized_exponents_Maple_format*(*A2*, *x*, *t*, infinity);

$$\left[\left[\frac{1}{x}=t^2, \frac{1}{t}, 1, \left[-\frac{5}{2}\right]\right], \left[x=t^2, 0, 1, [7]\right]\right]$$
$$\left[\left[\frac{1}{x}=t^2, \frac{1}{t}, 1, \left[-\frac{5}{2}\right]\right], \left[x=t^2, 0, 1, [7]\right]\right] \tag{2.2.4}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *-1*);
> *Generalized_exponents_Maple_format*(*A2*, *x*, *t*, *-1*);

$$[[x+1=t, 0, 3, 0]]$$

$$\left[\left[x+1=t, -\frac{2}{t}, 3, 0\right]\right] \quad (2.2.5)$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *5*);
    Generalized_exponents_Maple_format(*A2*, *x*, *t*, *5*);

$$[[x-5=t, 0, 3, 0]]$$

$$\left[\left[x-5=t, -\frac{1}{t^2}, 3, 0\right]\right] \quad (2.2.6)$$

Change of variable

> $H20 := generalized\_hyper\_sys\_coeff\left([7], \left[2, \frac{3}{2}\right], \mathrm{D}, Dx, x, 2, 1\right);$

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{7}{x^2} & \frac{x-3}{x^2} & -\frac{9}{2x} \end{bmatrix} \quad (2.3.1)$$

> $f := \frac{(x-2)}{(x-3)^2};\ A3 := changeOfVars\_matrix(H20, x, f);$

$$f := \frac{x-2}{(x-3)^2}$$

$$A3 := \begin{bmatrix} 0 & -\frac{x-1}{(x-3)^3} & 0 \\ 0 & 0 & -\frac{x-1}{(x-3)^3} \\ -\frac{7(x-3)(x-1)}{(x-2)^2} & \frac{(3x^2-19x+29)(x-1)}{(x-2)^2(x-3)} & \frac{9}{2}\frac{x-1}{(x-2)(x-3)} \end{bmatrix} \quad (2.3.2)$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *omega*, *0*);
    Generalized_exponents_Maple_format(*A3*, *x*, *omega*, *0*);

$$\left[\left[x=\omega, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right]$$

$$[[x=\omega, 0, 3, 0]] \quad (2.3.3)$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, infinity);
    Generalized_exponents_Maple_format(*A3*, *x*, *t*, infinity);

$$\left[\left[\frac{1}{x}=t^2, \frac{1}{t}, 1, \left[\ -\frac{5}{2}\ \right]\right], \left[x=t^2, 0, 1, \left[\ 7\ \right]\right]\right]$$

$$\left[\left[\frac{1}{x}=t, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right] \quad \textbf{(2.3.4)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *2*);
> *Generalized_exponents_Maple_format*(*A3*, *x*, *t*, *2*);

$$[[x-2=t, 0, 3, 0]]$$

$$\left[\left[x-2=t, 0, 3, \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\right]\right] \quad \textbf{(2.3.5)}$$

> *Generalized_exponents_Maple_format*(*H20*, *x*, *t*, *3*);
> *Generalized_exponents_Maple_format*(*A3*, *x*, *t*, *3*);

$$[[x-3=t, 0, 3, 0]]$$

$$\left[\left[x-3=t, \frac{2}{t}, 1, \left[\ -4\ \right]\right], \left[x=t, -\frac{2}{t}, 1, \left[\ -4\ \right]\right], \left[x=t, 0, 1, \left[\ 14\ \right]\right]\right] \quad \textbf{(2.3.6)}$$

>

Possible scenarios

What comes from where?

> *H20*;

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{7}{x^2} & \frac{x-3}{x^2} & -\frac{9}{2x} \end{bmatrix} \quad \textbf{(3.1.1)}$$

> *A1*; *## new singularities from Gauge*

(3.1.2)

$$\begin{bmatrix} -\frac{1}{x+1} & 0 & \frac{1}{x} \\ \frac{7x^3+15x^2+3x-3}{(x+1)^2 x^2} & -\frac{11}{2x} & \frac{1}{2}\,\frac{2x^2-13x-13}{(x+1)\,x^2} \\ \frac{2}{(x+1)^2} & 1 & \frac{1}{x+1} \end{bmatrix} \qquad \textbf{(3.1.2)}$$

> *A2*; ## *new singularities from exp*

$$\left[\left[-\frac{3x^5-42x^4+198x^3-239x^2-175x-125}{x^2\,(x+1)^2\,(x-5)^3}, 1, 0\right], \left[0, -\frac{3x^5-42x^4+198x^3-239x^2-175x-125}{x^2\,(x+1)^2\,(x-5)^3}, 1\right], \left[\frac{7}{x^2}, \frac{x-3}{x^2}, -\frac{1}{2}\,\frac{9x^6-111x^5+330x^4+486x^3-2053x^2-1475x-250}{x^2\,(x+1)^2\,(x-5)^3}\right]\right] \qquad \textbf{(3.1.3)}$$

> *A3*; ## *new singularities from change of variable*

$$\begin{bmatrix} 0 & -\frac{x-1}{(x-3)^3} & 0 \\ 0 & 0 & -\frac{x-1}{(x-3)^3} \\ -\frac{7\,(x-3)\,(x-1)}{(x-2)^2} & \frac{(3x^2-19x+29)\,(x-1)}{(x-2)^2\,(x-3)} & \frac{9}{2}\,\frac{x-1}{(x-2)\,(x-3)} \end{bmatrix} \qquad \textbf{(3.1.4)}$$

>

>

Disappearance of singularities corresponding to f

> $H20 := generalized\_hyper\_sys\_coeff\left([\,], \left[\frac{3}{2}, \frac{2}{3}\right], \mathrm{D}, Dx, x, 2, 0\right);$

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{1}{x^2} & -\frac{19}{6x} \end{bmatrix} \qquad \textbf{(3.2.1)}$$

> $f := \frac{(x-1)^6}{(x-3)^2};\ A5 := changeOfVars\_matrix(H20, x, f);$

$$f := \frac{(x-1)^6}{(x-3)^2}$$

$$A5 := \begin{bmatrix} 0 & \frac{4\,(x-1)^5\,(x-4)}{(x-3)^3} & 0 \\ 0 & 0 & \frac{4\,(x-1)^5\,(x-4)}{(x-3)^3} \\ \frac{4\,(x-4)\,(x-3)}{(x-1)^7} & -\frac{4\,(x-4)\,(x-3)}{(x-1)^7} & -\frac{38}{3}\,\frac{x-4}{(x-3)\,(x-1)} \end{bmatrix} \qquad (3.2.2)$$

> $T := \Big[\Big[(x-1)^6 - \frac{355}{4}\,x - \frac{5837}{2}\,x^3 - 27\,x^{15} + 4635\,x^{12} + \frac{5235}{8}\,x^2 + \frac{45}{32}\,x^{16} - \frac{5059}{4}\,x^{13} + \frac{169}{32} + \frac{276103}{8}\,x^6 - 45870\,x^7 - \frac{80613}{4}\,x^5 + \frac{35809}{4}\,x^4 + \frac{1893}{8}\,x^{14} - \frac{24777}{2}\,x^{11} + \frac{200321}{8}\,x^{10} - \frac{156453}{4}\,x^9 + \frac{764577}{16}\,x^8, (x-1)^{15}, \frac{3}{16}\,\big(5\,x^{14} - 66\,x^{13} + 405\,x^{12} - 1532\,x^{11} + 3993\,x^{10} - 7590\,x^9 + 10857\,x^8 - 11880\,x^7 + 9999\,x^6 - 6446\,x^5 + 3135\,x^4 - 1116\,x^3 + 275\,x^2 - 42\,x + 3\big)\,(x-1)^5\Big],$
$\Big[-\frac{355}{4}\,x - \frac{5837}{2}\,x^3 - 27\,x^{15} + 4635\,x^{12} + \frac{5235}{8}\,x^2 + \frac{45}{32}\,x^{16} - \frac{5059}{4}\,x^{13} + \frac{169}{32} + \frac{276103}{8}\,x^6 - 45870\,x^7 - \frac{80613}{4}\,x^5 + \frac{35809}{4}\,x^4 + \frac{1893}{8}\,x^{14} - \frac{24777}{2}\,x^{11} + \frac{200321}{8}\,x^{10} - \frac{156453}{4}\,x^9 + \frac{764577}{16}\,x^8, (x-1)^{15}, \frac{3}{16}\,\big(5\,x^{14} - 66\,x^{13} + 405\,x^{12} - 1532\,x^{11} + 3993\,x^{10} - 7590\,x^9 + 10857\,x^8 - 11880\,x^7 + 9999\,x^6 - 6446\,x^5 + 3135\,x^4 - 1116\,x^3 + 275\,x^2 - 42\,x + 3\big)\,(x-1)^5\Big],$
$\Big[\frac{27}{2} - 9\,x + \frac{3}{2}\,x^2, 0, (x-1)^5\Big]\Big]:$

> $A6 := gauge\_transf\_matrix(A5, T, x):$

> $A7 := exp\_transf\_matrix\left(A6, -\frac{15}{x-1}, x\right):$

> $map(denom, A7);$ ## (x-1) disappeared!!!

$$\begin{bmatrix} 8\,(x-3)^3 & (x-3)^3 & 4\,(x-3)^3 \\ 128\,(x-3)^3 & 4\,(x-3)^3 & 64\,(x-3)^3 \\ 16\,x-48 & x-3 & 24\,x-72 \end{bmatrix} \qquad (3.2.3)$$

>

Classification of singularities: Relation to p and q?

> $H10 := generalized\_hyper\_sys\_coeff\left(\left[7, \frac{2}{3}\right], \left[-\frac{1}{2}\right], \mathrm{D}, Dx, x, 1, 2\right);$

(3.3.1)

$$H10 := \begin{bmatrix} 0 & 1 \\ -\frac{14}{3\,x\,(x-1)} & -\frac{1}{6}\,\frac{52\,x+3}{x\,(x-1)} \end{bmatrix} \quad \textbf{(3.3.1)}$$

> $A4 := exp\_transf\_matrix\left(H10, \frac{2}{(x-5)^6}, x\right);$

5 is introduced by an exp-product transf and so A4 corresponds to pFq's with reg sing and not irreg ones

$$A4 := \left[\left[-\frac{2}{(x-5)^6}, 1\right], \left[-\frac{14}{3\,x\,(x-1)}, -\frac{1}{6}\,\frac{1}{x\,(x-1)\,(x-5)^6}\left(52\,x^7 - 1557\,x^6 + 19410\,x^5 - 128875\,x^4 + 480000\,x^3 - 946863\,x^2 + 756238\,x + 46875\right)\right]\right] \quad \textbf{(3.3.2)}$$

>

>

How to recover f's, E's, and T's?

> $A7;$

 # Our input matrices are complicated !!! What kind of info will lead us to f's,E's, and T's?

$$\left[\left[\frac{1}{8}\,\frac{1}{(x-3)^3}\left(45\,x^{16} - 999\,x^{15} + 10029\,x^{14} - 60731\,x^{13} + 249477\,x^{12} - 740235\,x^{11} + 1646777\,x^{10} - 2809983\,x^9 + 3725667\,x^8 - 3858789\,x^7 + 3116983\,x^6 - 1945569\,x^5 + 920519\,x^4 - 318761\,x^3 + 75771\,x^2 - 10997\,x + 892\right), \frac{4\,(x^7 - 10\,x^6 + 39\,x^5 - 80\,x^4 + 95\,x^3 - 66\,x^2 + 25\,x - 4)\,(x-1)^8}{(x-3)^3},\right.\right. \quad \textbf{(3.4.1)}$$

$$\frac{1}{4}\,\frac{1}{(x-3)^3}\left(15\,x^{19} - 318\,x^{18} + 3129\,x^{17} - 19092\,x^{16} + 81276\,x^{15} - 257160\,x^{14} + 628404\,x^{13} - 1215552\,x^{12} + 1891266\,x^{11} - 2390388\,x^{10} + 2466750\,x^9 - 2080104\,x^8 + 1428492\,x^7 - 792456\,x^6 + 350132\,x^5 - 120352\,x^4 + 30791\,x^3 - 5246\,x^2 + 385\,x + 28\right)\Big],$$

$$\left[\frac{1}{128}\,\frac{1}{(x-3)^3}\left(945\,x^{15} - 18774\,x^{14} + 166383\,x^{13} - 880320\,x^{12}\right.\right.$$

$$+ 3140977 x^{11} - 8082690 x^{10} + 15633303 x^9 - 23320500 x^8 + 27217707 x^7$$

$$- 24967290 x^6 + 17922885 x^5 - 9945304 x^4 + 4179747 x^3 - 1277014 x^2$$

$$+ 239989 x - 1468\Big), \frac{1}{4} \frac{1}{(x-3)^3} \Big(21 x^{14} - 329 x^{13} + 2310 x^{12} - 9786 x^{11}$$

$$+ 28259 x^{10} - 59367 x^9 + 94248 x^8 - 115500 x^7 + 110203 x^6 - 81687 x^5$$

$$+ 46526 x^4 - 19994 x^3 + 6285 x^2 - 1337 x + 148\Big), \frac{1}{64} \frac{1}{(x-3)^3} \Big(315 x^{18}$$

$$- 5943 x^{17} + 51702 x^{16} - 278166 x^{15} + 1044246 x^{14} - 2920218 x^{13}$$

$$+ 6331962 x^{12} - 10923822 x^{11} + 15243468 x^{10} - 17372280 x^9 + 16237278 x^8$$

$$- 12441186 x^7 + 7778026 x^6 - 3931310 x^5 + 1583542 x^4 - 499882 x^3$$

$$+ 122825 x^2 - 23353 x + 2796\Big)\Big],$$

$$\Big[-\frac{3}{16} \frac{1}{x-3} \Big(45 x^{11} - 774 x^{10} + 5709 x^9 - 23996 x^8 + 64442 x^7$$

$$- 117060 x^6 + 147778 x^5 - 130384 x^4 + 79161 x^3 - 31622 x^2 + 7473 x - 676\Big),$$

$$-\frac{6 (x-1)^3 \left(x^7 - 10 x^6 + 39 x^5 - 80 x^4 + 95 x^3 - 66 x^2 + 25 x - 4\right)}{x-3},$$

$$-\frac{1}{24} \frac{1}{x-3} \Big(135 x^{14} - 2187 x^{13} + 15876 x^{12} - 69228 x^{11} + 204039 x^{10}$$

$$- 432135 x^9 + 680724 x^8 - 812592 x^7 + 740421 x^6 - 513513 x^5 + 267300 x^4$$

$$- 101412 x^3 + 26541 x^2 - 4293 x + 244\Big)\Big]\Big]$$

>

Removable singularities

Irregular singularity

Totally removable: irreg---> ordinary

> $H30 := generalized\_hyper\_sys\_coeff\left([7], \left[2, \frac{3}{2}, -\frac{1}{5}\right], D, Dx, x, 3, 1\right);$

(4.1.1.1)

$$H30 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{x^3} & \frac{1}{5} & \frac{5x+3}{x^3} & -\frac{33}{5x^2} & -\frac{63}{10x} \end{bmatrix} \quad \textbf{(4.1.1.1)}$$

> $A9 := exp\_transf\_matrix\left(H30, \frac{1}{(x-5)^3} + \frac{1}{x^2}, x\right);$

$$A9 := \left[\left[-\frac{x^3 - 14x^2 + 75x - 125}{x^2 (x-5)^3}, 1, 0, 0\right], \left[0, -\frac{x^3 - 14x^2 + 75x - 125}{x^2 (x-5)^3}, 1, 0\right], \left[0, 0, -\frac{x^3 - 14x^2 + 75x - 125}{x^2 (x-5)^3}, 1\right], \left[\frac{7}{x^3}, \frac{1}{5}, \frac{5x+3}{x^3}, -\frac{33}{5x^2}, -\frac{1}{10} \frac{63x^4 - 935x^3 + 4585x^2 - 7125x - 1250}{x^2 (x-5)^3}\right]\right] \quad \textbf{(4.1.1.2)}$$

>

> $Generalized\_exponents\_Maple\_format(H30, x, t, 5);$
$Generalized\_exponents\_Maple\_format(A9, x, t, 5);$

$$[[x - 5 = t, 0, 4, 0]]$$

$$\left[\left[x - 5 = t, -\frac{1}{t^2}, 4, 0\right]\right] \quad \textbf{(4.1.1.3)}$$

> $E0, T0, T0inv, A10 := RemovSing\_p(A9, x, x - 5);$

"The roots of ", $x - 5$,

"were removable singularities under the exp-product transformation",

$$-\frac{1}{(x-5)^3}$$

$$E0, T0, T0inv, A10 := -\frac{1}{(x-5)^3}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textbf{(4.1.1.4)}$$

$$\left[\begin{array}{cccc} -\frac{1}{x^2} & 1 & 0 & 0 \\ 0 & -\frac{1}{x^2} & 1 & 0 \\ 0 & 0 & -\frac{1}{x^2} & 1 \\ \frac{7}{x^3} \quad \frac{1}{5}\,\frac{5x+3}{x^3} & -\frac{33}{5x^2} & -\frac{1}{10}\,\frac{63x+10}{x^2} & \end{array}\right]$$

>

Non-removable

> *H30*;

$$\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{x^3} & \frac{1}{5}\,\frac{5x+3}{x^3} & -\frac{33}{5x^2} & -\frac{63}{10x} \end{array}\right] \quad \textbf{(4.1.2.1)}$$

> $f := \frac{(x-2)^2}{(x-3)^4}$; $A12 := changeOfVars\_matrix(H30, x, f)$;

$$f := \frac{(x-2)^2}{(x-3)^4}$$

$$A12 := \left[\left[0, -\frac{2(x-2)(x-1)}{(x-3)^5}, 0, 0\right], \left[0, 0, -\frac{2(x-2)(x-1)}{(x-3)^5}, 0\right], \left[0, 0, 0, -\frac{2(x-2)(x-1)}{(x-3)^5}\right], \left[-\frac{14(x-1)(x-3)^7}{(x-2)^5}, -\frac{2}{5}\,\frac{(x-1)\left(3x^4-36x^3+167x^2-344x+263\right)(x-3)^3}{(x-2)^5}, \frac{66}{5}\,\frac{(x-1)(x-3)^3}{(x-2)^3}, \frac{63}{5}\,\frac{x-1}{(x-2)(x-3)}\right]\right] \quad \textbf{(4.1.2.2)}$$

> *Generalized_exponents_Maple_format*(*A12*, *x*, *t*, *3*);

(4.1.2.3)

$$\left[\left[x-3=-64\,t^3,\ \frac{1}{6\,t}-\frac{1}{64\,t^4},\ 1,\ -\frac{104}{15}\right],\ \left[x=t^3,\ 0,\ 1,\ \left[\ \frac{83}{3}\ \right]\right]\right] \quad \textbf{(4.1.2.3)}$$

>

Partially removable: irreg ---> reg

> *H30*;

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{x^3} & \frac{1}{5} & \frac{5\,x+3}{x^3} & -\frac{33}{5\,x^2} & -\frac{63}{10\,x} \end{bmatrix} \quad \textbf{(4.1.3.1)}$$

> $A10 := exp\_transf\_matrix\left(H30, \frac{1}{x^2}, x\right)$;

$$A10 := \begin{bmatrix} -\frac{1}{x^2} & 1 & 0 & 0 \\ 0 & -\frac{1}{x^2} & 1 & 0 \\ 0 & 0 & -\frac{1}{x^2} & 1 \\ \frac{7}{x^3} & \frac{1}{5}\,\frac{5\,x+3}{x^3} & -\frac{33}{5\,x^2} & -\frac{1}{10}\,\frac{63\,x+10}{x^2} \end{bmatrix} \quad \textbf{(4.1.3.2)}$$

> *Generalized_exponents_Maple_format*(*H30*, *x*, omega, *0*);
Generalized_exponents_Maple_format(*A10*, *x*, *omega*, *0*);

$$\left[\left[x=\omega,\ 0,\ 4,\ \begin{bmatrix} -\frac{9}{5} & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right]\right]$$

$$\left[\left[x=\omega,\ -\frac{1}{\omega},\ 4,\ \begin{bmatrix} -\frac{9}{5} & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right]\right] \quad \textbf{(4.1.3.3)}$$

> $A11 := exp\_transf\_matrix\left(A10, -\frac{1}{x^2}, x\right);$
> #####E0, T0, T0inv,A10:=RemovSing_p(A10, x, x);

$$A11 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{x^3} & \frac{1}{5} & \frac{5x+3}{x^3} & -\frac{33}{5x^2} & -\frac{63}{10x} \end{bmatrix} \quad (4.1.3.4)$$

> Generalized_exponents_Maple_format(A10, x, t, 0);
> Generalized_exponents_Maple_format(A11, x, t, 0);

$$\left[\left[x = t, -\frac{1}{t}, 4, \begin{bmatrix} -\frac{9}{5} & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right]\right]$$

$$\left[\left[x = t, 0, 4, \begin{bmatrix} -\frac{9}{5} & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right]\right] \quad (4.1.3.5)$$

>

>

Regular singularity

Non-removable

> Generalized_exponents_Maple_format(A11, x, t, 0);

$$\left[\left[x = t, 0, 4, \begin{bmatrix} -\frac{9}{5} & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right]\right] \quad (4.2.1.1)$$

>

Removable

> ###H30;

> ###$T := Matrix\left(4, 4, \left[\frac{1}{x-2}, 0, 0, 0, 3, 1, 0, 0, 0, 0, \frac{1}{x-2}, 0, 0, 4, 0, x-2\right]\right)$; $A13 := gauge\_transf\_matrix(H30, T, x)$;

> $A13 := \left[\left[\frac{3x^2-12x+13}{x-2}, x-2, 0, 0\right], \left[-\frac{3(3x^2-12x+13)}{x-2}, -3x+6, \frac{1}{x-2}, 0\right], \left[0, 4x-8, \frac{1}{x-2}, (x-2)^2\right], \left[\frac{1}{2}\frac{72x^5-288x^4+312x^3+6x^2-3x-4}{(x-2)^2 x^3}, \frac{1}{2}\frac{24x^4-48x^3-48x^2+2x+3}{x^3(x-2)}, -\frac{1}{4}\frac{16x^2+21}{(x-2)^2x^2}, -\frac{7x-12}{(x-2)x}\right]\right]$:

> $Generalized\_exponents\_Maple\_format(A13, x, t, 2)$;

$$\left[\left[x-2=t, 0, 4, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right]\right] \quad \textbf{(4.2.2.1)}$$

> $E0, T0, T0inv, A14 := RemovSing\_p(A13, x, x-2)$;

"The roots of ", $x-2$, "were removable singularities under the exp-product",

$-\frac{1}{x-2}$, "and a gauge transformation"

$$E0, T0, T0inv, A14 := -\frac{1}{x-2}, \begin{bmatrix} 0 & \frac{85}{206}(x-2)^2 & (x-2)^2 & 0 \\ 0 & 0 & 0 & x-2 \\ 0 & (x-2)^2 & 0 & 0 \\ 1 & 0 & -\frac{103}{8}x+\frac{103}{4} & -\frac{309}{56}x+\frac{309}{28} \end{bmatrix}, \quad \textbf{(4.2.2.2)}$$

$$\left[\begin{array}{cccc} \frac{103}{8(x-2)} & \frac{309}{56} & -\frac{85}{16(x-2)} & 1 \\ 0 & 0 & \frac{1}{(x-2)^2} & 0 \\ \frac{1}{(x-2)^2} & 0 & -\frac{85}{206(x-2)^2} & 0 \\ 0 & \frac{1}{x-2} & 0 & 0 \end{array}\right], \Bigg[\Bigg[$$

$$-\frac{1}{16}\,\frac{85x^2-170x+96}{x},$$

$$-\frac{1}{11536}\,\frac{1}{x^3}\left((x-2)^2\left(236385x^4-827985x^3+15141x^2+19306x+2380\right)\right),$$

$$-\frac{1}{896}\,\frac{1}{x^3}\left(44496x^6-395125x^5+1061380x^4-989732x^3+135744x^2+1344x+1792\right),$$

$$\frac{1}{896}\,\frac{11433x^5-42484x^4+73844x^3-80832x^2+896x+1344}{x^3}\Bigg],$$

$$\left[1, 0, -\frac{103}{8}x+\frac{103}{4}, -\frac{309}{56}x+\frac{421}{28}\right],$$

$$\left[-\frac{85}{206}, \frac{255}{206}x-\frac{255}{103}, -\frac{133}{8}+\frac{133}{16}x, -\frac{30017}{5768}+\frac{255}{112}x\right],$$

$$\left[0, -\frac{765}{206}x^2+\frac{1530}{103}x-\frac{3109}{206}, -9x^2+36x-39, -3x+6\right]\Bigg]$$

>

Recovering the pullback function

Logarithmic singularities

Example 1

> $H30 := generalized\_hyper\_sys\_coeff\left([\,], \left[3, \frac{2}{3}\right], D, Dx, x, 2, 0\right);$

$$H30 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{2}{x^2} & -\frac{14}{3x} \end{bmatrix} \tag{5.1.1.1}$$

> $f := \frac{4\cdot(x-2)^2\cdot(x-1)\cdot(x+4)}{(x-3)^2}; A15 := changeOfVars\_matrix(H30, x, f);$

$$f := \frac{4(x-2)^2(x-1)(x+4)}{(x-3)^2}$$

$$A15 := \left[\left[0, \frac{4(x-2)\left(2x^3-9x^2-9x+26\right)}{(x-3)^3}, 0\right], \left[0, 0, \frac{4(x-2)\left(2x^3-9x^2-9x+26\right)}{(x-3)^3}\right], \left[\frac{1}{4}\frac{\left(2x^3-9x^2-9x+26\right)(x-3)}{(x+4)^2(x-1)^2(x-2)^3}, -\frac{1}{2}\frac{\left(2x^3-9x^2-9x+26\right)(x-3)}{(x+4)^2(x-1)^2(x-2)^3}, -\frac{14}{3}\frac{2x^3-9x^2-9x+26}{(x-2)(x-1)(x+4)(x-3)}\right]\right] \tag{5.1.1.2}$$

> $Generalized\_exponents\_Maple\_format(A15, x, t, 1);$
$Generalized\_exponents\_Maple\_format(A15, x, t, 2);$
$Generalized\_exponents\_Maple\_format(A15, x, t, 3);$
$Generalized\_exponents\_Maple\_format(A15, x, t, -4);$
$Generalized\_exponents\_Maple\_format(A15, x, t, \text{infinity});$

$$\left[\left[x-1=t, 0, 3, \begin{bmatrix} -\frac{5}{3} & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}\right]\right]$$

$$\left[\left[x-2=t, 0, 3, \begin{bmatrix} -\frac{10}{3} & 0 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -8 \end{bmatrix}\right]\right]$$

$$\left[\left[x-3=-\frac{1}{448}t^3, -\frac{448}{t^2}, 1, \left[\frac{16}{9}\right]\right]\right]$$

$$\left[\left[x+4=t, 0, 3, \begin{bmatrix} -\frac{5}{3} & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}\right]\right]$$

$$\left[\left[\frac{1}{x} = -\frac{1}{32} t^3, -\frac{32}{t^2}, 1, \left[\frac{16}{9}\right]\right]\right] \quad \textbf{(5.1.1.3)}$$

> *log_f*(x, 2, 0, omega, t, [1, 2, -4], [3, infinity], *A15*); ## *this should not be empty*

$$\left\{-\frac{4\,(x-2)^2\,(x-1)\,(x+4)}{(x-3)^2}\right\} \quad \textbf{(5.1.1.4)}$$

>

Example 2

> $f := \frac{(x-2)^5 \cdot (x+4)^2}{(x-3)^2 \cdot (x+1)}$; *A15* := *changeOfVars_matrix*(*H30*, *x*, *f*);

$$f := \frac{(x-2)^5\,(x+4)^2}{(x-3)^2\,(x+1)}$$

$$A15 := \left[\left[0, \frac{(x-2)^4\,(x+4)\,\left(4x^3-3x^2-27x-56\right)}{(x-3)^3\,(x+1)^2}, 0\right],\right.$$
$$\left[0, 0, \frac{(x-2)^4\,(x+4)\,\left(4x^3-3x^2-27x-56\right)}{(x-3)^3\,(x+1)^2}\right],$$
$$\left[\frac{(x-3)\,\left(4x^3-3x^2-27x-56\right)}{(x+4)^3\,(x-2)^6},\right.$$
$$-\frac{2\,(x-3)\,\left(4x^3-3x^2-27x-56\right)}{(x+4)^3\,(x-2)^6},$$
$$\left.\left.-\frac{14}{3}\,\frac{4x^3-3x^2-27x-56}{(x+4)\,(x-2)\,(x-3)\,(x+1)}\right]\right] \quad \textbf{(5.1.2.1)}$$

> *log_f*(x, 2, 0, omega, t, [2, -4], [3, -1, infinity], *A15*); ## *this should not be empty*

$$\left\{-\frac{(x-2)^5\,(x+4)^2}{(x-3)^2\,(x+1)}\right\} \quad \textbf{(5.1.2.2)}$$

Example 3

> $f := \frac{-7}{(x+4)}$; *A16* := *changeOfVars_matrix*(*H30*, *x*, *f*);

$$f := -\frac{7}{x+4}$$

$$A16 := \begin{bmatrix} 0 & \frac{7}{(x+4)^2} & 0 \\ 0 & 0 & \frac{7}{(x+4)^2} \\ \frac{1}{7} & -\frac{2}{7} & \frac{14}{3\,(x+4)} \end{bmatrix} \quad \textbf{(5.1.3.1)}$$

> log_f(x, 2, 0, omega, t, [infinity], [-4], A16);

$$\left\{\frac{7}{x+4}\right\} \quad \textbf{(5.1.3.2)}$$

Example 4

> $f := \frac{(x-2)\cdot(x-1)}{(x-3)^2}$; A17 := changeOfVars_matrix(H30, x, f);

$$f := \frac{(x-2)\,(x-1)}{(x-3)^2}$$

$$A17 := \quad \textbf{(5.1.4.1)}$$

$$\left[\left[0, -\frac{3x-5}{(x-3)^3}, 0\right], \left[0, 0, -\frac{3x-5}{(x-3)^3}\right], \left[-\frac{(x-3)\,(3x-5)}{(x-1)^2\,(x-2)^2}, \frac{2\,(x-3)\,(3x-5)}{(x-1)^2\,(x-2)^2}, \frac{14}{3}\ \frac{3x-5}{(x-1)\,(x-2)\,(x-3)}\right]\right]$$

> log_f(x, 2, 0, omega, t, [1, 2], [3], A17);

$$\left\{-\frac{(x-2)\,(x-1)}{(x-3)^2}\right\} \quad \textbf{(5.1.4.2)}$$

>

Irrational singularity

>

Example 1

> $H30 := generalized\_hyper\_sys\_coeff\left([2, 3], \left[\text{sqrt}(3), \frac{1}{3}, \frac{1}{4}\right], D, Dx, x, 3, 2\right)$;

$$H30 := \quad \textbf{(5.2.1.1)}$$

$$\left[\left[0, 1, 0, 0\right],\right.$$

$$\left[0, 0, 1, 0\right],$$

$$\left[0, 0, 0, 1\right],$$

$$\left.\left[\frac{6}{x^3}, -\frac{1}{12}\frac{\sqrt{3}-72x}{x^3}, -\frac{1}{12}\frac{19\sqrt{3}+20-12x}{x^2}, -\frac{1}{12}\frac{12\sqrt{3}+43}{x}\right]\right]$$

> *Generalized_exponents_Maple_format*(*H30*, *x*, *t*, 0);

$$\left[\left[x=t, 0, 4, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{9}{4} & 0 & 0 \\ 0 & 0 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3}-2 \end{bmatrix}\right]\right] \quad \textbf{(5.2.1.2)}$$

> $f := \frac{7\cdot(x-5)^2}{(x-3)^2}$; *A18* := *changeOfVars_matrix*(*H30*, *x*, *f*);

$$f := \frac{7\,(x-5)^2}{(x-3)^2}$$

$$A18 := \left[\left[0, \frac{28\,(x-5)}{(x-3)^3}, 0, 0\right],\right. \quad \textbf{(5.2.1.3)}$$

$$\left[0, 0, \frac{28\,(x-5)}{(x-3)^3}, 0\right],$$

$$\left[0, 0, 0, \frac{28\,(x-5)}{(x-3)^3}\right],$$

$$\left[\frac{24}{49}\frac{(x-3)^3}{(x-5)^5},\right.$$

$$-\frac{1}{147}\frac{\left(x^2\sqrt{3}-6x\sqrt{3}-504x^2+9\sqrt{3}+5040x-12600\right)(x-3)}{(x-5)^5},$$

$$-\frac{1}{21}\frac{19x^2\sqrt{3}-114x\sqrt{3}-64x^2+171\sqrt{3}+720x-1920}{(x-3)(x-5)^3},$$

$$\left.\left.-\frac{1}{3}\frac{12\sqrt{3}+43}{(x-5)(x-3)}\right]\right]$$

> *Generalized_exponents_Maple_format*(*A18*, *x*, *t*, 3);

Generalized_exponents_Maple_format(*A18*, *x*, *t*, 5);

$$\left[\left[x-3=t, \frac{\mathit{RootOf}(\_Z^2-112)}{t}, 1, \left[\sqrt{3}-\frac{71}{12}\right]\right], \left[x=t, 0, 2, \begin{bmatrix}4 & 1\\ 0 & 4\end{bmatrix}\right]\right]$$

$$\left[\left[x-5=t, 0, 4, \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -\frac{14}{3} & 0 & 0\\ 0 & 0 & -\frac{9}{2} & 0\\ 0 & 0 & 0 & -2\sqrt{3}-4\end{bmatrix}\right]\right] \quad \textbf{(5.2.1.4)}$$

> *irrat_f*(*x*, 1, 0, omega, *t*, [5], [3], *A18*);

$$\left\{-\frac{14\,(x-5)}{(x-3)^2}, \frac{7\,(x-5)^2}{(x-3)^2}\right\} \quad \textbf{(5.2.1.5)}$$

>

Example 2

> $f := \frac{3\cdot(x-\mathrm{sqrt}(2))^2\cdot(x+\mathrm{sqrt}(2))}{(x-3)^2\cdot(x-4)}$; *A19* := *changeOfVars_matrix*(*H30*, *x*, *f*);

$$f := \frac{3\,(x-\sqrt{2})^2\,(x+\sqrt{2})}{(x-3)^2\,(x-4)}$$

$$A19 := \left[\left[0, -\frac{3\,(-x+\sqrt{2})\,(42\,x+\sqrt{2}\,x^2-22-7\sqrt{2}\,x-10\,x^2+12\sqrt{2})}{(x-3)^3\,(x-4)^2}, 0, 0\right], \quad \textbf{(5.2.2.1)}$$

$$\left[0, 0, -\frac{3\,(-x+\sqrt{2})\,(42\,x+\sqrt{2}\,x^2-22-7\sqrt{2}\,x-10\,x^2+12\sqrt{2})}{(x-3)^3\,(x-4)^2}, 0\right],$$

$$\left[0, 0, 0, -\frac{3\,(-x+\sqrt{2})\,(42\,x+\sqrt{2}\,x^2-22-7\sqrt{2}\,x-10\,x^2+12\sqrt{2})}{(x-3)^3\,(x-4)^2}\right],$$

$$\left[-\frac{2}{3}\,\frac{(x-4)\,(x-3)^3\left(42\,x+\sqrt{2}\,x^2-22-7\,\sqrt{2}\,x-10\,x^2+12\,\sqrt{2}\right)}{\left(x+\sqrt{2}\right)^3\left(-x+\sqrt{2}\right)^5},\right.$$

$$\frac{1}{108}\,\frac{1}{\left(x+\sqrt{2}\right)^3\left(-x+\sqrt{2}\right)^5}\left(\left(42\,x+\sqrt{2}\,x^2-22-7\,\sqrt{2}\,x\right.\right.$$

$$\left.-10\,x^2+12\,\sqrt{2}\right)\left(x^3\,\sqrt{3}-10\,x^2\,\sqrt{3}-432\,\sqrt{2}+432\,x+216\,\sqrt{2}\,x^2\right.$$

$$\left.\left.-216\,x^3+33\,x\,\sqrt{3}-36\,\sqrt{3}\right)(x-3)\right),\;\frac{1}{36}\left(\left(19\,x^3\,\sqrt{3}\right.\right.$$

$$-190\,x^2\,\sqrt{3}-72\,\sqrt{2}+732\,x+36\,\sqrt{2}\,x^2-16\,x^3+627\,x\,\sqrt{3}-200\,x^2$$

$$\left.\left.-684\,\sqrt{3}-720\right)\left(42\,x+\sqrt{2}\,x^2-22-7\,\sqrt{2}\,x-10\,x^2+12\,\sqrt{2}\right)\right)\Big/$$

$$\left(\left(x+\sqrt{2}\right)^2\left(-x+\sqrt{2}\right)^3(x-4)\,(x-3)\right),$$

$$\left.\left.-\frac{1}{12}\,\frac{\left(12\,\sqrt{3}+43\right)\left(42\,x+\sqrt{2}\,x^2-22-7\,\sqrt{2}\,x-10\,x^2+12\,\sqrt{2}\right)}{(x-3)\,(x-4)\left(x^2-2\right)}\right]\right]$$

>

> *irrat_f*(*x*, 1, 0, omega, *t*, [sqrt(2), -sqrt(2)], [3, 4], *A19*);

$$\left\{\frac{3\left(-x+\sqrt{2}\right)^2\left(x+\sqrt{2}\right)}{(x-3)^2\,(x-4)}\right\} \qquad \textbf{(5.2.2.2)}$$

Example 3

> $f := \dfrac{11\cdot(x-\mathrm{sqrt}(2))^2\cdot(x+\mathrm{sqrt}(2))}{(x-3)^2}$: A20 := *changeOfVars_matrix*(*H30*, *x*, *f*);

$$A20 := \left[\left[0,\,-\frac{11\left(-x+\sqrt{2}\right)\left(4+\sqrt{2}\,x+x^2-3\,\sqrt{2}-9\,x\right)}{(x-3)^3},\,0,\,0\right],\right. \qquad \textbf{(5.2.3.1)}$$

$$\left[0,\,0,\,-\frac{11\left(-x+\sqrt{2}\right)\left(4+\sqrt{2}\,x+x^2-3\,\sqrt{2}-9\,x\right)}{(x-3)^3},\,0\right],$$

$$\left[0,\,0,\,0,\,-\frac{11\left(-x+\sqrt{2}\right)\left(4+\sqrt{2}\,x+x^2-3\,\sqrt{2}-9\,x\right)}{(x-3)^3}\right],$$

$$\left[-\frac{6}{121}\,\frac{\left(4+\sqrt{2}\,x+x^2-3\,\sqrt{2}-9\,x\right)(x-3)^3}{\left(x+\sqrt{2}\right)^3\left(-x+\sqrt{2}\right)^5},\right.$$

$$\frac{1}{1452}\frac{1}{\left(x+\sqrt{2}\right)^3\left(-x+\sqrt{2}\right)^5}\left(\left(4+\sqrt{2}\,x+x^2-3\sqrt{2}-9x\right)\left(x^2\sqrt{3}-1584\sqrt{2}+1584x+792\sqrt{2}\,x^2-792x^3-6x\sqrt{3}+9\sqrt{3}\right)(x-3)\right),$$

$$\frac{1}{132}\frac{1}{\left(x+\sqrt{2}\right)^2\left(-x+\sqrt{2}\right)^3(x-3)}\left(\left(4+\sqrt{2}\,x+x^2-3\sqrt{2}-9x\right)\left(19x^2\sqrt{3}-264\sqrt{2}+144x+132\sqrt{2}\,x^2-132x^3-114x\sqrt{3}+20x^2+171\sqrt{3}+180\right)\right),$$

$$-\frac{1}{12}\frac{\left(4+\sqrt{2}\,x+x^2-3\sqrt{2}-9x\right)\left(12\sqrt{3}+43\right)}{(x-3)\left(x^2-2\right)}\Bigg]\Bigg]$$

> *irrat_f*(x, 1, 0, omega, t, [sqrt(2), -sqrt(2)], [3, infinity], *A20*);

$$\left\{\frac{11\left(-x+\sqrt{2}\right)^2\left(x+\sqrt{2}\right)}{(x-3)^2}\right\} \quad \textbf{(5.2.3.2)}$$

Example 4

> *H30* := *generalized_hyper_sys_coeff*$\left([2], \left[2\cdot \mathrm{sqrt}(3), \frac{1}{3}\right], D, Dx, x, 2, 1\right)$;

$$H30 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{x^2} & -\frac{1}{3}\,\frac{2\sqrt{3}-3x}{x^2} & -\frac{2}{3}\,\frac{3\sqrt{3}+2}{x} \end{bmatrix} \quad \textbf{(5.2.4.1)}$$

> $f := \frac{12\cdot(x-1)\cdot(x-5)}{(x-3)^2}$; *A21* := *changeOfVars_matrix*(*H30*, x, f);

$$f := \frac{12\,(x-1)\,(x-5)}{(x-3)^2}$$

$$A21 := \left[\left[0, \frac{96}{(x-3)^3}, 0\right], \quad \textbf{(5.2.4.2)}$$

$$\left[0, 0, \frac{96}{(x-3)^3}\right],$$

$$\left[\frac{1}{72}\,\frac{96x-288}{(x-1)^2\,(x-5)^2},\right.$$

$$-\frac{4}{9}\,\frac{x^2\sqrt{3}-6x\sqrt{3}-18x^2+9\sqrt{3}+108x-90}{(x-5)^2\,(x-1)^2\,(x-3)},$$

$$-\frac{16}{3}\,\frac{3\sqrt{3}+2}{(x-3)\,(x-1)\,(x-5)}\Big]\Big]$$

> *Generalized_exponents_Maple_format*(*A21*, *x*, *t*, *1*);
> *Generalized_exponents_Maple_format*(*A21*, *x*, *t*, *5*);
> *Generalized_exponents_Maple_format*(*A21*, *x*, *t*, *3*);

$$\left[\left[x-1=t,\,0,\,3,\,\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{4}{3} & 0\\ 0 & 0 & -2\sqrt{3}-1\end{bmatrix}\right]\right]$$

$$\left[\left[x-5=t,\,0,\,3,\,\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{4}{3} & 0\\ 0 & 0 & -2\sqrt{3}-1\end{bmatrix}\right]\right]$$

$$\left[\left[x-3=t,\,\frac{RootOf\left(\_Z^2+192\right)}{t},\,1,\,\left[-\frac{13}{6}+2\sqrt{3}\right]\right],\,\left[x=t,\,0,\,1,\,\left[\,4\,\right]\right]\right] \quad \textbf{(5.2.4.3)}$$

> *irrat_f*(*x*, 1, 0, omega, *t*, [1, 5], [3], *A21*);

$$\left\{\frac{12\,(x-1)\,(x-5)}{(x-3)^2}\right\} \quad \textbf{(5.2.4.4)}$$

Rational singularity

Example 1

> $H30 := generalized\_hyper\_sys\_coeff\left([\,],\left[\frac{3}{2},\frac{2}{3}\right],\mathrm{D},Dx,x,2,0\right);$

$$H30 := \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \frac{1}{x^2} & -\frac{1}{x^2} & -\frac{19}{6x}\end{bmatrix} \quad \textbf{(5.3.1.1)}$$

> $f := \frac{(x-4)^2\cdot(x-1)\cdot(x-2)^6}{(x-3)};\ A22 := changeOfVars\_matrix(H30, x, f);$

$$f := \frac{(x-4)^2\,(x-1)\,(x-2)^6}{x-3}$$

$$A22 := \left[\left[0,\,\frac{2\,(x-4)\,(x-2)^5\,\left(4x^3-31x^2+74x-50\right)}{(x-3)^2},\,0\right],\right. \quad \textbf{(5.3.1.2)}$$

$$\left[0, 0, \frac{2\,(x-4)\,(x-2)^5\,\left(4\,x^3-31\,x^2+74\,x-50\right)}{(x-3)^2}\right],$$

$$\left[\frac{2\,\left(4\,x^3-31\,x^2+74\,x-50\right)}{(x-2)^7\,(x-1)^2\,(x-4)^3}, -\frac{2\,\left(4\,x^3-31\,x^2+74\,x-50\right)}{(x-2)^7\,(x-1)^2\,(x-4)^3},\right.$$

$$\left.-\frac{19}{3}\,\frac{4\,x^3-31\,x^2+74\,x-50}{(x-2)\,(x-4)\,(x-1)\,(x-3)}\right]\Bigg]$$

> *Generalized_exponents_Maple_format*(*A22*, *x*, *t*, *1*);
> *Generalized_exponents_Maple_format*(*A22*, *x*, *t*, *2*);
> *Generalized_exponents_Maple_format*(*A22*, *x*, *t*, *4*);
> *Generalized_exponents_Maple_format*(*A22*, *x*, *t*, *3*);

$$\left[\left[x-1=t, 0, 3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & -\frac{5}{3} \end{bmatrix}\right]\right]$$

$$\left[\left[x-2=t, 0, 3, \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}\right]\right]$$

$$\left[\left[x-4=t, 0, 3, \begin{bmatrix} -5 & 0 & 0 \\ 0 & -\frac{10}{3} & 0 \\ 0 & 0 & -5 \end{bmatrix}\right]\right]$$

$$\left[\left[x-3=\frac{1}{4}\,t^3, -\frac{2}{t}, 1, \left[\frac{1}{18}\right]\right]\right] \qquad \textbf{(5.3.1.3)}$$

> *rat_f*(*x*, 2, 0, omega, *t*, [1, 4], [3, infinity], *A22*, { });

$$\left\{-\frac{(2-x)^6\,(x-1)\,(x-4)^2}{x-3}, -\frac{(x-4)^2\,(x-1)\,(x-2)^6}{x-3}\right\} \qquad \textbf{(5.3.1.4)}$$

Example 2

> $f := \frac{(x-1)^6}{(x-3)^2}$: *A23* := *changeOfVars_matrix*(*H30*, *x*, *f*);

(5.3.2.1)

$$A23 := \begin{bmatrix} 0 & \frac{4\,(x-1)^5\,(x-4)}{(x-3)^3} & 0 \\ 0 & 0 & \frac{4\,(x-1)^5\,(x-4)}{(x-3)^3} \\ \frac{4\,(x-4)\,(x-3)}{(x-1)^7} & -\frac{4\,(x-4)\,(x-3)}{(x-1)^7} & -\frac{38}{3}\,\frac{x-4}{(x-3)\,(x-1)} \end{bmatrix} \quad \textbf{(5.3.2.1)}$$

> *Generalized_exponents_Maple_format*(*A23, x, t, 1*);
> *Generalized_exponents_Maple_format*(*A23, x, t, 3*);
> *Generalized_exponents_Maple_format*(*A23, x, t,* infinity);

$$\left[\left[x-1=t, 0, 3, \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}\right]\right]$$

$$\left[\left[x-3=-\frac{1}{512}\,t^3, -\frac{512}{t^2}, 1, \left[\frac{7}{9}\right]\right]\right]$$

$$\left[\left[\frac{1}{x}=-64\,t^3, -\frac{1}{64\,t^4}, 1, \left[\frac{11}{9}\right]\right]\right] \quad \textbf{(5.3.2.2)}$$

> *rat_f*(*x*, 2, 0, omega, *t*, [ ], [3, infinity], *A23*, { });

$$\left\{-\frac{(-x+1)^6}{(x-3)^2}, -\frac{(-x+5)^6}{(x-3)^2}, -\frac{(x-5)^6}{(x-3)^2}, -\frac{(x-1)^6}{(x-3)^2}\right\} \quad \textbf{(5.3.2.3)}$$

>

Example 3

> $f := \frac{5\cdot(x-4)^2\cdot(x-1)\cdot(3\cdot x-2)^6}{(x-3)}$: *A24* := *changeOfVars_matrix*(*H30, x, f*);

$$A24 := \left[\left[0, \frac{10\,(x-4)\,(3\,x-2)^5\,(12\,x^3-89\,x^2+202\,x-122)}{(x-3)^2}, 0\right],\right.$$
$$\left[0, 0, \frac{10\,(x-4)\,(3\,x-2)^5\,(12\,x^3-89\,x^2+202\,x-122)}{(x-3)^2}\right],$$
$$\left[\frac{2}{5}\,\frac{12\,x^3-89\,x^2+202\,x-122}{(3\,x-2)^7\,(x-1)^2\,(x-4)^3}, -\frac{2}{5}\,\frac{12\,x^3-89\,x^2+202\,x-122}{(3\,x-2)^7\,(x-1)^2\,(x-4)^3},\right.$$
$$\left.\left.-\frac{19}{3}\,\frac{12\,x^3-89\,x^2+202\,x-122}{(3\,x-2)\,(x-4)\,(x-1)\,(x-3)}\right]\right] \quad \textbf{(5.3.3.1)}$$

> *Generalized_exponents_Maple_format*(*A24, x, t, 1*);
> $Generalized\_exponents\_Maple\_format\left(A24, x, t, \frac{2}{3}\right)$;

$Generalized\_exponents\_Maple\_format(A24, x, t, 4);$
$Generalized\_exponents\_Maple\_format(A24, x, t, 3);$

$$\left[\left[x-1=t, 0, 3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & -\frac{5}{3} \end{bmatrix}\right]\right]$$

$$\left[\left[x-\frac{2}{3}=t, 0, 3, \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}\right]\right]$$

$$\left[\left[x-4=t, 0, 3, \begin{bmatrix} -5 & 0 & 0 \\ 0 & -\frac{10}{3} & 0 \\ 0 & 0 & -5 \end{bmatrix}\right]\right]$$

$$\left[\left[x-3=\frac{1}{1384128720100}\,t^3, -\frac{1176490}{t}, 1, \left[\frac{1}{18}\right]\right]\right] \tag{5.3.3.2}$$

> $rat\_f(x, 2, 0, \mathrm{omega}, t, [1, 4], [3, \mathrm{infinity}], A24, \{\,\});$

$$\left\{-\frac{5\,(2-3\,x)^6\,(x-1)\,(x-4)^2}{x-3}, -\frac{5\,(x-4)^2\,(x-1)\,(3\,x-2)^6}{x-3}\right\} \tag{5.3.3.3}$$

>

Our approach

Example 1

> $H20 := generalized\_hyper\_sys\_coeff\left([\,], \left[\frac{1}{3}, \frac{-2}{3}\right], \mathrm{D}, Dx, x, 2, 0\right);$

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & \frac{2}{9\,x^2} & -\frac{2}{3\,x} \end{bmatrix} \tag{6.1.1}$$

> $f := \frac{11}{7\cdot(x-2)^2};\ A25 := changeOfVars\_matrix(H20, x, f);$

$$f := \frac{11}{7\,(x-2)^2}$$

$$A25 := \begin{bmatrix} 0 & -\frac{22}{7\,(x-2)^3} & 0 \\ 0 & 0 & -\frac{22}{7\,(x-2)^3} \\ -\frac{14}{11}x+\frac{28}{11} & -\frac{28}{99}x+\frac{56}{99} & \frac{4}{3\,(x-2)} \end{bmatrix} \quad \textbf{(6.1.2)}$$

> $A25 := exp\_transf\_matrix\left(A25, -\frac{5}{(x+6)^3}, x\right);$

$$A25 := \begin{bmatrix} \frac{5}{(x+6)^3} & -\frac{22}{7\,(x-2)^3} & 0 \\ 0 & \frac{5}{(x+6)^3} & -\frac{22}{7\,(x-2)^3} \\ -\frac{14}{11}x+\frac{28}{11} & -\frac{28}{99}x+\frac{56}{99} & \frac{1}{3}\,\frac{4x^3+72x^2+447x+834}{(x-2)\,(x+6)^3} \end{bmatrix} \quad \textbf{(6.1.3)}$$

> *Generalized_exponents_Maple_format*(A25, x, t, 2);
Generalized_exponents_Maple_format(A25, x, t, -6);
Generalized_exponents_Maple_format(A25, x, t, infinity);

$$\left[\left[x-2=-\frac{7}{88}t^3, -\frac{88}{7\,t^2}, 1, \left[-\frac{8}{9}\right]\right]\right]$$

$$\left[\left[x+6=t, \frac{5}{t^2}, 3, 0\right]\right]$$

$$\left[\left[\frac{1}{x}=t, 0, 3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 1 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}\right]\right] \quad \textbf{(6.1.4)}$$

>

> $E, T, Tinv, BB := RemovSing\_p(A25, x, (x+6)) : map(denom, BB);$

"The roots of ", $x+6$,

"were removable singularities under the exp-product transformation", $\frac{5}{(x+6)^3}$

$$\begin{bmatrix} 1 & 7\,(x-2)^3 & 1 \\ 1 & 1 & 7\,(x-2)^3 \\ 11 & 99 & 3x-6 \end{bmatrix} \quad \textbf{(6.1.5)}$$

> $E, T, Tinv, BB := RemovSing\_p(A25, x, (x-2)) :$

>

> log_f(x, 2, 0, omega, t, [infinity], [2], A25, { });

$$\left\{-\frac{11}{7\,(x-2)^2}\right\} \tag{6.1.6}$$

> L := recover_H0_candidates$\left(\left[\left[\left[\frac{1}{x}=\omega, 0, 3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 1 \\ 0 & 0 & -\frac{8}{3}\end{bmatrix}\right]\right]\right], \left[\left[x-2=-\frac{7}{88}\,\omega^3, -\frac{88}{7\,\omega^2}, 1, \left[-\frac{8}{9}\right]\right]\right]\right]$, [infinity], [2], [ ], [ ], [2], [2], x, D, Dx, a, b, 2, 3);

$$L := \left\{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{16}{9\,x^2} & -\frac{11}{3\,x}\end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{22}{9\,x^2} & -\frac{25}{6\,x}\end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{22}{9\,x^2} & -\frac{25}{6\,x}\end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{121}{36\,x^2} & -\frac{14}{3\,x}\end{bmatrix}\right\} \tag{6.1.7}$$

> Candidate_1 := L[1] : M := changeOfVars_matrix$\left(H20, x, \frac{11}{7\,(x-2)^2}\right)$;

$$M := \begin{bmatrix} 0 & -\frac{22}{7\,(x-2)^3} & 0 \\ 0 & 0 & -\frac{22}{7\,(x-2)^3} \\ -\frac{14}{11}\,x+\frac{28}{11} & -\frac{28}{99}\,x+\frac{56}{99} & \frac{4}{3\,(x-2)}\end{bmatrix} \tag{6.1.8}$$

> M_E := exp_transf_matrix$\left(M, -\frac{5}{(x+6)^3}, x\right)$;

(6.1.9)

$$M\_E := \begin{bmatrix} \frac{5}{(x+6)^3} & -\frac{22}{7(x-2)^3} & 0 \\ 0 & \frac{5}{(x+6)^3} & -\frac{22}{7(x-2)^3} \\ -\frac{14}{11}x+\frac{28}{11} & -\frac{28}{99}x+\frac{56}{99} & \frac{1}{3}\frac{4x^3+72x^2+447x+834}{(x-2)(x+6)^3} \end{bmatrix} \quad (6.1.9)$$

> $T := find\_gauge\_rational(M\_E, A25, x);$

$$T := \begin{bmatrix} \_c_1 & 0 & 0 \\ 0 & \_c_1 & 0 \\ 0 & 0 & \_c_1 \end{bmatrix} \quad (6.1.10)$$

> *### Rmk: The other candidates are also equivalent*

>

Example 2

> $H20 := generalized\_hyper\_sys\_coeff\left([\,], \left[\frac{3}{2}, \frac{2}{3}\right], \mathrm{D}, Dx, x, 2, 0\right);$

$$H20 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x^2} & -\frac{1}{x^2} & -\frac{19}{6x} \end{bmatrix} \quad (6.2.1)$$

> $f := \frac{(x-4)^2\cdot(x-1)\cdot(x-2)^6}{(x-3)}; A25 := changeOfVars\_matrix(H20, x, f);$

$$f := \frac{(x-4)^2 (x-1) (x-2)^6}{x-3}$$

$$A25 := \left[\left[0, \frac{2(x-4)(x-2)^5(4x^3-31x^2+74x-50)}{(x-3)^2}, 0\right], \left[0, 0, \frac{2(x-4)(x-2)^5(4x^3-31x^2+74x-50)}{(x-3)^2}\right], \left[\frac{2(4x^3-31x^2+74x-50)}{(x-2)^7(x-1)^2(x-4)^3}, -\frac{2(4x^3-31x^2+74x-50)}{(x-2)^7(x-1)^2(x-4)^3}, -\frac{19}{3}\frac{4x^3-31x^2+74x-50}{(x-2)(x-4)(x-1)(x-3)}\right]\right] \quad (6.2.2)$$

> $Generalized\_exponents\_Maple\_format(\mathrm{A25, x, t}, 2);$
> $Generalized\_exponents\_Maple\_format(A25, x, t, 4);$

```
Generalized_exponents_Maple_format(A25, x, t, 1);
Generalized_exponents_Maple_format(A25, x, t, 3, 1);
Generalized_exponents_Maple_format(A25, x, t, infinity);
```

$$\left[\left[x-2=t, 0, 3, \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}\right]\right]$$

$$\left[\left[x-4=t, 0, 3, \begin{bmatrix} -5 & 0 & 0 \\ 0 & -\frac{10}{3} & 0 \\ 0 & 0 & -5 \end{bmatrix}\right]\right]$$

$$\left[\left[x-1=t, 0, 3, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & -\frac{5}{3} \end{bmatrix}\right]\right]$$

$$\left[\left[x-3=\frac{1}{4}t^3, -\frac{2}{t}, 1, \left[\frac{1}{18}\right]\right]\right]$$

$$\left[\left[\frac{1}{x}=-\frac{1}{512}t^3, -\frac{134217728}{t^8}-\frac{983040}{t^5}-\frac{1280}{t^2}, 1, \frac{28}{9}\right]\right] \tag{6.2.3}$$

> $E, T, Tinv, BB := RemovSing\_p(A25, x, (x-2)) : map(denom, BB);$

"The roots of ", $x-2$, "were removable singularities under the exp-product", $-\frac{15}{x-2}$,

"and a gauge transformation"

$$\begin{bmatrix} 64\,(x-3)^2 & (x-3)^2 & (x-3)^2 \\ 64\,(x-3)^2\,(x-1)^2\,(x-4)^3 & (x-3)^2 & (x-3)^2\,(x-4)\,(x-1) \\ 8192\,(x-3)^2\,(x-1)^2\,(x-4)^3 & 64\,(x-3)^2 & 192\,(x-3)^2\,(x-4)\,(x-1) \end{bmatrix} \tag{6.2.4}$$

> $rat\_f(x, 2, 0, \text{omega}, t, [1, 4], [3, \text{infinity}], A25, \{\});$

$$\left\{-\frac{(2-x)^6\,(x-1)\,(x-4)^2}{x-3}, -\frac{(x-4)^2\,(x-1)\,(x-2)^6}{x-3}\right\} \tag{6.2.5}$$

> $recover\_H0\_candidates\left([\,], \left[\left[x-3=\frac{1}{4}\omega^3, -\frac{2}{\omega}, 1, \left[\frac{1}{18}\right]\right]\right], \left[\frac{1}{x}=-\frac{1}{512}\omega^3, -\frac{1280}{\omega^2}-\frac{134217728}{\omega^8}-\frac{983040}{\omega^5}, 1, \frac{28}{9}\right]\right], [\,], [\,], [2], [6], [3, \text{infinity}], [1, 8], x, D, Dx, a, b, 2, 3\right);$

$$\left\{\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{1}{x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{3}{2x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{4}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{7}{6x}\end{bmatrix},\right. \tag{6.2.6}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{5}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{2}{9x^2}&-\frac{2}{x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{5}{18x^2}&-\frac{13}{6x}\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{5}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{3x^2}&-\frac{13}{6x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{2}{9x^2}&-\frac{2}{x}\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{9x^2}&-\frac{11}{6x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{4}{9x^2}&-\frac{7}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{5}{9x^2}&-\frac{5}{2x}\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{11}{6x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{3}{2x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{4x^2}&-\frac{2}{x}\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{6x^2}&-\frac{11}{6x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{12x^2}&-\frac{5}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{3x^2}&-\frac{13}{6x}\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{5}{12x^2}&-\frac{7}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&0&-\frac{4}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{6x^2}&-\frac{11}{6x}\end{bmatrix},$$

$$\left.\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{9x^2}&-\frac{5}{3x}\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\\frac{1}{x^2}&-\frac{1}{18x^2}&-\frac{3}{2x}\end{bmatrix}\right\}$$

> *##...... recover coeffs for each and test equivalence for each*