Some Recent Advances in Ramanujan-type Series for $1/\pi$

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SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN
Established in collaboration with MIT

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But it is bigger than Andorra, Malta, Liechtenstein, San Marino, Monaco, ...

Outline

1. Introduction – why Ramanujan series are special, and how to prove them
   - What they are not
   - How to prove them
2. Generalizations – some recent developments, and how they relate to generating functions
   - General theorem
   - Proof
3. Computer algebra – leading to some examples that cannot be explained by classical (modular) theory
   - Wilf-Zeilberger
   - Legendre's relation

Hypergeometric series

Pochhammer symbol: $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.

Generalized *hypergeometric series* ${}_pF_q$:

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

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Complete elliptic integral K and E:

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle| k^2\right), \quad E(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle| k^2\right).$$

$$k' := \sqrt{1-k^2}, \quad K'(k) := K(k'), \quad E'(k) := E(k').$$

Ramanujan series for $1/\pi$ – what they are

S. Ramanujan (1914) found innovative ways to write $1/\pi$ as a series involving algebraic (sometimes rational) summands:

$$\sum_{n=0}^{\infty} \frac{(s)_n (\frac{1}{2})_n (1-s)_n}{n!^3} (a+bn)\, z_0^n = \frac{1}{\pi},$$

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In terms of a hypergeometric series:

$${}_4F_3\left(\begin{matrix} s, \frac{1}{2}, 1-s, 1+\frac{a}{b} \\ 1,\ 1,\ \frac{a}{b} \end{matrix} \middle| z_0\right) = \frac{1}{a\,\pi}.$$

Ramanujan series for $1/\pi$ – examples

First anticipated by G. Bauer (1859),

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (1+4n)(-1)^n = \frac{2}{\pi}.$$

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For instance, the series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{1}{2})_n (\frac{5}{6})_n}{n!^3} (13591409+545140134n) \left(\frac{-1}{53360^3} \right)^n = \frac{640320^{3/2}}{12\pi}$$

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Such computations contributed to the early development of high-performance computing.

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$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n^3}{n!^3} (1+12n)(-1)^n = \frac{3}{\pi}.$$

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We may expand a function f as

$$f(x) = \sum_{n=0}^{\infty} a_n\, P_n(x), \quad \text{where } a_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)\, f(x)\, \mathrm{d}x.$$

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E. g. picking $f(x) = (1-x^2)^{m-1/2}$ then setting $x=0$ gives

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(1+4n)(-1)^n \frac{(n+\frac{1}{2})_{-m}}{(n+1)_m} = \frac{(-1)^m}{(\frac{1}{2})_m^2}\frac{2}{\pi}.$$

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Setting $m=0$ gives Bauer's series. This approach is also limited.

Ramanujan series for $1/\pi$ – how they work

The Ramanujan series

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$$a\,{}_3F_2\left(\left.\begin{matrix} s, \frac{1}{2}, 1-s \\ 1,\, 1 \end{matrix}\right| z_0\right) + (b\,z_0)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_3F_2\left(\left.\begin{matrix} s, \frac{1}{2}, 1-s \\ 1,\, 1 \end{matrix}\right| z\right)\bigg|_{z=z_0} = \frac{1}{\pi}.$$

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Clausen's formula (1828) states that

$${}_3F_2\left(\left.\begin{matrix} s, \frac{1}{2}, 1-s \\ 1,\ 1 \end{matrix}\right| 4k^2(1-k^2)\right) = {}_2F_1\left(\left.\begin{matrix} s, 1-s \\ 1 \end{matrix}\right| k^2\right)^2.$$

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So let $z_0 = 4k^2(1-k^2)$, but not just for any k.

Theta functions

When $s = 1/2$,

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$$ \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \;\; \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \;\; \theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. $$

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Fact: with the (unexpected) parametrization $-\log(q) = \pi\frac{K'(k)}{K(k)}$:

$$ k(q) = \frac{\theta_2^2(q)}{\theta_3^2(q)}, \quad k'(q) = \frac{\theta_4^2(q)}{\theta_3^2(q)}, \quad K(k(q)) = \frac{\pi}{2}\theta_3^2(q). $$

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Consequently, for $p \in \mathbb{N}$, $k(q^p)$ is an algebraic function of $k(q)$; alternatively, there is a polynomial P_p in two variables with integer coefficients such that

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Moreover, $M_p := K(k(q^p))/K(k(q))$ is also an algebraic function of $k(q)$, called the multiplier of order p.

Modular machinery

Due to θ function transformations, $k(e^{-\pi\sqrt{p}}) = k'(e^{\pi/\sqrt{p}})$. Since $k(e^{-\pi\sqrt{p}})$ and $k(e^{-\pi/\sqrt{p}})$ are related by the pth order modular equation, we can solve for them.

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In fact $K(k_p)$ can be evaluated in closed form (Selberg-Chowla), moreover

$$E(k_p) = \left(1 - \frac{\alpha_p}{\sqrt{p}}\right) K(k_p) + \frac{\pi}{4\sqrt{p}\, K(k_p)},$$

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J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, Wiley, New York, 1987.

Derivatives

The final pieces we need are

$$\frac{\mathrm{d}}{\mathrm{d}k}K(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \qquad \frac{\mathrm{d}}{\mathrm{d}k}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}.$$

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As we've seen, a Ramanujan series is a linear combination of $\frac{4}{\pi^2}K(k)^2$ and its derivative, $\frac{8}{\pi^2}\left[\frac{E(k)K(k)}{k(1-k^2)} - \frac{K(k)^2}{k}\right]$.

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($\frac{1}{\pi}$ is not that special, it's a by-product of a transformation.)

General result ($s = 1/2$)

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} \Big(\underbrace{\alpha_p - \sqrt{p}\, k_p^2}_{a} + \underbrace{\sqrt{p}(1 - 2k_p^2)}_{b}\, n \Big) \big(\underbrace{4k_p^2(1 - k_p^2)}_{z_0} \big)^n = \frac{1}{\pi}.$$

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The coefficients in modular equations grow quickly as p increases. For instance, the Chudnovsky series uses $p = 163$, but no one so far has exhibited P_{163}, so the series has no satisfactory proof.

General result ($s = 1/2$)

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} \Big(\underbrace{\alpha_p - \sqrt{p}\, k_p^2}_{a} + \underbrace{\sqrt{p}(1-2k_p^2)}_{b}\, n\Big) \big(\underbrace{4k_p^2(1-k_p^2)}_{z_0}\big)^n = \frac{1}{\pi}.$$

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Open question 1

Find an efficient algorithm to generate the pth order modular equation for any $p \in \mathbb{N}$.

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   - What they are not
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   - General theorem
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Generalizations of Ramanujan series

More recently, several authors (B. Berndt, H. H. Chan, S. Cooper, W. Zudilin, et al.) have proven new series of Ramanujan type:

$$\sum_{n=0}^{\infty} h(n)\,(a+bn)\,z_0^n = \frac{1}{\pi},$$

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From here we can apply the same procedure: use Clausen's formula to relate this to K, evaluate at singular values, etc.

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Theorem (J. W., W. Zudilin 2011)

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There are 14 special triplets of (a,b,c) for which $\{u_n\}$ is an integer sequence (Apéry-like sequence).

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$$= (1-cXY)\underbrace{\left\{\sum_{n=0}^{\infty} u_n X^n\right\}}_{\sim K(x)} \underbrace{\left\{\sum_{n=0}^{\infty} u_n Y^n\right\}}_{\sim K(y)}.$$

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The RHS involves K's of different arguments, but can be simplified if we pick $x = k(q^p)$, $y = k(q)$, then use the multiplier M_p:

$$K(x)K(y) = M_p\, K(y)^2.$$

Example

With $(a, b, c) = (7, 2, -8)$, $p = 2$, $q = \exp(-4\sqrt{2/3}\,\pi)$:

$$\sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{n} \binom{n}{k}^3}_{u_n} P_n\left(\frac{1+\sqrt{3}}{\sqrt{6}}\right) (\underbrace{7-2\sqrt{3}}_{a} + \underbrace{18}_{b}\, n) \left(\underbrace{\frac{2-\sqrt{3}}{2\sqrt{6}}}_{z_0}\right)^n = \frac{27+11\sqrt{3}}{\sqrt{2}\,\pi}.$$

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The moral seems to be: as long as a sequence has an ogf expressible as a product of K's, it can produce Ramanujan type series for $1/\pi$. We still need the singular values, but do not require the sequence to be modular.

Proof of the Theorem

The Theorem was discovered with help from a computer search.

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- The Theorem has been generalized even further by H. H. Chan and Y. Tanigawa in 2013.

Open questions

Open question 2

Find a closed form for

$$\sum_{n=0}^{\infty} P_n(x)^3 \, z^n,$$

where x and z may be related in some way.

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Open question 3

Suppose the generating function $\sum_{n=0}^{\infty} A_n(x) \, z^n$ satisfies a 2nd order linear differential equation. What can we say about

$$\sum_{n=0}^{\infty} \binom{2n}{n} A_n(x) \, z^n?$$

Again, x and z may be related in some way.

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So $\displaystyle\sum_n F(m,n) = \frac{\Gamma(m+3/2)}{\Gamma(3/2)\Gamma(m+1)}$.

Finally, set $m=-\frac{1}{2}$; equality still holds due to Carlson's theorem.

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Examples of Guillera's series obtained using WZ:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n^2 (\frac{3}{4})_n^2}{(\frac{1}{2})_n \, n!^3} \frac{48n^2+32n+3}{2n+1} \frac{1}{4^n} = \frac{8\sqrt{2}}{\pi},$$

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Open question 4

In all the series proven by WZ so far, the only prime factors of z_0 are 2 and 3. Why?

Legendre's relation

The four functions K, E, K', E' are not independent:

Legendre (1811)

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2}.$$

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Let z_0 be a point such that $\alpha(z_0)^2 = 1 - \beta(z_0)^2$, so the RHS $= K(\alpha(z_0))\, K'(\alpha(z_0))$.

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For some undetermined coefficients $A_i(z_0)$, compute

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J. Wan, Series for $1/\pi$ using Legendre's relation, *Integr. Transf. Spec. F.* **25** (2014), 1–14.

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With $\rho = (1 - 2xz + z^2)^{1/2}$, $\alpha = (1 - \rho - z)/2$, $\beta = (1 - \rho + z)/2$,

$$\underbrace{\sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{n!^2} P_n(x) z^n}_{G(z)} = {}_2F_1\left(\left.\begin{matrix} s, 1-s \\ 1 \end{matrix}\right| \alpha\right) {}_2F_1\left(\left.\begin{matrix} s, 1-s \\ 1 \end{matrix}\right| \beta\right).$$

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Let $s = 1/4$ and apply the transformation below to both ${}_2F_1$'s:

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix} \middle| z^2\right) = \frac{1}{\sqrt{1+z}} \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{2z}{1+z}\right).$$

A lot of computer algebra

We require

$$\frac{2\sqrt{\alpha}}{1+\sqrt{\alpha}} + \frac{2\sqrt{\beta}}{1+\sqrt{\beta}} = 1.$$

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As outlined, take a linear combination of derivatives of G.
Substitute in the above x_0 and z_0, and choose the coefficients such that the result is $EK' + E'K - KK'$.

The result

For *each* $q \in (0,1)$, we get a series for $1/\pi$:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{3}{4})_n}{n!^2} P_n(x_0) \left(C_0+C_1 n+C_2 n^2+C_3 n^3\right) {z_0}^n = \frac{3(3+q)(9-q)^{5/2}}{2\pi},$$

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where

$$\begin{aligned}
C_3 &= (1-q)^2(63+q)^2,\\
C_2 &= -48(1-q)(81+162q+13q^2),\\
C_1 &= -(9-41q)(9+106q+13q^2),\\
C_0 &= 96q^2(7+q).
\end{aligned}$$

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- This seems to eliminate the need for modular functions, except many of the transformations used seem modular in nature.
- Other authors (Guillera, Zudilin, ...) are independently using transformations to avoid modular functions.

Non-modular example

A draw back of this method is that the functions get complicated very quickly, and some form of automatic differentiation is needed.

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This method can produce series which look like they belong to a class of (modular) series, but are in fact not modular. E. g.

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 P_n\left(\frac{1}{2}\right)(3+14n)\left(\frac{3}{128}\right)^n = \frac{8\sqrt{2}}{\pi}.$$

It looks like a consequence of the Theorem, but it's not.

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Ramanujan's series stand out from other series for $1/\pi$ (or other transcendental constants).

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Thank you.