Computing the Rank Profile Matrix

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Inria SpecFun Seminar
Saclay, November 16, 2015.

Gaussian elimination in computer algebra

Swiss army knife for applications:

Matrix factorization **(LU decomposition)**

- Solving linear systems
- Computing determinants

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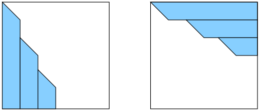
Computing linear dependencies **(Echelon structure)**

- Basis of vector spaces (Krylov iteration)
- Echelon structure of the Macaulay matrix (Gröbner basis)



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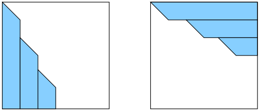

Rank profiles: how to select the first 3 linearly indep rows of

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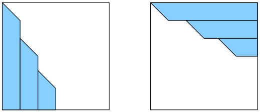

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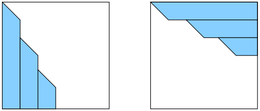

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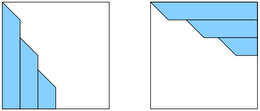

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Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal list of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Rank $= 3$
RowRP $= \{1,2,4\}$

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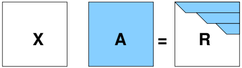
But $|A_{1,1}| = 0$

Relation to echelon forms

Transformation to echelon form

$\forall A\ \exists X$ non-singular s.t.

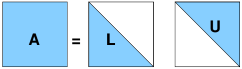


Relation to echelon forms:

- ColRP unchanged by left multiplication with an invertible matrix

ColRP = pivot columns in the **row** echelon form

Triangular Matrix decompositions and rank profiles

Decomposition	Exists for	Unique
	Generic rank profile	✓

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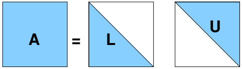
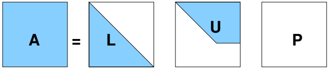
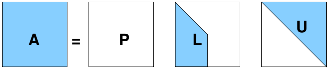
	Generic rank profile	✓
	Generic row rank profile	✗
	Generic col rank profile	✗

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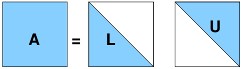
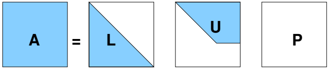
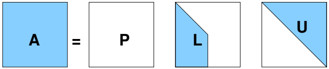
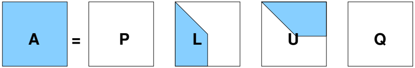
	Generic rank profile	✓
	Generic row rank profile	✗
	Generic col rank profile	✗
	Any matrix	✗

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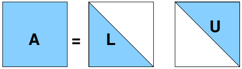
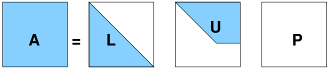
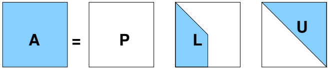
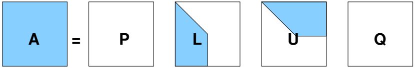
	Generic rank profile	✓
	Generic row rank profile	✗
	Generic col rank profile	✗
	Any matrix	✗

$\rightarrow P, Q$ may reveal row and/or col rank profiles.

Computing rank profiles

Via Gaussian elimination revealing row echelon forms:

[Ibarra, Moran and Hui 82]

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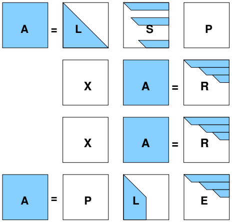

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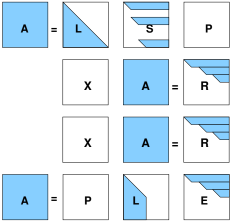
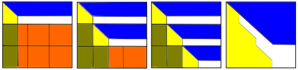


Lessons learned (or what we thought was necessary):

- treat rows in order
- exhaust all columns before considering the next row
- **slab** block splitting (recursive or iterative)



$\rightsquigarrow$ similar to partial pivoting

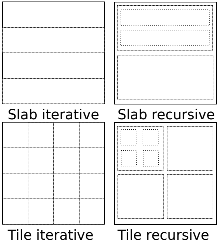
Motivation

Need more flexible blocking

Slab blocking

- leads to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead ?



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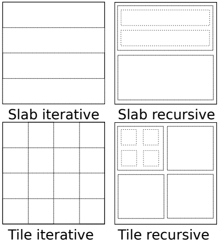


Gathering linear independence invariants

Two ways to look at a matrix (looking left or right):

- Row rank profile, column echelon form
- Column rank profile, row echelon form

Unique invariant?

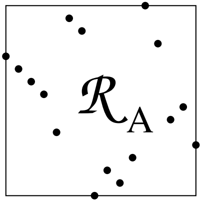
Outline

1. The rank profile Matrix
2. Computing the rank profile matrix
3. Algorithmic instances
4. Relations to other decompositions
5. Generalization over a Ring
6. The small rank case

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m\times n}$.
There exists a unique, $m \times n$, rank(A)-sub-permutation matrix $\mathcal{R}^A$ of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of A.



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Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Properties of the rank profile matrix

Properties

- A *invertible* $\Rightarrow \mathcal{R}^A$ *is a permutation matrix*
- A *is square with generic rank profile* $\Rightarrow \mathcal{R}^A = I_n$
- *RowRP*(A) = *RowSupport*$(\mathcal{R}^A)$
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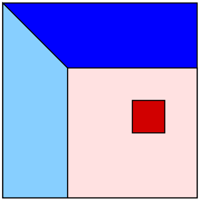

Four types of elementary operations:

Search: finding a pivot

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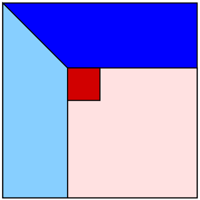

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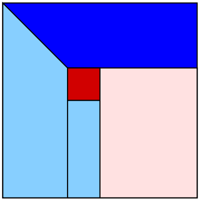

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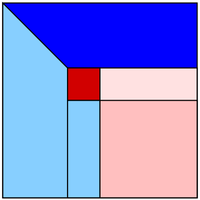

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Update: applying the elimination $a_{i,j} \leftarrow a_{i,j} - \frac{a_{i,k}a_{k,j}}{a_{k,k}}$

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

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Problem (Reformulation)

Under what conditions on the **Search** *and* **Permutation** *operations does a PLUQ decomposition algorithm reveals RowRP, ColRP or $\mathcal{R}^A$?*

The Pivoting matrix

Definition (The pivoting matrix)

Given a PLUQ decomposition $A = PLUQ$ with rank r, define

$$\Pi_{P,Q} = P \begin{bmatrix} I_r & \\ & \end{bmatrix} Q.$$

Locates the position of the pivots in the matrix A.

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- $\Pi_{P,Q} = \mathcal{R}^A$
- $RowSupp(\Pi_{P,Q}) = RowSupp(\mathcal{R}^A) = RowRP(A)$ *(Weaker)*
- $ColSupp(\Pi_{P,Q}) = ColSupp(\mathcal{R}^A) = ColRP(A)$ *(Weaker)*

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

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Example

Search: "Any non zero element on the topmost row":

$$A = \begin{bmatrix} 2&0&3&0\\ 1&0&0&0\\ 0&0&4&0\\ 0&2&0&1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix},$$

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Search: "Any non zero element on the topmost row":

$$A = \begin{bmatrix} 2&0&3&0\\ 1&0&0&0\\ 0&0&4&0\\ 0&2&0&1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}, \quad \Pi_{P,Q} = \begin{bmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}.$$

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

Search revealing rank profiles

- No stability issue over exact domains
- Intuition: must **minimize** some **ordering of the row/col indices** (notion of rank profile)

Example

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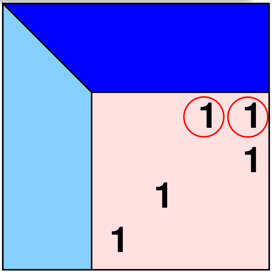
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row



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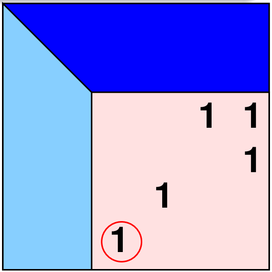


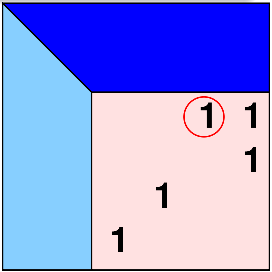
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex order: first non-zero on the first non-zero row



Pivoting and permutation strategies

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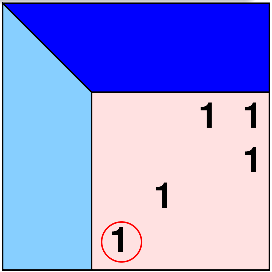

Pivoting and permutation strategies

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Row/Col order: any non-zero on the first non-zero row/col

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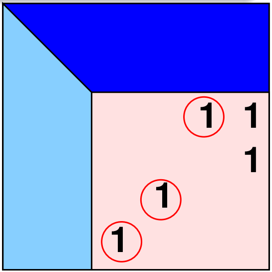
Product order: first non-zero in the (i, j) leading sub-matrix



Sufficient ?

Is lexicographic ordering sufficient to reveal both rank profiles?

Example

With a lexicographic ordering

1. $A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Pi_{P,Q}$

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2. But $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \rightsquigarrow \mathcal{R}^A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\Pi_{P,Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

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$\rightsquigarrow$ Pivot Swaps mix-up precedence between rows/cols.

$\rightsquigarrow$ **Permutations** also have to be considered

Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

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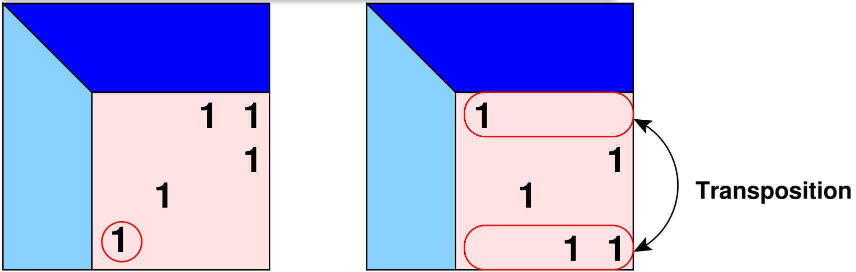
Permutation

- Transpositions



Pivoting and permutation strategies

Pivot Search

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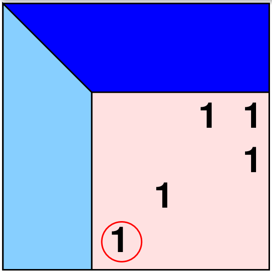
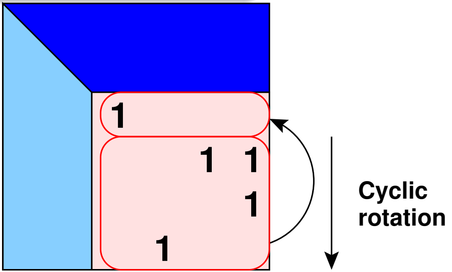
Permutation

- Transpositions
- Cyclic Rotations





Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order Col. order						
Lexico.						
Rev. lex.						
Product						

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order Col. order	Transposition	Transposition	✓			[IMH82] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \left[\, 1\ 2\ \ldots\ m \,\right] P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$

Pivoting strategies revealing rank profiles

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Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product						

- RowRP $= [\,1\ 2\ \ldots\ m\,]\, P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
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Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
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Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

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- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
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Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
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Lexico.	Transposition	Transposition	✓			[Sto00]
Lexico.	Transposition	Rotation	✓	✓	✓	[DPS15]
Lexico.	Rotation	Rotation	✓	✓	✓	[DPS15]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Rev. lex.	Rotation	Transposition	✓	✓	✓	[DPS15]
Rev. lex.	Rotation	Rotation	✓	✓	✓	[DPS15]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

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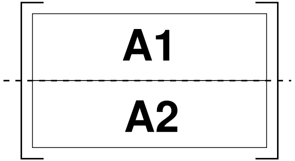
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

Outline

1. The rank profile Matrix
2. Computing the rank profile matrix
3. Algorithmic instances
4. Relations to other decompositions
5. Generalization over a Ring
6. The small rank case

The slab recursive algorithm

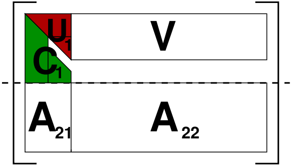
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1

The slab recursive algorithm

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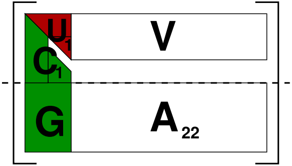

1. Split A Row-wise
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3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)

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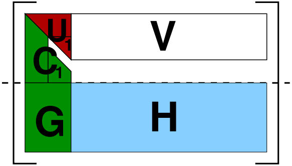

1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ ($\texttt{trsm}$)
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The slab recursive algorithm

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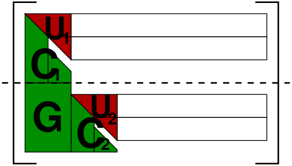

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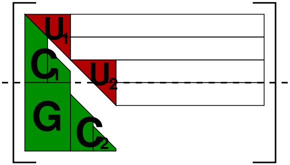

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6. Row permutations

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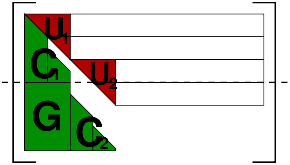

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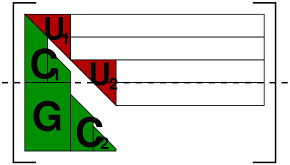
Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



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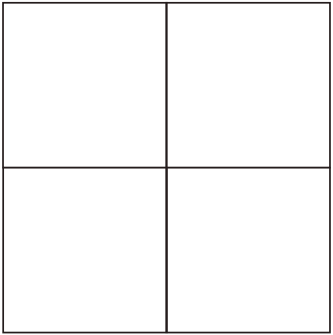
6. Row permutations

Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP
- Row Rotations : Computes $\mathcal{R}^A$ [DPS15]

The tiled recursive algorithm

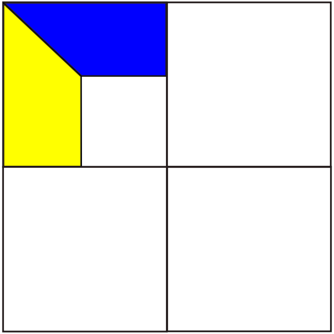
Dumas, P. and Sultan 13



2×2 block splitting

The tiled recursive algorithm

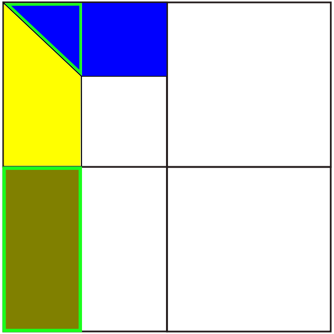
Dumas, P. and Sultan 13



Recursive call

The tiled recursive algorithm

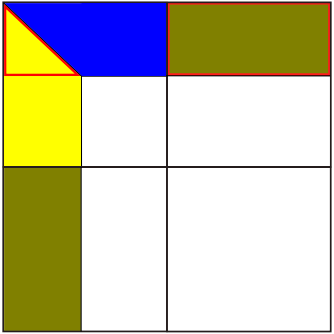
Dumas, P. and Sultan 13



TRSM: $B \leftarrow BU^{-1}$

The tiled recursive algorithm

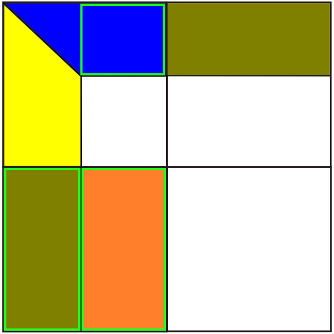
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

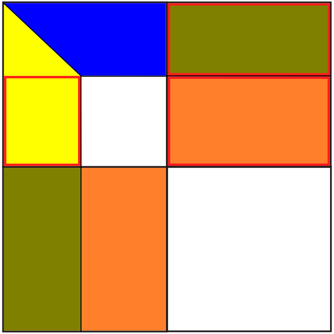
Dumas, P. and Sultan 13



MatMul: $C \leftarrow C - A \times B$

The tiled recursive algorithm

Dumas, P. and Sultan 13



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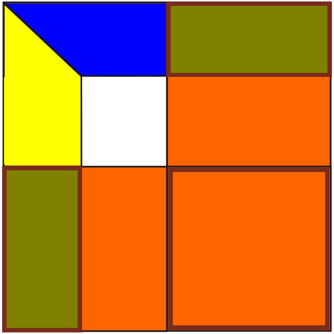
The tiled recursive algorithm

Dumas, P. and Sultan 13



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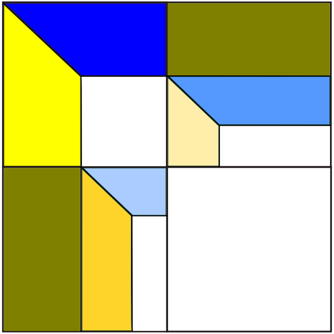
The tiled recursive algorithm

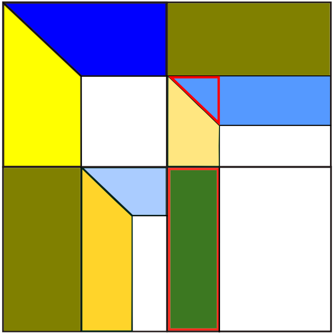
Dumas, P. and Sultan 13



2 independent recursive calls (compatible with the **product order**)

The tiled recursive algorithm

Dumas, P. and Sultan 13



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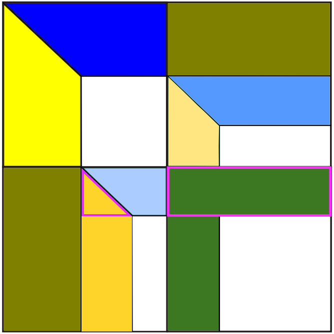
The tiled recursive algorithm

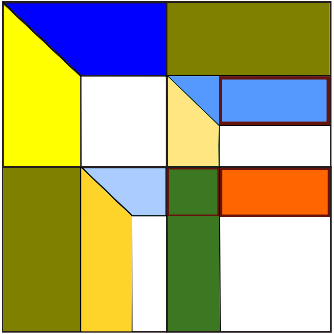
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

Dumas, P. and Sultan 13



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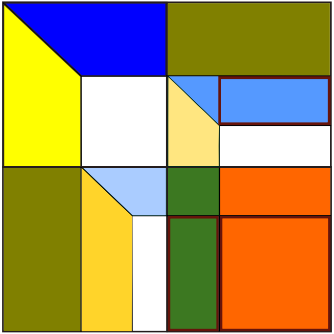
The tiled recursive algorithm

Dumas, P. and Sultan 13



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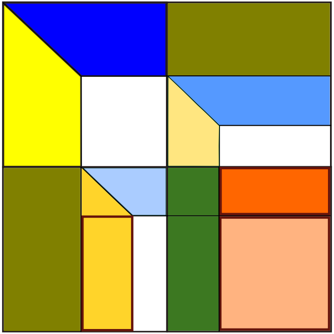
The tiled recursive algorithm

Dumas, P. and Sultan 13



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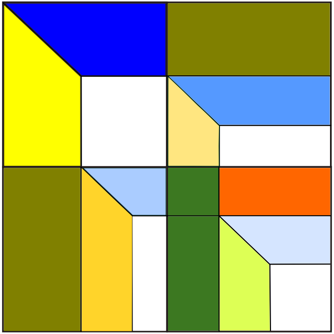
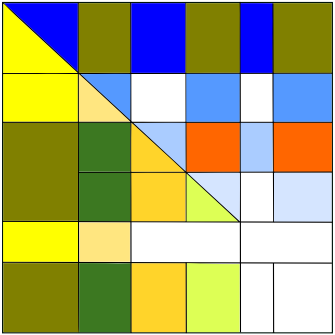


Recursive call

The tiled recursive algorithm

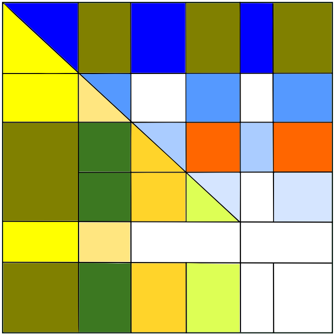
Dumas, P. and Sultan 13



Puzzle game (block **rotations**)

The tiled recursive algorithm

Dumas, P. and Sultan 13



- $O(mnr^{\omega-2})$ ($2/3n^3$ for $\omega = 3$)
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Iterative algorithms

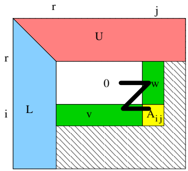
- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

Iterative algorithms

- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

Which base case algorithm?

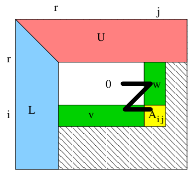
- Formerly [DPS13]: **product order** iterative algorithm
  - ✗ many permutations
  - ✗ many modular reductions



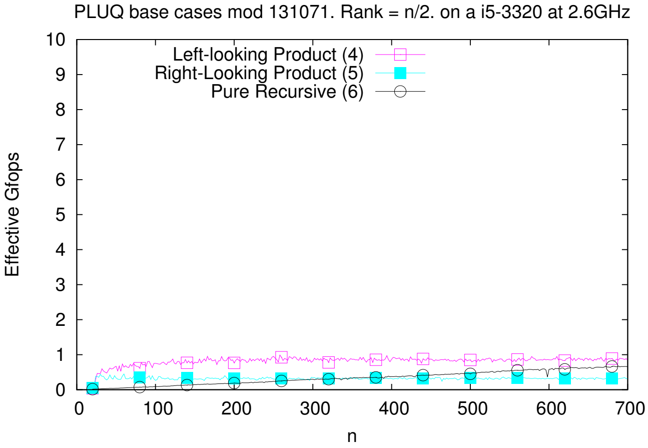
Iterative algorithms

- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

Which base case algorithm?



- Formerly [DPS13]: **product order** iterative algorithm
  - ✗ many permutations
  - ✗ many modular reductions
- [DPS15]: Simply use the schoolbook algorithm (Lexico+Rotations)
  - ✓ fewer permutations
  - ✓ modular reductions delayed more easily
  - ✓ Crout variant: better data access pattern


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Left-looking Product (4)
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Pure Recursive (6)
Effective Gfops
n
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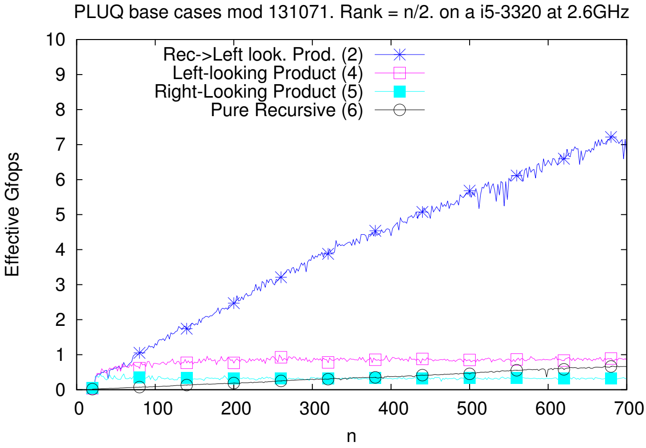
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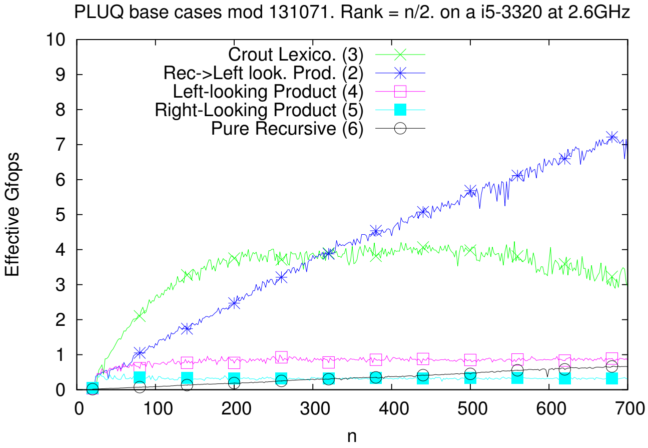
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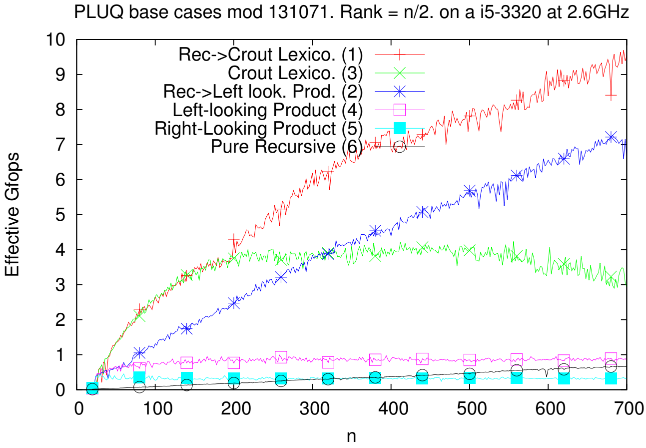

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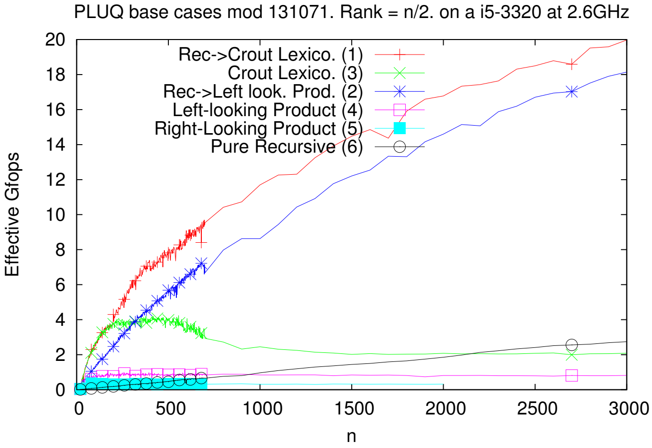

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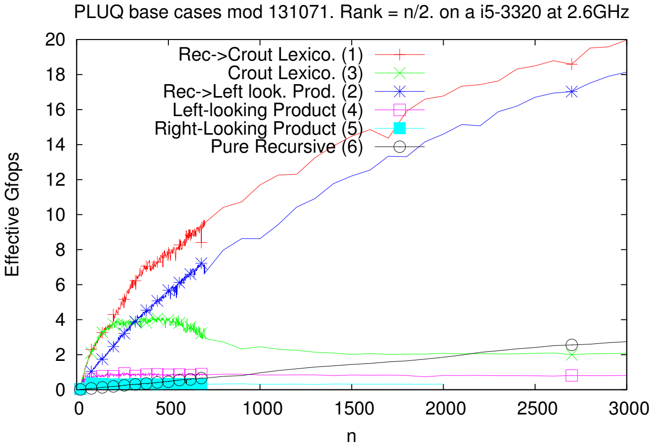

- > 2 Gfops improvement
- Implemented in `FFLAS-FFPACK` (kernel of `LinBox`).

Outline

1. The rank profile Matrix
2. Computing the rank profile matrix
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Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
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- $\frac{17}{2^{\omega}-4} MM(m,n)$ with $m = n = 2^k$.
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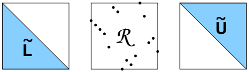
$$E = \mathcal{R}^A$$



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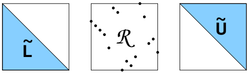

$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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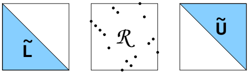

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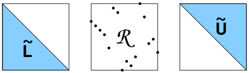

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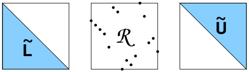

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With appropriate pivoting: $\Pi_{P,Q} = \mathcal{R}(A)$

LUP and PLU decompositions

LUP

If A has generic RowRP

- $LUP(A)$ with Lex order and col. rot.: $\rightsquigarrow \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} P = \mathcal{R}^A$

In particular, if A has full row rank and $m = n$: $\rightsquigarrow P = \mathcal{R}^A$

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PLU

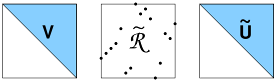
If A has generic ColRP

- $PLU(A)$ with RevLex order and row rot. $\rightsquigarrow P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} = \mathcal{R}^A$

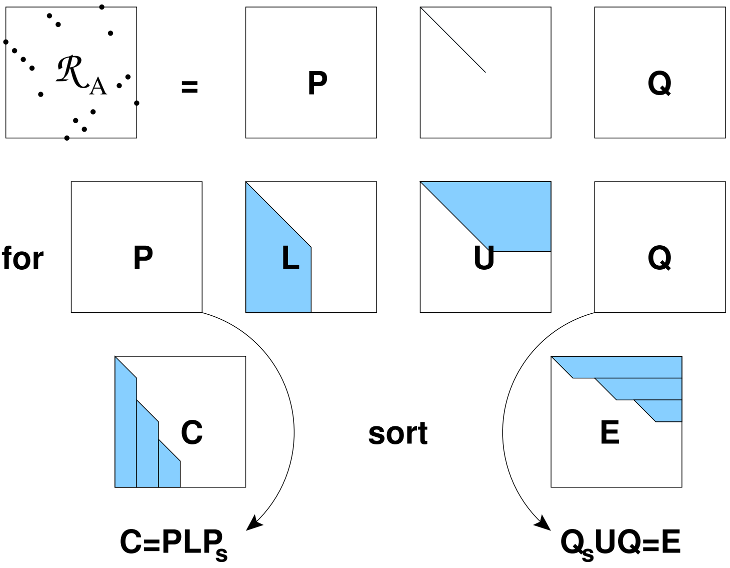
In particular, if A has full column rank and $m = n$: $\rightsquigarrow P = \mathcal{R}^A$

Bruhat decomposition

- If $A = \tilde{L}\mathcal{R}^A\tilde{U}$, then
  - For J_n the unit anti-diagonal matrix,
  - $V = J_n\tilde{L}J_n$ is upper triangular
  - $\tilde{\mathcal{R}} = J_n\mathcal{R}^A$ is a rank r sub-permutation
  - $A = V\tilde{\mathcal{R}}\tilde{U}$ (Bruhat decomposition)



Echelon forms



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The rank profile matrix over a Ring R

Notion of rank over a Ring

Spanning rank:

$$s_R(A) = \min\{r : A = BC \text{ where } B \text{ is } m \times r \text{ and } C \text{ is } r \times n\}.$$

McCoy's rank:

- $\mathcal{M}_R(A) =$ max size of a non zero minor in A
- largest number of cols of A with right nullspace $= \{0\}$

Smith's rank: (over a PIR) $\mathcal{S}_R =$ number of unit Smith invariants $= \mathcal{M}_R$

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Over a Principal Ideal Domain

Define K_R, the field of fractions of R. $\rightsquigarrow$ notion of field rank r_{K_R}.

Fact: $s_R = \mathcal{M}_R = r_{K_R}$.

$\rightsquigarrow$ same rank profile matrix

The rank profile matrix over a Ring with zero divisors

Example

Over $\mathbb{Z}/4\mathbb{Z}$, consider $A = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$

- $s_R(A) = 1$ as $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$
- Then $\mathcal{R}^A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ with $b, c \neq 0$.
- But $d \neq 0$ as $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} z & t \end{bmatrix} \Rightarrow x, y, z, t \neq 0 \Rightarrow \begin{cases} x = z = 2, \\ y, t = -1 \end{cases}$

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But $\mathcal{M}_R(\begin{bmatrix} 0 & 2 \end{bmatrix}) = 0$, hence $\mathcal{R}(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ with McCoy's rank.

The rank profile matrix over a Ring

Theorem

Over an arbitrary ring, the Rank profile matrix can always be defined using McCoy's rank.

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Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
(Compressed sensing applications)

[Cheung Kwok Lau'12]: Compute the rank r and r linearly independent rows in $\tilde{O}(r^\omega + mn)$ probabilistic

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Can the rank profile matrix be computed in such complexities?

[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
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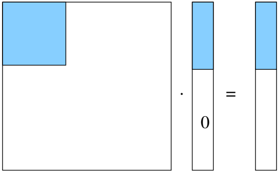

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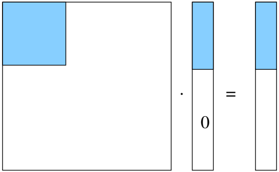

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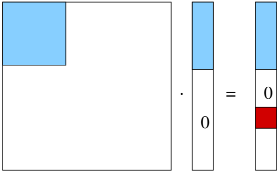

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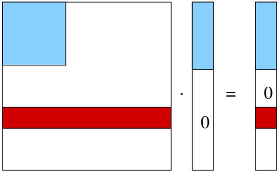

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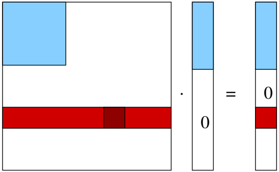

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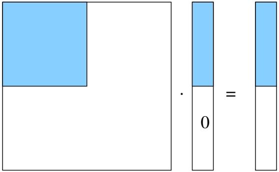

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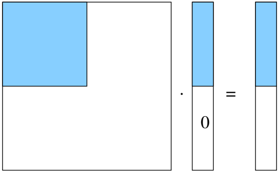

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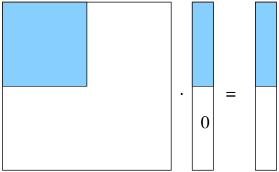

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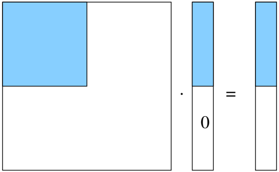

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Lexico. search with rotations $\rightsquigarrow$ computes $\mathcal{R}^A$

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $\tilde{O}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
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5. Recover $\mathcal{R}^A$ by inflating $\mathcal{R}^{A_r} = P$ with zeroes.

Perspective

- Application to F5 elimination (Gröbner basis) [Sun Lin Wang'14]
- Communication avoiding variants [Demmel & Al.'12]
- How to accomodate sparse elimination constraints ?
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Thank you!

Bibliography

[Malaschonok'10]: $A = LEU$

- first instance of $\mathcal{R}^A$.
- no consideration on rank profile nor echelon form

[DSP'13]: $A = PLUQ$

Computed only via a product order pivoting,
Rank sensitive $O(r^{\omega-2}mn)$, any $m \times n$ matrix of any rank r.

[DPS'15]

- Conditions for any PLUQ alg. to reveal $\mathcal{R}^A$
- New pivoting strategies $\rightsquigarrow$ faster base case

[DPS'XX in preparation]

- $\mathcal{R}^A$ in $\tilde{O}(r^\omega + mn)$
- generalization of $\mathcal{R}^A$ to rings