# Explicit Generating Series for Small-Step Walks in the Quarter Plane

**Frédéric Chyzak**

April 27, 2015

Joint work with A. Bostan, M. Kauers, L. Pech, and M. van Hoeij

# Why Lattice Paths?

## Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

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### A history and a survey of lattice path enumeration

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Reflection principle
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ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

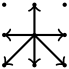
In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

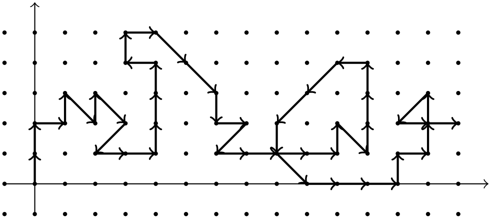
# Enumerative Combinatorics of Lattice Walks

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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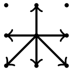
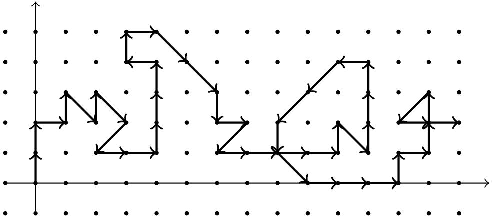
$\mathfrak{S} =$ 



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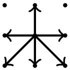
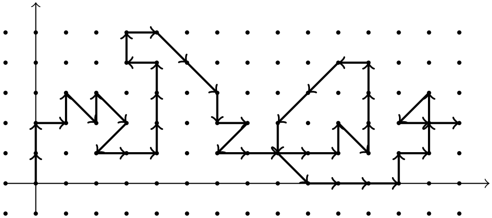
$\mathfrak{S} =$ 



▷ $f_{n;i,j}$ = number of walks of length $n$ ending at $(i,j)$.

▷ Special, combinatorially meaningful specializations:

- $f_{n;0,0}$ counts walks of length $n$ returning to the origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ counts all walks with prescribed length $n$.

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Algebraic nature of $F$: algebraic? transcendental?
- Explicit form: of $F$? of $f$?
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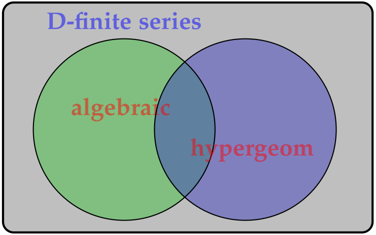
Our goal: Use computer algebra to give computational answers.

# Important Classes of Univariate Power Series



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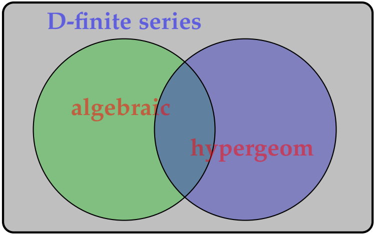
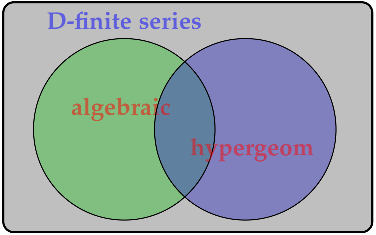



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

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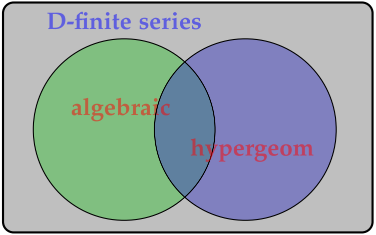


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▷ *Hypergeometric*: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}\,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1), $$

$$ t(1-t)S''(t) + \big(c - (a+b+1)t\big)S'(t) - abS(t) = 0. $$

# Small-Step Models of Interest

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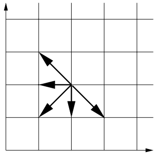

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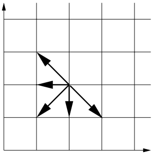

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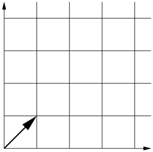

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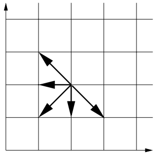

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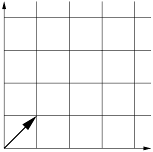

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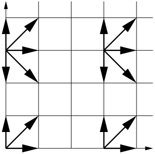

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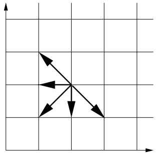

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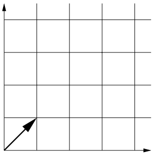

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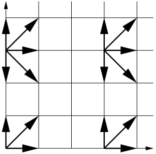

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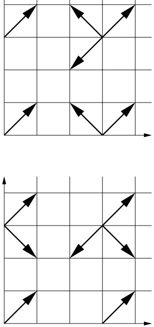

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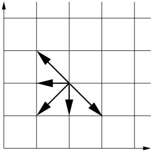

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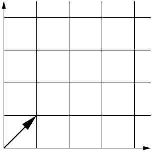

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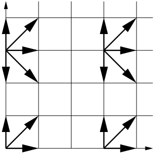

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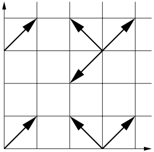

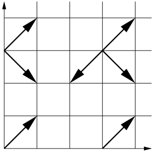

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One is left with 79 interesting distinct models.

# Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

| | OEIS | $\mathfrak{S}$ | alg | equiv | | OEIS | $\mathfrak{S}$ | alg | equiv |
|---|---|---|---|---|---|---|---|---|---|
| 1 | A005566 |  | N | $\frac{4}{\pi}\frac{4^n}{n}$ | 13 | A151275 |  | N | $\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$ |
| 2 | A018224 |  | N | $\frac{2}{\pi}\frac{4^n}{n}$ | 14 | A151314 |  | N | $\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$ |
| 3 | A151312 |  | N | $\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$ | 15 | A151255 |  | N | $\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$ |
| 4 | A151331 |  | N | $\frac{8}{3\pi}\frac{8^n}{n}$ | 16 | A151287 |  | N | $\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$ |
| 5 | A151266 |  | N | $\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$ | 17 | A001006 |  | Y | $\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$ |
| 6 | A151307 |  | N | $\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$ | 18 | A129400 |  | Y | $\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$ |
| 7 | A151291 |  | N | $\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$ | 19 | A005558 |  | N | $\frac{8}{\pi}\frac{4^n}{n^2}$ |
| 8 | A151326 |  | N | $\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$ | | | | | |
| 9 | A151302 |  | N | $\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$ | 20 | A151265 |  | Y | $\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$ |
| 10 | A151329 |  | N | $\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$ | 21 | A151278 |  | Y | $\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$ |
| 11 | A151261 |  | N | $\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$ | 22 | A151323 |  | Y | $\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$ |
| 12 | A151297 |  | N | $\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$ | 23 | A060900 |  | Y | $\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$ |

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

PROVE THIS TABLE!

# Previous Work on the 23 Cases

▷ Human proof of D-finiteness/algebraicity for cases 1–22 in [Bousquet-Mélou & Mishna, 2010]:

- based on averaging over a group of rational invariant transformations,
- but implied algorithm too expensive to exhibit ODEs!

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▷ Human proofs of asymptotics $f_n \sim \kappa n^\alpha \rho^n$:

- $\rho$ for all cases in [Fayolle & Raschel, 2012];
- $(\alpha, \rho)$ for cases 1–4,17–23 (zero drift) using [Denisov & Wachtel, 2013];
- $(\kappa, \alpha, \rho)$ for cases 1–4 (2 axes of sym.) in [Melczer & Mishna, 2014];
- $(\kappa, \alpha, \rho)$ for cases 17–22 in [Bousquet-Mélou & Mishna, 2010];
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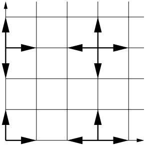
# The Kernel Equation [≤ Knuth, 1968]: an Example



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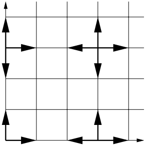

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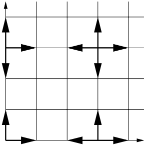

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Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

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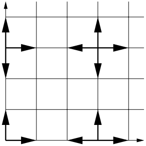

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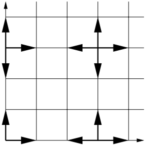
$$f_{n+1;i,j} x^i y^j t^{n+1} = \left(f_{n;i+1,j} x^{i+1} y^j t^n\right) \times \bar{x} t + [\![0 < j]\!] \left(f_{n;i,j-1} x^i y^{j-1} t^n\right) \times y t + [\![0 < i]\!] \left(f_{n;i-1,j} x^{i-1} y^j t^n\right) \times x t + \left(f_{n;i,j+1} x^i y^{j+1} t^n\right) \times \bar{y} t,$$

Notation: $\bar{x} = \frac{1}{x}, \quad \bar{y} = \frac{1}{y}.$



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$$F(t;x,y) - 1 = \big(F(t;x,y) - F(t;0,y)\big) \times \bar{x}t + F(t;x,y) \times yt + F(t;x,y) \times xt + \big(F(t;x,y) - F(t;x,0)\big) \times \bar{y}t,$$

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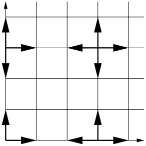

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walk of length $n$ followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

Functional ("kernel") equation:

$$((x + \bar{x} + y + \bar{y})\, t - 1)\, F(t; x, y) = \bar{y} t F(t; x, 0) + \bar{x} t F(t; 0, y) - 1.$$

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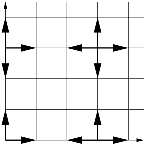

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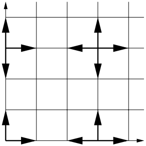
Remarks:

- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.

# D-Finiteness via the Finite Group: an Example



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of $(x, y)$ into, respectively:

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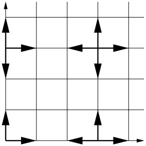

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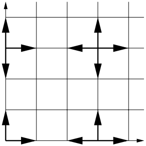

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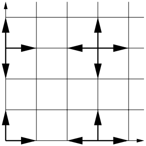

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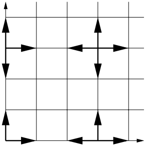

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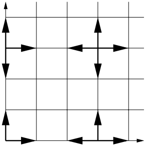

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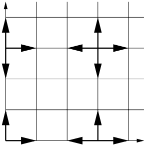

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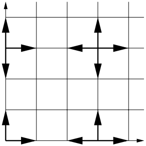

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# Cases 1–19 are D-Finite

**Theorem** [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

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▷ Remark: The formula provides no direct information for $x = y = 1$.

# Explicit Expressions for the Cases 1–19

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Obtained by obtaining and solving:

$$t^2(4t+1)(8t-1)(2t-1)(t+1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' + (1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$

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1. (New) If $R = \sum_\theta \frac{(-1)^\theta\,\theta(xy)}{J(t;x,y)}$, then $[x^>][y^>]R = \mathrm{Res}_u\,(\mathrm{Res}_v\,H)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.

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   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
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For $F(t;x,y)$, run whole process for $F(t;0,0)$, $F(t;x,0)$, and $F(t;0,y)$, then substitute into kernel equation!

# Computing Positive Parts as Residues

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \stackrel{?}{=} \frac{1}{1-w} \stackrel{?}{=} 1 + w + w^2 + \cdots$$

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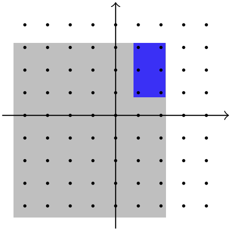

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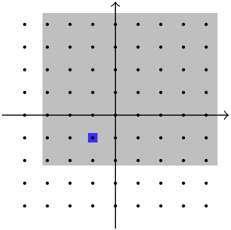

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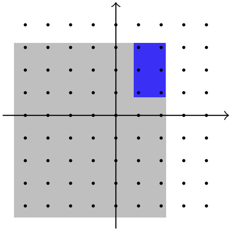

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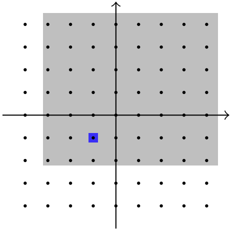

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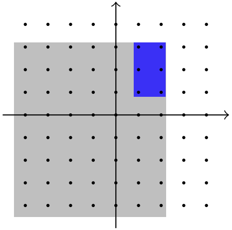
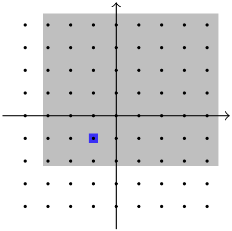
$$\boxed{Z \in \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \Rightarrow \left(\bar{u}\bar{v}[\bar{u}^<\bar{v}^<]Z(\bar{u},\bar{v})\right)_{u=a,v=b} = \mathrm{Res}_{u,v} \frac{Z(u,v)}{(1-au)(1-bv)}}$$

Our solution:



$[\bar{u}^{<}\bar{v}^{<}] : \mathbb{Q}[[\bar{u},\bar{v}]][u,v] \to uv\mathbb{Q}[u,v]$



$\mathrm{Res}_{u,v} : \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \to \mathbb{Q}$

Finally, observe $R \in \mathbb{Q}(x)[y,\bar{y}][[t]]$, so that:

$$F(t;x,y) = \frac{1}{xy}[x^{>}][y^{>}]R = \bar{x}\bar{y}[\bar{x}^{<}\bar{y}^{<}]R$$

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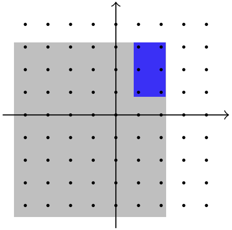

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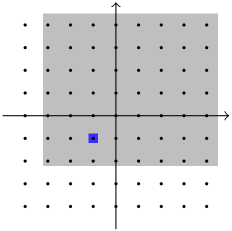

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$$F(t;x,y) = \frac{1}{xy}[x^{>}][y^{>}]R = \bar{x}\bar{y}[\bar{x}^{<}\bar{y}^{<}]R = \mathrm{Res}_{u,v}\,\frac{R(1/u,1/v)}{(1-xu)(1-yv)}.$$

# Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \ldots$, search for rational $\eta_{i,j}^k$ and $\Phi^k$ s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t, u) D_t^i D_u^j (H)(t, u, v) = D_v\big(\Phi^k(t, u, v) H(t, u, v)\big)$$

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  by a variant of Abramov's algorithm

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$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

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  by uncoupling + a variant of Abramov's algorithm, or by a variant of Barkatou's algorithm

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$$\to \sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint \oint H(t,u,v)\,du\,dv.$$

- Recombine: from the action on $\hat{H}$, there are $G^k$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i = D_u \hat{\Phi}(t,u,D_t,D_u) + \sum_k G^k(t,u,D_t,D_u) \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j.$$

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- Recombine: there are $G^k$ s. t., upon application to $H$,

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (H) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) H\big) + \sum_k G^k(t,u,D_t,D_u) D_v(\Phi^k H).$$

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- Recombine: there are $G^k$ s. t., upon application to $H$,

$$\left(\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i\right)(H) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) H\big) + D_v\left(\sum_k G^k(t,u,D_t,D_u)(\Phi^k H)\right).$$

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- Recombine: there are $L \in \mathbb{Q}'[t]\langle D_t \rangle$ and $U, V \in \mathbb{Q}'(t,u,v)$ s. t.

$$L(H) = D_u(U) + D_v(V).$$

# Solving $L(y(t)) = 0$ in Terms of Rational Pullbacks of ${}_2F_1$ Series

**Ordinary or regular-singular $p$**

- $f(t) = \sum (t-\xi)^e \ln^i(t-\xi)\phi_{e,i}(t)$ $\qquad$ ($e \in \mathbb{C}$, $i \in \mathbb{N}$, $\phi_{e,i}$ analytic)
- $L$ of order 2 and irreducible: $d_\xi := \pm(e_{1,\xi} - e_{2,\xi})$ $\quad$ "exponent difference"

**Removable singularity $\xi$**

$\exists$ solution basis of $L$ of the form $(gf_1, gf_2)$ for any $g$ and $f_1, f_2$ analytic at $\xi$.

**Hypergeometric and "standard" equations**

- $L_c^{a,b}\left({}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}\middle| z\right)\right) = 0 \quad \to$

| $e_0$ | $e_1$ | $e_\infty$ |
|---|---|---|
| $\pm(1-c)$ | $\pm(c-a-b)$ | $\pm(a-b)$ |

- $L_{c;w}^{a,b}(h(t)) = 0$ where $h(t) = {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}\middle| w(t)\right) \quad \to$

$$w(t) \sim \lambda(t-\xi)^m \implies d_\xi = me_0$$
$$w(t) \sim 1 + \lambda(t-\xi)^m \implies d_\xi = me_1$$
$$w(t) \sim \lambda(t-\xi)^{-m} \implies d_\xi = me_\infty$$

**Solution $(r_0 h + r_1 h')\exp(\int r)$ with $r, r_0, r_1 \in \mathbb{C}(t)$: $L$ and $L_{c;w}^{a,b}$ need to have**

Same non-removable singularities + Same exponent differences modulo $\mathbb{Z}$

# Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

| | hyp$_1$ | hyp$_2$ | $w$ | | hyp$_1$ | hyp$_2$ | $w$ |
|---|---|---|---|---|---|---|---|
| 1 | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | $16t^2$ | 10 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$ |
| 2 | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | | $16t^2$ | 11 | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{16t^2}{4t^2+1}$ |
| 3 | ${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | | $\frac{16t}{(2t+1)(6t+1)}$ | 12 | ${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64t^3(2t+1)}{(8t^2-1)^2}$ |
| 4 | ${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | | $\frac{16t(1+t)}{(1+4t)^2}$ | 13 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64t^2(t^2+1)}{(16t^2+1)^2}$ |
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| 6 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64t^3(1+t)}{(1-4t^2)^2}$ | 15 | ${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | $64t^4$ |
| 7 | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | $\frac{16t^2}{4t^2+1}$ | 16 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64t^3(1+t)}{(1-4t^2)^2}$ |
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| 9 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\,\middle\vert\, w\right)$ | $\frac{64t^2(t^2+1)}{(16t^2+1)^2}$ | 19 | ${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 2\end{matrix}\,\middle\vert\, w\right)$ | $16t^2$ |

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| 4 | ${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\vert w\right)$ | | $\frac{16t(1+t)}{(1+4t)^2}$ | 13 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$ | $\frac{64t^2(t^2+1)}{(16t^2+1)^2}$ |
| 5 | ${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 1\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\vert w\right)$ | $64t^4$ | 14 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\vert w\right)$ | $\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$ |
| 6 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$ | $\frac{64t^3(1+t)}{(1-4t^2)^2}$ | 15 | ${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 2\end{matrix}\middle\vert w\right)$ | $64t^4$ |
| 7 | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 1\end{matrix}\middle\vert w\right)$ | $\frac{16t^2}{4t^2+1}$ | 16 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\vert w\right)$ | $\frac{64t^3(1+t)}{(1-4t^2)^2}$ |
| 8 | ${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | $\frac{64t^3(2t+1)}{(8t^2-1)^2}$ | | | | |
| 9 | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$ | $\frac{64t^2(t^2+1)}{(16t^2+1)^2}$ | 19 | ${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$ | ${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 2\end{matrix}\middle\vert w\right)$ | $16t^2$ |

Observations: Pairs of related hyps $+{}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix}\middle\vert w\right)$ with $m = c - (a+b) \in \mathbb{Z}$.

# Proofs of Algebraicity/Transcendence of $F(t;x,y)$ and $F(t;1,1)$

**Theorem**

- In cases 1–19, $F(t;x,y)$ is transcendental since $F(t;0,0)$ is.
- In cases 1–16 and 19, $F(t;1,1)$ is transcendental.
- Specific simplifications prove algebraicity of $F(t;1,1)$ in cases 17–18.

*Proof*: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- $F$ is algebraic $\Longrightarrow$ $G$ is algebraic.
- Computing a few coefficients of $G$ shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to $L_2$ decides whether $L_2$ has nonzero algebraic solutions.

# Coefficient Asymptotics $\kappa n^\alpha \rho^n$ (*in progress*)

Local theory of D-finite functions $\longrightarrow$
Systematic method for coefficient asymptotics
(Flajolet and Odlyzko's singularity analysis)

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \longrightarrow \qquad f_n \sim \cdots$$

**Singularity analysis [Flajolet & Odlyzko (1990)]**

- Determine dominant singularities of the *complex-analytic function* $f$.
- Find asymptotic expansion

$$f(z) =_{z \to s} \sum_{\alpha,\gamma} c_{\alpha,\gamma} (s-z)^\alpha \left( \ln \frac{1}{s-z} \right)^\gamma .$$

- Syntactic transfer into an asymptotic expansion for $f_n$.

# Three Formulas from (DLMF 15.8) on ${}_2F_1$s

▷ To ensure that $c - a - b \in \mathbb{N}$: for $m \in \mathbb{N}$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b-m \end{matrix} \middle| z\right) = (1-z)^{-m} {}_2F_1\left(\begin{matrix} a-m \quad b-m \\ (a-m)+(b-m)+m \end{matrix} \middle| z\right). $$

▷ To bring $\pm\infty$ at $1^-$: for $z < 1/2$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| z\right) = (1-2z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a \quad \frac{1}{2}a+\frac{1}{2} \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| \frac{4z(z-1)}{(1-2z)^2}\right). $$

▷ Local logarithmic behaviour at 1: for $m \in \mathbb{N}$, $z \in D(1,1) \setminus [0,1]$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+m \end{matrix} \middle| z\right) = $$

polynomial of degree $m-1$ in $1-z$

$+$ term in $(1-z)^m \ln(1-z) +$ higher order terms.

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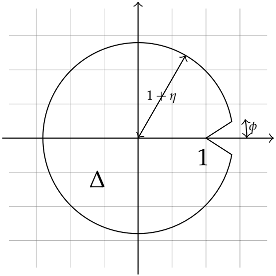
$$ \frac{1}{\Gamma(a+m)\Gamma(b+m)} \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k (b)_k (m-k-1)!}{k!} (1-z)^k $$

$$ - (-1)^m \frac{(1-z)^m}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a+m)_k (b+m)_k}{k!(k+m)!} (1-z)^k \Big( \ln(1-z) $$

$$ - \psi(k+1) - \psi(k+m+1) + \psi(a+k+m) + \psi(b+k+m) \Big). $$

# Transfer Theorems [Flajolet & Odlyzko 1990]




For $f(z) = \sum_{n=0}^{\infty} f_n z^n$ analytic in $\Delta \setminus \{1\}$:

| $f(z)$ | $f_n$ | assumptions |
|---|---|---|
| $O\big((1-z)^\alpha\big)$ | $O\big(n^{-(\alpha+1)}\big)$ | $\alpha \in \mathbb{R}$ |
| $o\big((1-z)^\alpha\big)$ | $o\big(n^{-(\alpha+1)}\big)$ | $\alpha \in \mathbb{R}$ |
| $\sim C(1-z)^\alpha$ | $\sim \frac{Cn^{-(\alpha+1)}}{\Gamma(-\alpha)}$ | $\alpha \in \mathbb{R} \setminus \mathbb{N}$ |
| $\sum_{j=0}^{m-1} c_j(1-z)^{\alpha_j} + O\big((1-z)^A\big)$ | $\sum_{j=0}^{m-1} \frac{c_j n^{-(\alpha_j+1)}}{\Gamma(-\alpha_j)} + O\big(n^{-(A+1)}\big)$ | $\alpha_1 \leq \dots \leq \alpha_{m-1} < A$ |
| $O\big((1-z)^\alpha(\ln(1-z)^{-1})^\gamma\big)$ | $O\big(n^{-(\alpha+1)}(\ln n)^\gamma\big)$ | $\alpha, \gamma \in \mathbb{R}$ |
| $o\big((1-z)^\alpha(\ln(1-z)^{-1})^\gamma\big)$ | $o\big(n^{-(\alpha+1)}(\ln n)^\gamma\big)$ | $\alpha, \gamma \in \mathbb{R}$ |
| $\sim C(1-z)^\alpha(\ln(1-z)^{-1})^\gamma$ | $\sim \frac{Cn^{-(\alpha+1)}(\ln n)^\gamma}{\Gamma(-\alpha)}$ | $\alpha, \gamma \in \mathbb{R} \setminus \mathbb{N}$ |
| $\vdots$ | $\vdots$ | |

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t\to\frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \quad \longrightarrow \quad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t\to-\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \quad \longrightarrow \quad (-1)^n\frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

# Example of Asymptotic Behaviour Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}\,{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, w\right)$$

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# Example of Behaviour Not Driven by the ${}_2F_1$: at $(1,1)$

$$F(t;1,1) = \frac{1}{t(1-t)} \int \frac{t(4+\int f)}{(1-4t)^{3/2}} \quad \text{where}$$

$$f = \frac{(1+2t)(1-4t)^{1/2}}{2t^2} \left(1 + \frac{1}{2t(1+2t)(1+4t^2)^{1/2}} h\right) = \frac{1}{t^2} + O(1),$$

$$h = (1+t)(1-4t+8t^2)\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right) - (1-t)\,{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right),$$

$$w = \frac{16t^2}{1+4t^2} = 1 - \frac{1-12t^2}{1+4t^2}.$$

Singularities: $\frac{1}{4}$, $-\frac{1}{2}$, $\pm\frac{i}{2}$, $1$, $w=1$, $w=\infty$ $\longrightarrow$ Dominant singularity = $\frac{1}{4}$.

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$$w = \frac{16t^2}{1+4t^2} = 1 - \frac{1-12t^2}{1+4t^2}.$$

Singularities: $\frac{1}{4}$, $-\frac{1}{2}$, $\pm\frac{i}{2}$, $1$, $w=1$, $w=\infty \longrightarrow$ Dominant singularity = $\frac{1}{4}$.

$f_{n;1,1} \sim \frac{4}{3}\sqrt{\frac{1}{\pi}}\frac{4^n}{\sqrt{n}}$ holds under the conjecture $\int_0^{\frac{1}{4}} \left(f(t) - \frac{1}{t^2}\right) dt = 2.$

# Conclusions

Summary: three kinds of conjectures now proved:

- differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients (in progress).

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Summary: three kinds of conjectures now proved:

- differential operators that witness D-finiteness,
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A succession of equations of several types:

- recurrence relation on $f_{n;i,j}$,
- kernel equation on $F(t;x,y)$,
- ODE on $F(t;1,1)$.