Formal Verification of ODE-Solvers (and an Application to the Lorenz Attractor)

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SpecFun Seminar "Computations and Proofs"
2014-12-08

Ordinary Differential Equations (ODEs)

- ODEs: modelling continuous “real world”

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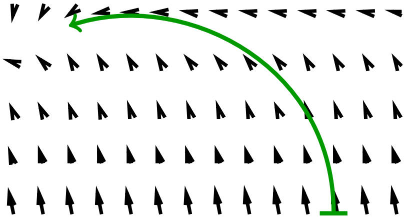

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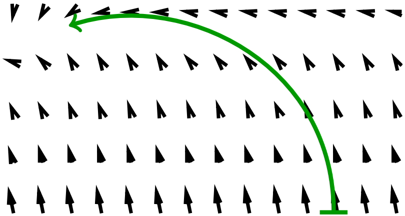

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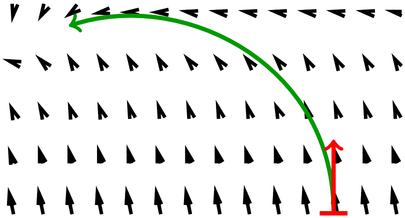

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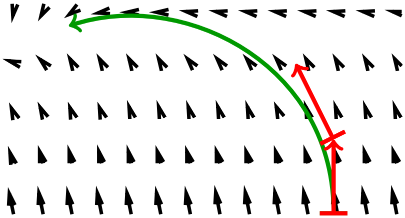

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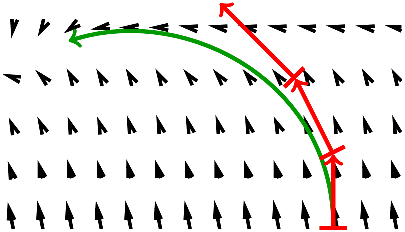

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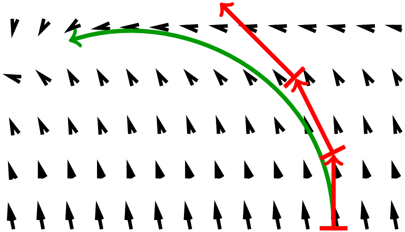

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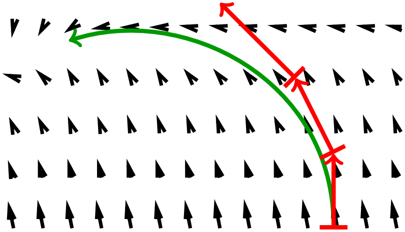

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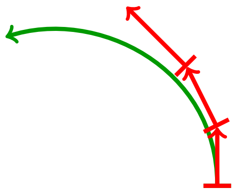
“Guaranteed Integration”

- computing with enclosures



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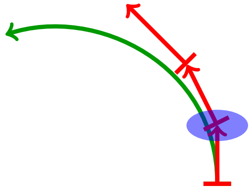

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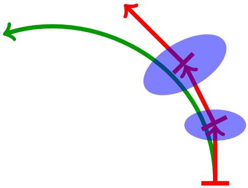

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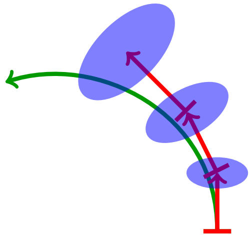

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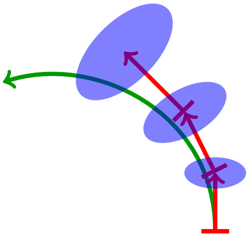

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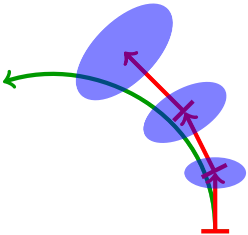

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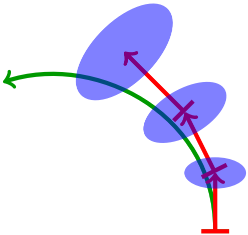

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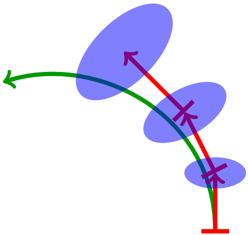

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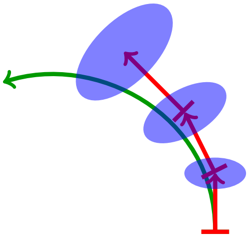

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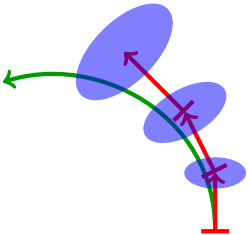

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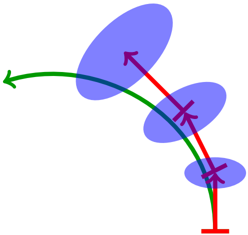

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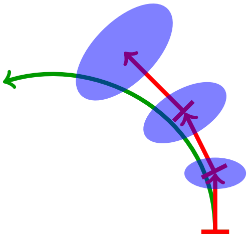

Contribution

verification of (Bouissou et al.'s) algorithm
in interactive theorem prover Isabelle/HOL

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Theorem (mechanically checked)

functional program computes enclosure for unique solution of ODE

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- application: Tucker's "computer-aided" proof for the Lorenz attractor

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- explicitly round m

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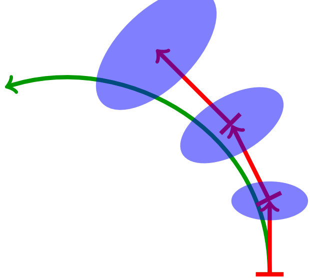

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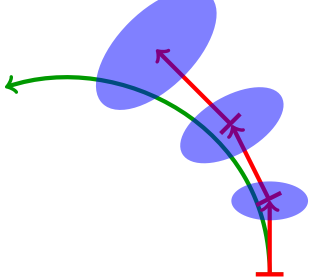

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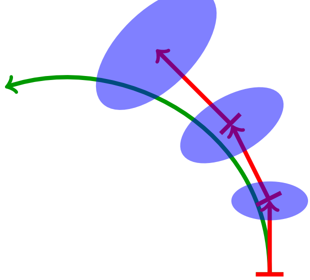

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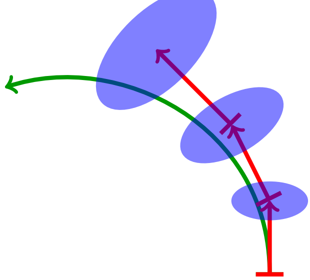

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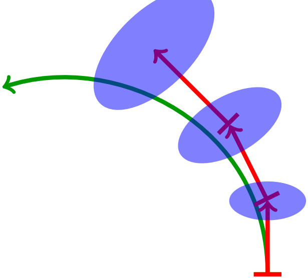

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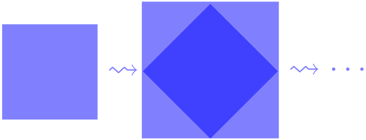

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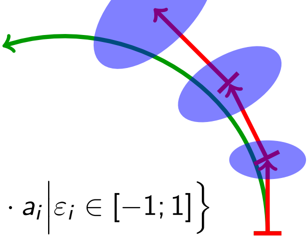

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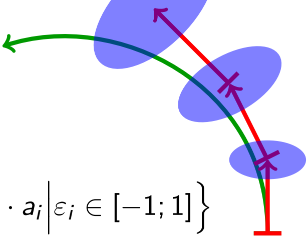

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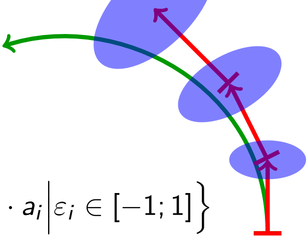

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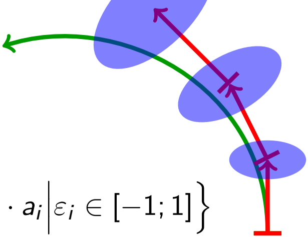

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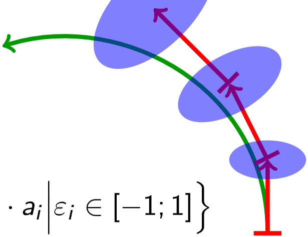

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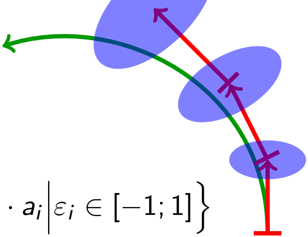

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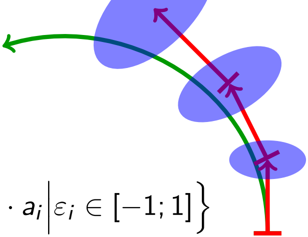

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implementation: sparse lists of sorted coefficients $(\mathbb{N} \times \mathbb{R}^n)$ *list*

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Theorem

If $x_0 \in X_0$ and certify-stepsize$(X_0) = (h, Y)$,
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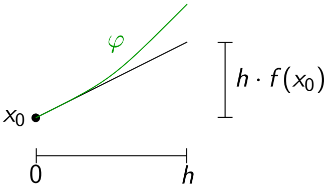

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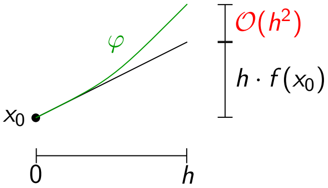

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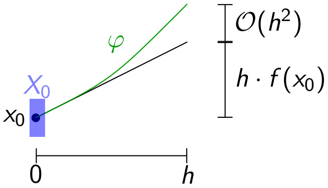

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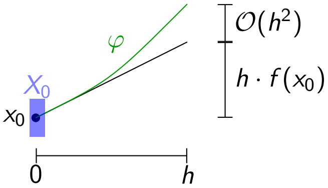

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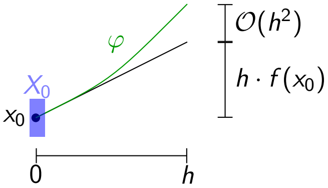

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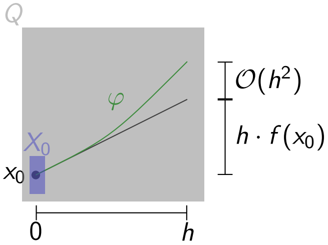

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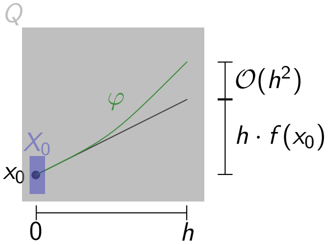

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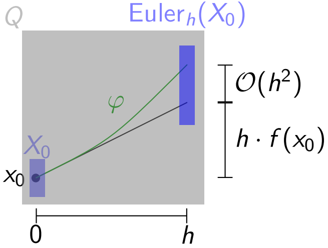

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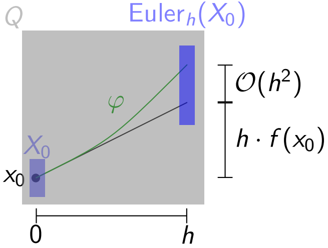

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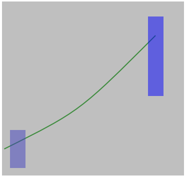
$\varphi(X_0, h) \subseteq \mathrm{Euler}_h(X_0)$

Discretization: One-Step Methods



Iterate:
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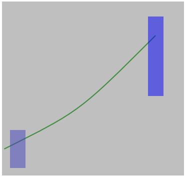

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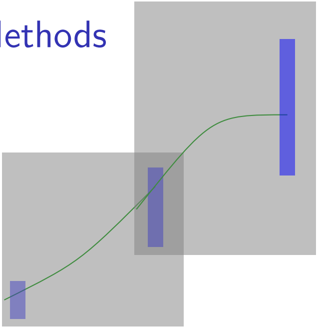

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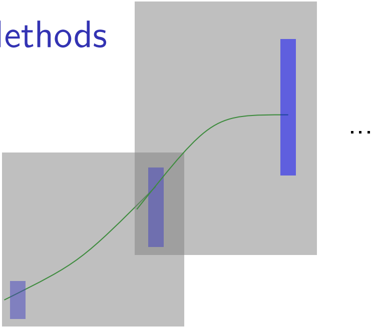

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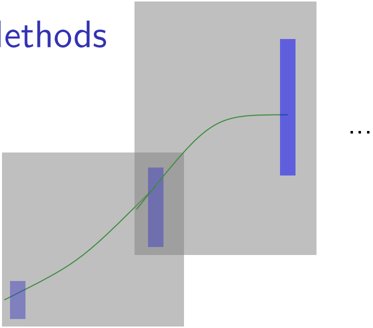

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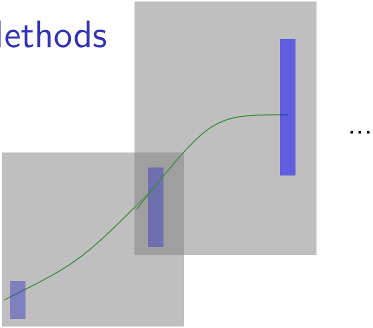

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Experiments

Isabelle's code generator: translation of executable fragment of HOL to SML

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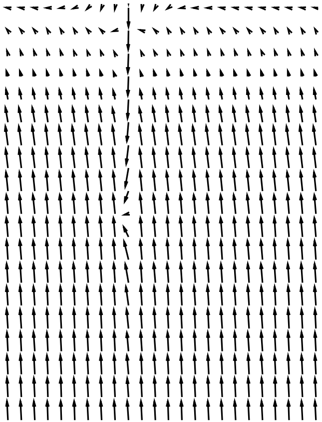

Oil reservoir problem

$y'(t) = z(t)$

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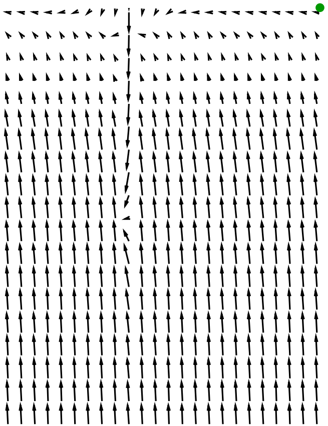

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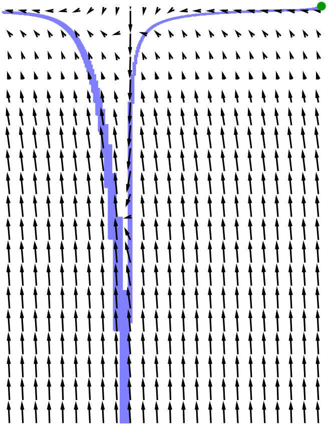

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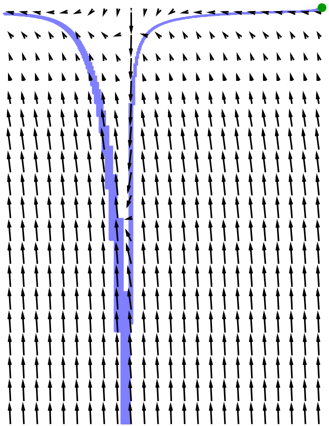

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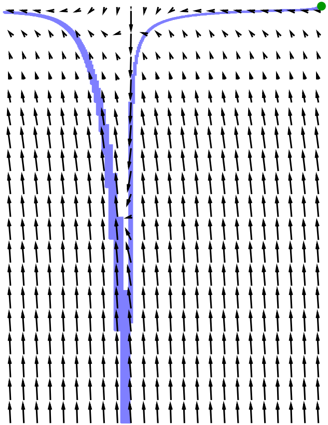

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results:

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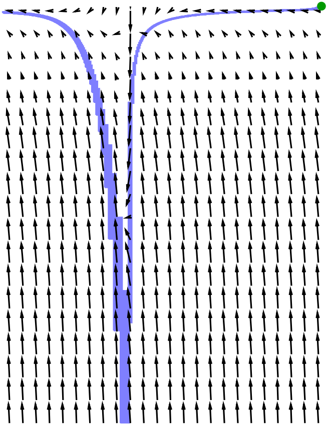
- verified algorithm – rigorous proof for enclosures



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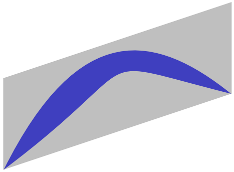
Further Optimizations

problem: affine arithmetic represents only convex sets

- splitting (fixed scale)



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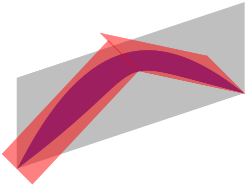

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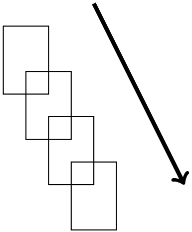

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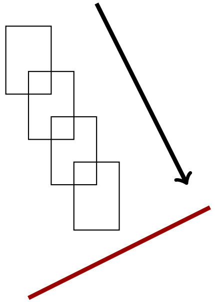

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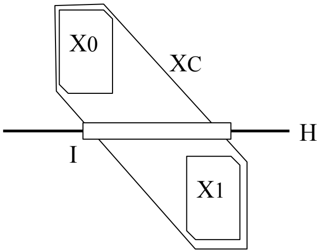
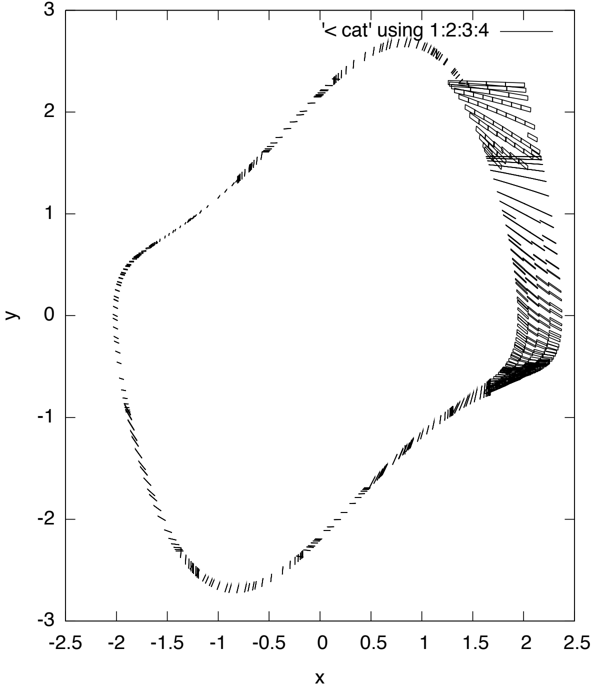


van der Pol:
$\dot{x} = y$
$\dot{y} = y(1 - x^2) - x$



Application: Existence of Lorenz attractor [Tucker 2002]

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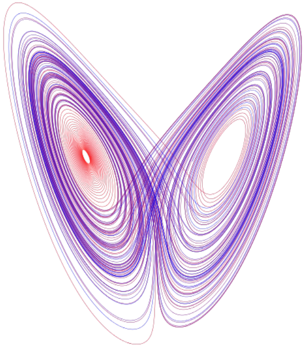

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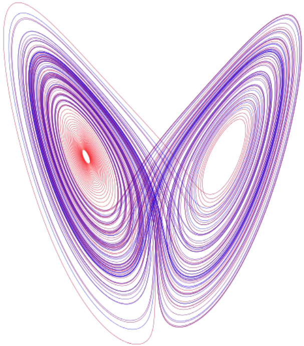

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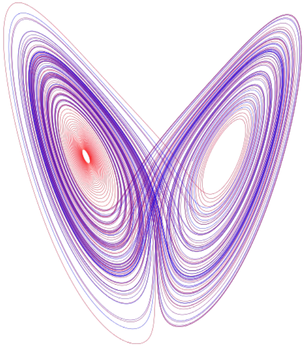

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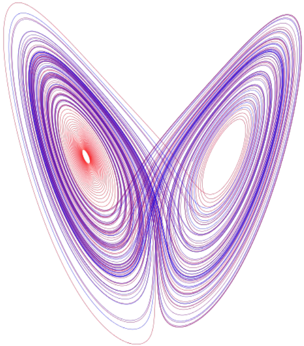

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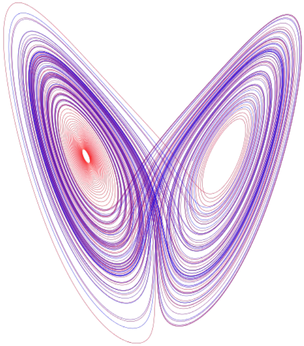

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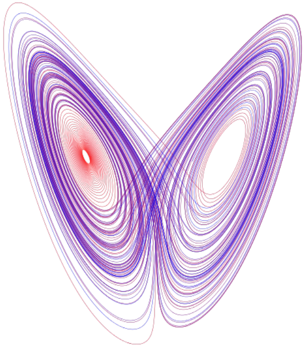

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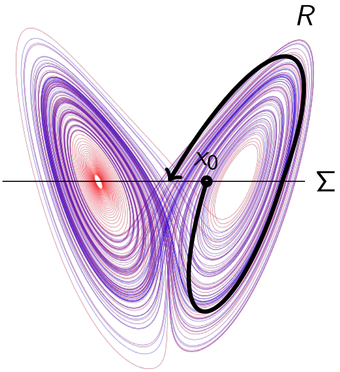
Computations for Tucker's Proof

Attractor studied via Poincaré map:
first return map R of dynamics on section $\Sigma \subseteq \mathbb{R}^2$



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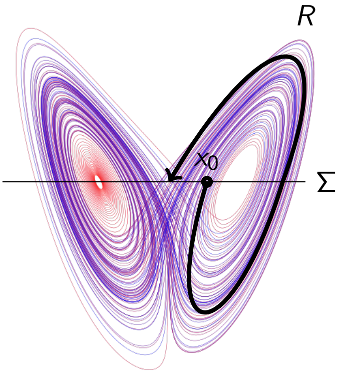

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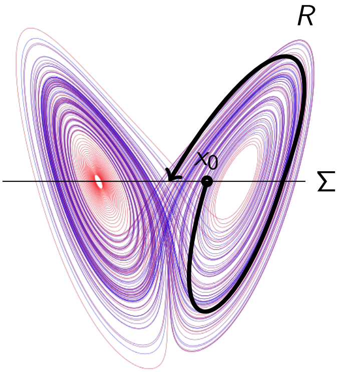
- iterate R until invariant set $R(N) \subseteq N \subseteq \Sigma$



Computations for Tucker's Proof

Attractor studied via Poincaré map:
first return map R of dynamics on section $\Sigma \subseteq \mathbb{R}^2$

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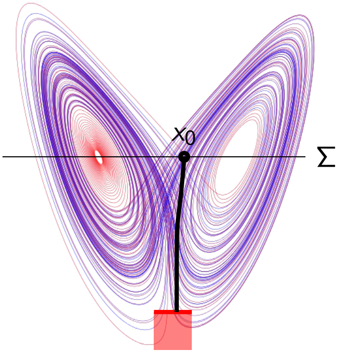

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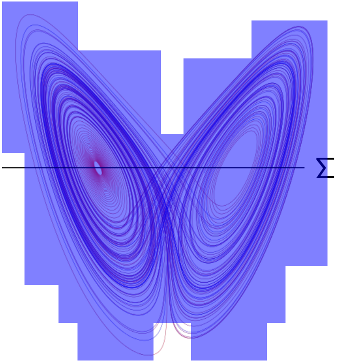

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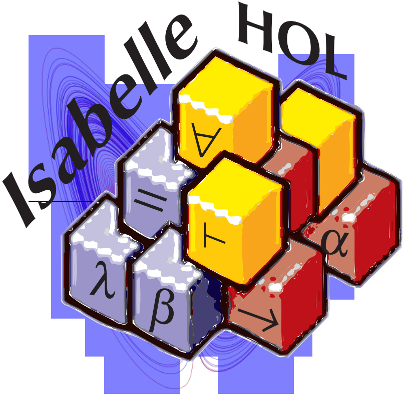
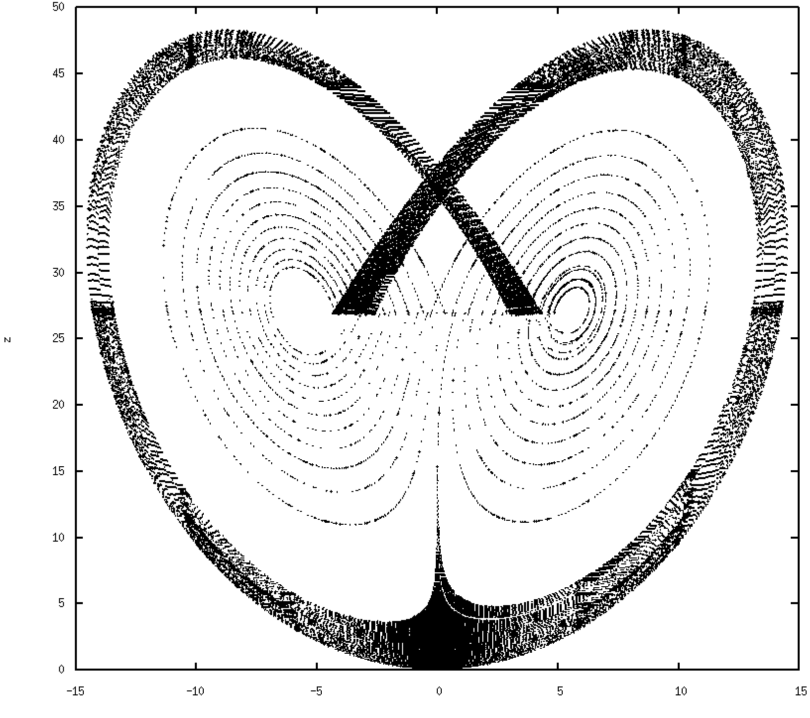
certified with

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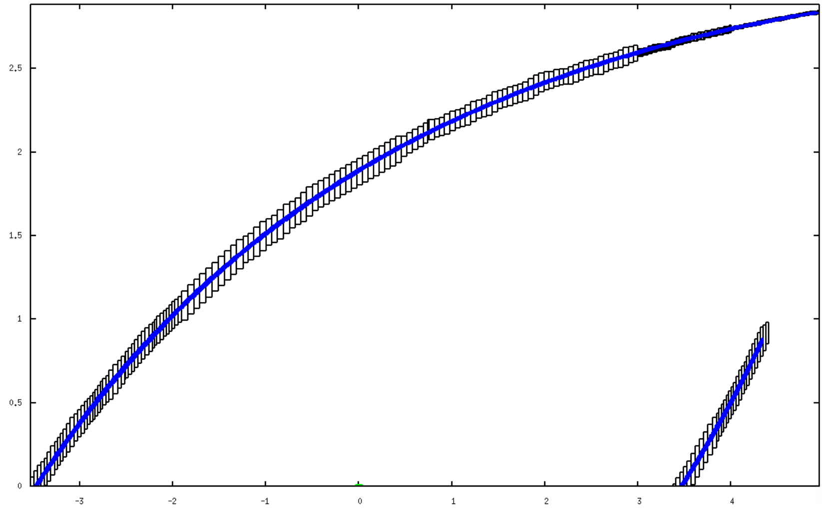
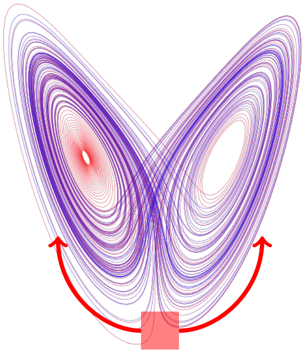
An Invariant Subset of the Return Plane Σ



Missing Computations for Tucker's Proof

Current/future work:
establish further properties used in Tucker's proof

- symbolic propagation close to origin: few additional computations



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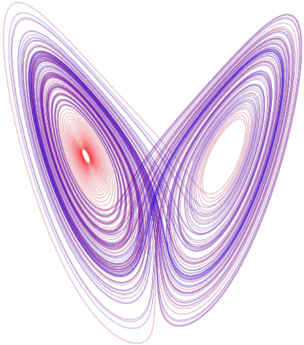

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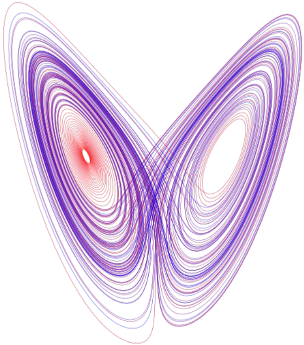

Is Tucker's proof then formalized?

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establish further properties used in Tucker's proof

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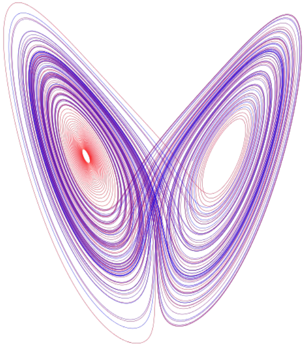

Is Tucker's proof then formalized? No:

Missing Computations for Tucker's Proof

Current/future work:
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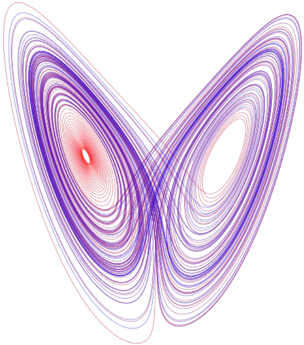

Is Tucker's proof then formalized? No:

- correctness of symbolic propagation

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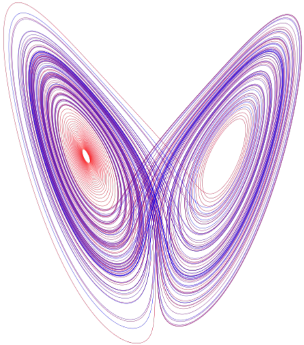

Is Tucker's proof then formalized? No:

- correctness of symbolic propagation
- no formalization of chaos theory

Summary

formalization in Isabelle/HOL

- rigorous numerics and enclosures with affine arithmetic

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- rigorous numerics and enclosures with affine arithmetic
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- reduction/splitting for nonlinear dynamics
- application: Lorenz attractor

Thank you for your attention

Formal Verification of ODE-Solvers (and an Application to the Lorenz Attractor)

Fabian Immler

Chair for Logic and Verification (Tobias Nipkow)
Institut für Informatik, Technische Universität München

SpecFun Seminar "Computations and Proofs"
2014-12-08