Around sparse polynomials

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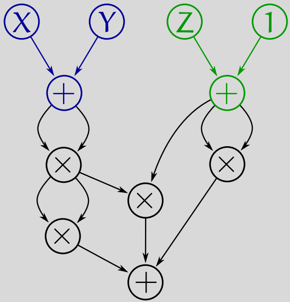
LIX — École Polytechnique

I. The real τ-conjecture

and its variants

$(X+Y)^4 + (Z+1)^2 + (X+Y)^2(Z+1)$



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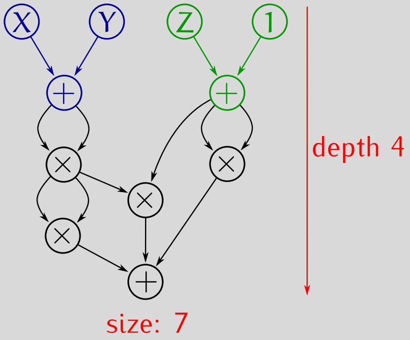

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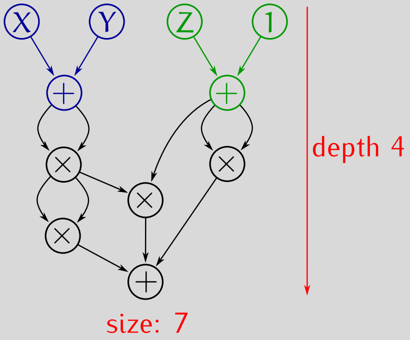

Complexity of a polynomial

$\tau(f) =$ size of its smallest circuit representation *without constant*

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Extended τ-conjecture $\implies$ Merel torsion theorem, ...

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- False for real roots (Chebyshev polynomials)

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- Case $k = 2$: Open.

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$f = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{ij}$: $\leqslant 2kt^m - 1$ real roots

Real τ-conjecture ⟹ Permanent is hard

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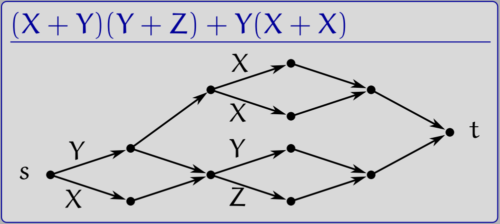
- Construct an equivalent Arithmetic Branching Program
  $\rightsquigarrow$ size $t^{\log 2d}+1$, depth $\delta = 3d-1$ **[Malod-Portier'08]**



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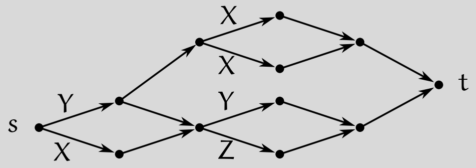

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τ-conjectures (implying $\text{PER} \notin \mathrm{VP}^0$)

$\forall f \in \mathrm{SPS}(k,m,t)$,

- $\exists \mathbb{L} \in \{\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \ldots\}$, [Phillipson-Rojas'13]
  $\#\{x \in \mathbb{L} : f(x) = 0\} \leqslant \mathrm{poly}(kmt)$;

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- Valid with $2^{(m+\log(kt))^c}$ instead of $\mathrm{poly}(kmt)$.

Known upper bounds

$$\mathrm{SPS}(k, m, t, A) = \left\{ \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}} : f_j\text{'s are } t\text{-sparse, } \alpha_{ij} \leqslant A \right\}$$

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- kt^m many edges; number of monomials
- $kt^{2m/3}$ many edges. [Koiran-Portier-Tavenas-Thomassé'13]

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Conclusion

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- What is the maximum number of real roots of $fg + 1$?

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- Best known lower bounds: $\mathcal{O}(t)$;
- The Newton polygon of $fg+1$ has at most $t^{4/3}$ many edges.

II. Factoring lacunary polynomials

joint work with

A. Chattopdhyay, P. Koiran, N. Portier & Y. Strozecki

Factorization of a polynomial P

Find $F_1, \ldots, F_t$, irreducible, s.t. $P = F_1 \times \cdots \times F_t$.

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Complexity

Polynomial in the **degree** of the polynomials

$\downarrow$

$\mathbb{Q}(\alpha)[X_1, \ldots, X_n]$
[Kaltofen'85, A. Lenstra'87]

$$X^{102}Y^{101} + X^{101}Y^{102} - X^{101}Y^{101} - X - Y + 1$$

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Gap Theorem [Cucker-Koiran-Smale'98]

Let

$$P(X) = \underbrace{\sum_{j=1}^{\ell} a_j X^{\alpha_j}}_{Q} + \underbrace{\sum_{j=\ell+1}^{k} a_j X^{\alpha_j}}_{R} \in \mathbb{Z}[X]$$

with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k$. Suppose that

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$$-9 + X^2 + 6X^7 + 2X^8 = -9 + X^2 + X^7(6 + 2X)$$

Factorization of lacunary polynomials

Theorems

There exist deterministic polynomial-time algorithms computing

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Let $f_1, \dots, f_\ell \in \mathbb{K}[X]$. Then

$$\mathrm{wr}(f_1, \dots, f_\ell) = \det \begin{bmatrix} f_1 & f_2 & \dots & f_\ell \\ f_1' & f_2' & \dots & f_\ell' \\ \vdots & \vdots & & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \dots & f_\ell^{(\ell-1)} \end{bmatrix}.$$

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Proposition [Bôcher, 1900]

$\mathrm{wr}(f_1, \ldots, f_\ell) \neq 0 \iff$ the f_j's are linearly independent.

Lemma

$$\mathrm{val}(\mathrm{wr}(f_1,\dots,f_\ell)) \geqslant \sum_{j=1}^{\ell} \mathrm{val}(f_j) - \binom{\ell}{2}$$

Proof.

$$\begin{array}{c|cccc|} & \mathrm{val}(f_1) & \mathrm{val}(f_2) & \dots & \mathrm{val}(f_\ell) \\ 0 & f_1 & f_2 & \dots & f_\ell \\ -1 & f_1' & f_2' & \dots & f_\ell' \\ \vdots & \vdots & \vdots & & \vdots \\ -(\ell-1) & f_1^{(\ell-1)} & f_2^{(\ell-1)} & \dots & f_\ell^{(\ell-1)} \end{array}$$

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Let $f_j = X^{\alpha_j}(uX + v)^{\beta_j}$, $uv \neq 0$, linearly independent, and s.t. $\alpha_j, \beta_j \geqslant \ell$. Then

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$$X^{2\ell-3} = (1+X)^{2\ell+3} - 1 - \sum_{j=3}^{\ell} \frac{2\ell-3}{2j-5}\binom{\ell+j-5}{2j-6} X^{2j-5}(1+X)^{\ell-1-j}$$

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Observation + Gap Theorem (recursively)

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- Independent from u and v
- X does not play a special role

Example

$$P = X^{31}Y^6 - 2X^{30}Y^7 + X^{29}Y^8 - X^{29}Y^6 + X^{18}Y^{13} - X^{16}Y^{15} + X^{17}Y^{13} + X^{16}Y^{14} + X^{10}Y^2 - X^9Y^3 + X^9Y^2 - X^5Y^6 + X^3Y^8 - 2X^3Y^7 + X^3Y^6$$

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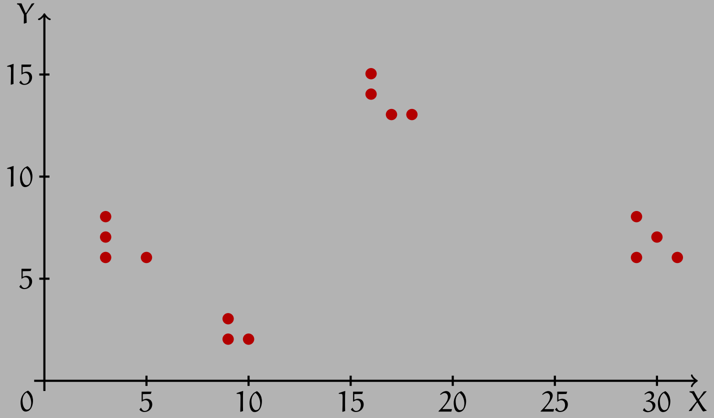

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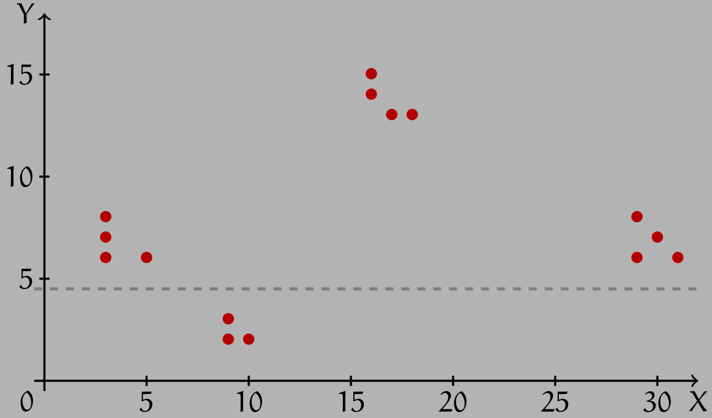

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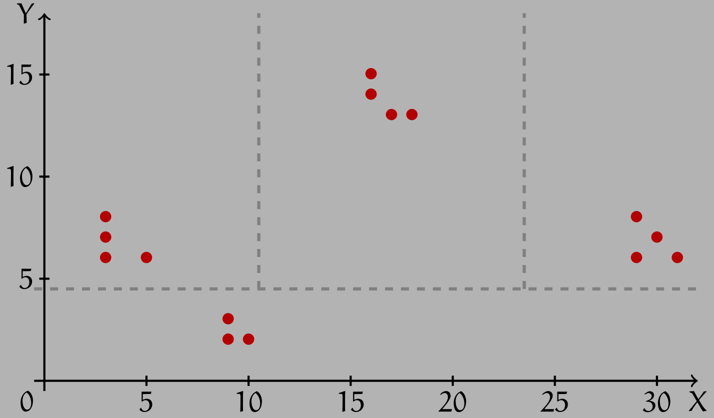

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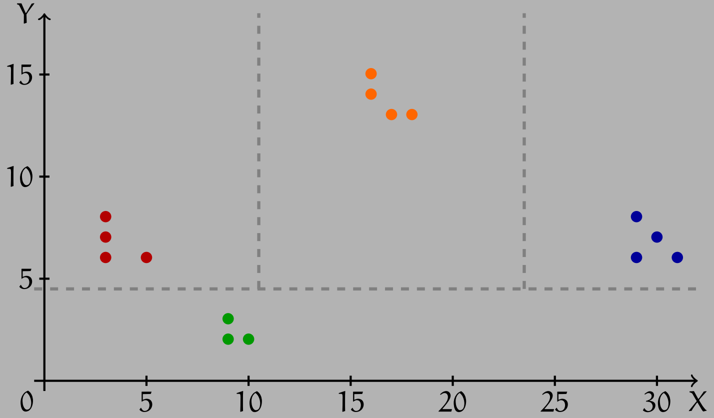

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$$P_1 = X^3Y^6(-X^2 + Y^2 - 2Y + 1)$$

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$$P_4 = X^{29}Y^6(X + Y - 1)(X - Y + 1)$$

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$\implies$ linear factors of P: $(X - Y + 1, 1)$, $(X, 3)$, $(Y, 2)$

Complete algorithm

Find linear factors of $P(X,Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

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↓ monomials

$(X, \min_j \alpha_j)$
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monomials

$(X, \min_j \alpha_j)$
$(Y, \min_j \beta_j)$

binomials

$(X - a)$
Factors of $\sum_j a_j X^{\alpha_j}$

$(Y - uX)$
Roots of $u \mapsto \sum_j a_j u^{\beta_j}$

Univariate lacunary factorization
[H. Lenstra'99]

Find linear factors of $P(X,Y) = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j}$

- **monomials** → $(X, \min_j \alpha_j)$, $(Y, \min_j \beta_j)$
- **binomials** →
  - $(X - a)$: Factors of $\sum_j a_j X^{\alpha_j}$
  - $(Y - uX)$: Roots of $u \mapsto \sum_j a_j u^{\beta_j}$

  Univariate lacunary factorization [H. Lenstra'99]
- **trinomials** → Common factors of $P_t = \sum_{j=j_t}^{j_t+\ell_t-1} a_j X^{\alpha_j} Y^{\beta_j}$ $(\deg(P_t) \leqslant \mathcal{O}(\ell_t^2))$

  Low-degree factorization [Kaltofen'82, ..., Lecerf'07]

Let $P = \sum_{j=1}^{k} a_j X^{\alpha_j} Y^{\beta_j} \in \mathbb{Q}(\alpha)[X, Y]$ be given in lacunary representation. There exists a **deterministic polynomial-time** algorithm to compute its linear factors, with multiplicities.

monomials — binomials — trinomials

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$(Y, \min_j \beta_j)$

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$h_P = \max_j |a_j|$ if $P \in \mathbb{Z}[X, Y]$

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  - [Kaltofen-Koiran] Apply k times the algorithm
- Algebraic number field only: based on [H. Lenstra'99]

- **Multilinear** factors, with a new Gap Theorem:

$$\mathrm{val}\left(\sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+w)^{\beta_j}(vX+t)^{\gamma_j}\right) \leqslant \alpha_1 + 2\binom{\ell}{2};$$

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- Multilinear factors with $\geqslant 3$ monomials over
  - $\overline{\mathbb{Q}}$: absolute factorization;
  - $\mathbb{R}$, $\mathbb{C}$: approximate factorization;
  - ...

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Theorem

Let $P = \sum_{j=1}^{\ell} a_j X^{\alpha_j}(uX+v)^{\beta_j} \in \mathbb{F}_{p^s}[X]$, where $p > \max_j(\alpha_j + \beta_j)$.

Then $\mathrm{val}(P) \leqslant \max_j\left(\alpha_j + \binom{\ell+1-j}{2}\right)$, provided $P \not\equiv 0$.

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Proposition

$\mathrm{wr}(f_1, \ldots, f_k) \neq 0 \iff f_j$'s linearly independent **over** $\mathbb{F}_{p^s}[X^p]$.

$$\text{Find multilinear factors of } P = \sum_{j=1}^{k} a_j X_1^{\alpha_{1,j}} \cdots X_n^{\alpha_{n,j}}$$
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[Gao'03, Lecerf'10]

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- monomials → $(X, \min_j \alpha_j)$, $(Y, \min_j \beta_j)$
- binomials → $(uX^{\beta} - \ldots$ … R… …variate la… polynomials
  - **NP-complete under BPP reductions**
  - [Kipnis-Shamir'99,Bi-Cheng-Rojas'13]
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- Correct bound for the valuation?

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- Practical efficiency