Fast Algorithms for Discrete Differential Equations

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Hadrien Notarantonio (Inria Saclay)

Joint work with:
Alin Bostan (Inria Saclay)
Mohab Safey El Din (Sorbonne Université)

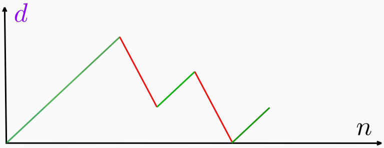
Motivation: Enumerating discrete structures...

Counting walks in $\mathbb{N}$ with steps in $\{+1, -2\}$

$c_n := \#\{n$ steps walks starting at 0 and ending at height $0\}$

$\updownarrow$

$G(t) := \sum_{n=0}^{\infty} c_n t^n$ generating function



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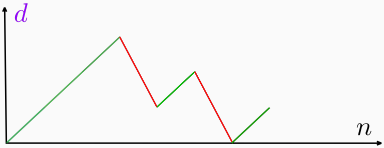

Enumeration refinement

$c_{n,d} := \# \{n$ steps walks starting at 0 and ending at height $d\}$

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$F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^{n} c_{n,d} u^d t^n$ complete generating function

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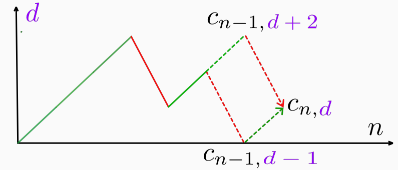

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DDE of order 2

$$c_{n,d} = c_{n-1,d-1} + c_{n-1,d+2}$$

$\updownarrow$

$$F(t,u) = 1 + t \cdot u \cdot F(t,u) + t \cdot \frac{F(t,u) - F(t,0) - u \cdot \partial_u F(t,0)}{u^2}$$

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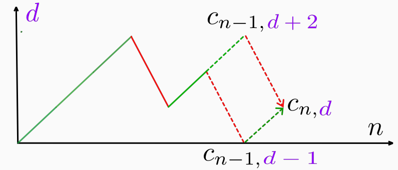

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$$c_{n,0} = c_n \implies F(t,0) = G(t)$$

... yields challenging computational problems

$\mathbb{K}$ effective field of characteristic 0. $\mathbb{K} = \mathbb{Q}, \mathbb{Q}(y), \ldots$

Starting point: $F \in \mathbb{K}[u][[t]]$, solution of the discrete differential equation of order **2**

$$F(t,u) = 1 + t \cdot u \cdot F(t,u) + t \cdot \Delta^{(2)} F(t,u),$$

where $\Delta F(t,u) := \frac{F(t,u) - F(t,0)}{u}$ and $\Delta^{(2)} F(t,u) = \frac{F(t,u) - F(t,0) - u \cdot \partial_u F(t,0)}{u^2}$.

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Classical: F, $F(t, 0)$ and $\partial_u F(t, 0)$ are algebraic. [Bousquet-Mélou, Jehanne '06]

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Goals:

- Compute a polynomial $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ such that $R(t, F(t,0)) = 0$.
- Estimate the size of R for such DDEs.
- Complexity estimates (ops. in $\mathbb{K}$) for the computation of R.

State of the art

Let $k \geq 1$, $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y_1, \ldots, y_k, t, u]$. For $F \in \mathbb{K}[u][[t]]$, define $\Delta(F) := (F - F(t,0))/u \in \mathbb{K}[u][[t]]$ and $\Delta^{(i)}(F) := \Delta \circ \Delta^{(i-1)}(F)$.

Theorem [Bousquet-Mélou, Jehanne '06]
There exists a unique solution $F \in \mathbb{K}[u][[t]]$ to

$$F(t,u) = f(u) + t \cdot Q(F, \Delta(F), \ldots, \Delta^{(k)}(F), t, u), \quad \textbf{(DDE)}$$

and moreover $F(t,u)$ is algebraic over $\mathbb{K}(t,u)$.

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[Tutte, Brown 60's], [Zeilberger '92]:	Guess-and-prove
[Gessel, Zeilberger '14]:	Guess-and-prove
[Brown '65], [Bender, Canfield '94]:	Quadratic method
[Knuth '68], [Banderier, Flajolet '02], [Bousquet-Mélou, Petkovšek '00]:	Kernel method (linear case)
[Bousquet-Mélou, Jehanne '06]:	Polynomial elimination
[Bostan, Chyzak, Notarantonio, Safey El Din '22]:	Polynomial elimination, Hybrid guess-and-prove

Modelization: from (DDE) of order k to structured polynomial systems

We write $P(u) \equiv P(F(t,u), F(t,0), \dots, \partial_u^{k-1} F(t,0), t, u)$ and $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \equiv \bigcup_{d \geq 1} \overline{\mathbb{K}}[[t^{\frac{1}{d}}]]$

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Discrete Differential Equation (DDE)

$\Downarrow$ numer

$$P(u) = 0$$

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General idea:

$\mathbf{U(t)} \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \setminus \overline{\mathbb{K}}$ solution in u of $\partial_1 P(u) = 0$

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$(x,u) = (F(t,\mathbf{U}), \mathbf{U})$ is a solution of

$$\begin{cases} P(x, F(t,0), \dots, \partial_u^{k-1}F(t,0), t, \mathbf{u}) = 0, \\ \partial_1 P(x, F(t,0), \dots, \partial_u^{k-1}F(t,0), t, \mathbf{u}) = 0, \\ \partial_u P(x, F(t,0), \dots, \partial_u^{k-1}F(t,0), t, \mathbf{u}) = 0. \end{cases}$$

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$\mathbf{U}_1(\mathbf{t}), \ldots, \mathbf{U}_\ell(\mathbf{t}) \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \setminus \overline{\mathbb{K}}$ distinct solutions in u of $\partial_1 P(u) = 0$ $\rightarrow$

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The **points**

$(x_1, \mathbf{u_1}) = (F(t, \mathbf{U_1}), \mathbf{U_1}), \ldots, (x_\ell, \mathbf{u}_\ell) = (F(t, \mathbf{U}_\ell), \mathbf{U}_\ell) \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]^2$ are solutions of the conditions:

$$\forall\, 1 \leq i \leq \ell, \quad \begin{cases} P(x_i, F(t,0), \ldots, \partial_u^{k-1}F(t,0), t, \mathbf{u_i}) = 0, \\ \partial_1 P(x_i, F(t,0), \ldots, \partial_u^{k-1}F(t,0), t, \mathbf{u_i}) = 0, \\ \partial_u P(x_i, F(t,0), \ldots, \partial_u^{k-1}F(t,0), t, \mathbf{u_i}) = 0. \end{cases}$$

and $\prod_{i\neq j} (\mathbf{u_i} - \mathbf{u_j}) \neq 0$.

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and $\prod_{i \neq j} (\mathbf{u_i} - \mathbf{u_j}) \neq 0$. $\ell = k \implies$ **$3k$ equations and $3k$ unknowns!**

Our contributions

$$F(t,u) = f(u) + t \cdot Q(F, \Delta(F), \ldots, \Delta^{(k)}(F), t, u), \qquad \textbf{(DDE)}$$

where $\Delta(F) := \frac{F(t,u) - F(t,0)}{u}$.

Input: $P :=$ numerator(DDE),

Goal: Compute $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ s.t. $R(t, F(t,0)) = 0$.

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1. **Geometric analysis** of Bousquet-Mélou and Jehanne's algorithm yielding:
   - Theoretical estimate for the degree of $R \in \mathbb{K}[t, z_0]$ s.t. $R(t, F(t,0)) = 0$,
   - Arithmetic complexity.
2. **New algorithm** based on **algebraic elimination** + Gröbner bases,
3. **Implementations** yielding practical improvements.

Example: (walks in $\mathbb{N}$ with steps in $\{+1, -2\}$)

We consider $P(F(t,u), F(t,0), \partial_u F(t,0), t, u) = 0$

$$u^2 - (u^2 - t(1+u^3)) \cdot F(t,u) - t \cdot F(t,0) - t \cdot u \cdot \partial_u F(t,0) = 0$$

An experimental observation... (Contribution 1)

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Study the solutions of $\partial_1 P(F(t,u), F(t,0), \partial_u F(t,0), t, u) = 0$

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Hence $(x_1, u_1, x_2, u_2) = (F(t, U_1), U_1, F(t, U_2), U_2)$ is a solution of the constraints $\mathcal{T}$

$$\mathcal{T}: \quad \text{For } 1 \le i \le 2, \begin{cases} P(x_i, F(t,0), \partial_u F(t,0), t, u_i) & = 0, \\ \partial_1 P(x_i, F(t,0), \partial_u F(t,0), t, u_i) & = 0, \\ \partial_u P(x_i, F(t,0), \partial_u F(t,0), t, u_i) & = 0. \end{cases} \quad m \cdot (u_1 - u_2) = 1,$$

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Permuting (x_1, u_1) and (x_2, u_2) does not change the solution set
$\implies \mathfrak{S}_2$ acts on $V(\mathcal{T})$ and preserves the $\{z_0, z_1, t\}$-coordinate space.

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Impact of this group action?

... yielding theoretical improvements (Contribution 1)

Denote by $\mathcal{I}$ the ideal generated by the k duplications of $(P, \partial_1 P, \partial_u P)$ and $m \cdot \prod_{i \neq j}(u_i - u_j) - 1 = 0$.

Assume that:

- there exist k distinct solutions $u = U_1, \ldots, U_k \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$ of $\partial_1 P(u) = 0$,
- $\mathcal{I}$ is radical and of dimension 0 over $\mathbb{K}(t)$. (3k equations and 3k unknowns)

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- $\mathcal{I}$ is radical and of dimension 0 over $\mathbb{K}(t)$. (3k equations and 3k unknowns)

Theorem [Bostan, N., Safey El Din '23]

- Let $\delta := \deg(P)$. There exists a nonzero polynomial $R \in \mathbb{K}[t, z_0]$ whose partial degrees are bounded by $\delta^{\mathbf{k}}(\delta - \mathbf{1})^{\mathbf{2k}}/\mathbf{k}!$ and such that $R(t, F(t,0)) = 0$.
- There exists an algorithm computing R in $\tilde{\mathbf{O}}(\delta^{\mathbf{6k}}(\mathbf{k}^{\mathbf{2}}\delta^{\mathbf{k+3}} + \delta^{\mathbf{1.89k}}/\mathbf{k}!))$ ops. in $\mathbb{K}$.

Ideas of the proof:

→ Bézout bound + $\mathfrak{S}_k$ acts on $V(\mathcal{I}_{\text{dup}})$ and preserves the z_0-coordinate space.

→ Parametric geometric resolution [Schost '03], [Giusti, Lecerf, Salvy '01]

$$z_0 = V(t,\lambda)/\partial_\lambda W(t,\lambda), W(t,\lambda) = 0$$

→ Change of monomial ordering:
Stickelberger's theorem [Cox '21] $\quad R = \text{Sqfree}(\text{Res}_\lambda(z_0 \cdot \partial_\lambda W - V, W))$
+ bivariate resultants [Villard '18], [van der Hoeven, Lecerf '21]

Summary of the initial problem: $\underline{z} \equiv z_0, \ldots, z_{k-1}$; $P =$ "numer"$(DDE) \in \mathbb{K}(t)[x, \mathbf{u}, \underline{z}]$

There exist k solutions $(x, \mathbf{u}) \in \overline{\mathbb{K}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} P(x, \mathbf{u}, F(t,0), \ldots, \partial_u^{k-1} F(t,0)) & = 0, \\ \partial_1 P(x, \mathbf{u}, F(t,0), \ldots, \partial_u^{k-1} F(t,0)) & = 0, \quad \mathbf{u} \neq 0, \\ \partial_u P(x, \mathbf{u}, F(t,0), \ldots, \partial_u^{k-1} F(t,0)) & = 0. \end{cases}$$

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Define

$$\pi_x : (x, \mathbf{u}, \underline{z}) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\mathbf{u}, \underline{z}) \in \overline{\mathbb{K}(t)}^{k+1},$$

$$\pi_u : (\mathbf{u}, \underline{z}) \in \overline{\mathbb{K}(t)}^{k+1} \mapsto (\underline{z}) \in \overline{\mathbb{K}(t)}^{k},$$

and consider $\mathbf{W} := \pi_x(V(P, \partial_1 P, \partial_u P) \setminus V(\mathbf{u}))$.

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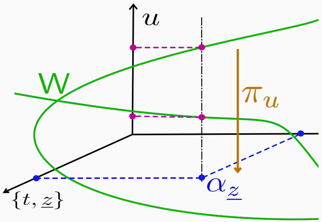

$$\# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap W = 2$$

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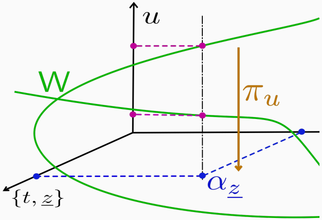

$\# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$

Objective:

Characterize with polynomial constraints

$\mathcal{F}_k := \{\alpha_{\underline{z}} \in \overline{\mathbb{K}(t)}^k \mid \# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq k\}$

Example: (Walks in $\mathbb{N}$ with steps in $\{+1, -2\}$)

$$P := (1-x)u^2 + tu^3x + t(x - z_0 - uz_1) \in \mathbb{K}(t)[x, u, z_0, z_1], \qquad k = 2.$$

G_u Gröbner basis of $\langle P, \partial_1 P, \partial_u P, mu - 1\rangle \cap \mathbb{K}[u, t, z_0, z_1]$ for $\{u\} \succ_{lex} \{t, z_0, z_1\}$:

Getting some intuition through our toy example... (Contribution 2)

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$$
\begin{aligned}
&\mathbf{B_0}: \quad \gamma_0 \\
&\mathbf{B_1}: \begin{cases} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{cases} \quad , \gamma_i, \beta_j \in \mathbb{K}[t, z_0, z_1] \\
&\mathbf{B_2}: \quad g_2 := u^2 + \beta_{r+1} \cdot u + \gamma_{r+1}
\end{aligned}
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Necessary condition:
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The pre-image of α by π_u is well-defined
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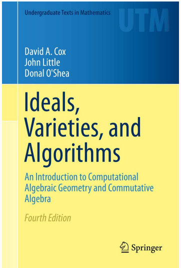
After adding these constraints to G_u and eliminating u and z_1:
$\mathbf{R(t, z_0) = t^3 z_0^3 - z_0 + 1}$ satisfies $\mathbf{R(t, F(t, 0)) = 0}$

… yields an algorithm based on elimination theory (Contribution 2)

- **Projecting**: Elimination theorem
- **Lifting points of the projections**: Extension theorems

Cardinality conditions on the fibers:

- Extension theorem (Gröbner bases version)
- $g(u, \alpha_{\underline{z}}) \in \overline{\mathbb{K}(t)}[u]$ of degree $k+j$ has at least k distinct roots $\iff$ One of the $(k \times k)$-minors of the Hermite quadratic form associated with g does not vanish at $\alpha_{\underline{z}}$



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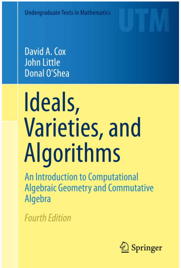

[Bostan, N., Safey El Din '23] :
Conjunctions of polynomial equations and inequations whose zero set is $\mathcal{F}_k$

First implementations yielding practical improvements (Contribution 3)

3-Tamari, $k = 3$

A	T	d_t	d_{z_0}
D	2d2h	5	16
DE	2m	5	16
HGP + DE	1h40m	5	16

5-constellations, $k = 4$

A	T	d_t	d_{z_0}
D	∞	—	—
DE	∞	26 •	53 •
HGP + DE	6h7m	2	5

•: data obtained after a computation mod $p = 65521$.

- **A**: Algorithm used (D: duplication, DE: direct elimination, HGP: Hybrid Guess-and-Prove [Bostan, Chyzak, N., Safey El Din '22]),
- **T**: total timing needed to obtain an output in $\mathbb{Q}[t, z_0]$,
- $\mathbf{d_Z}$: degree in $Z \in \{t, z_0\}$ of output $R \in \mathbb{Q}[t, z_0]$ s.t. $R(t, F(t, a)) = 0$,

Intel® Xeon® Gold CPU 6246R v4 @ 3.40GHz and 1.5TB of RAM with a single thread.
Gröbner bases computations are performed using the C library msolve, and all guessing computations are performed using the gfun Maple package.

Conclusion and future works

Conclusion

- New geometric interpretations of the problem “solving a DDE”,
- New algorithm based on algebraic elimination and Gröbner bases,
- Some promising practical results,
- (In the preprint) New geometric algorithm based on Stickelberger’s theorem.

Future works

- Study the *minimality* and the *genericity* of the introduced assumptions,
- Provide a maple package for solving DDEs, together with a tutorial paper.

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Input: $P(F(t,u), F(t,0), \ldots, \partial_u^{k-1}F(t,0), t, u) = 0$, $\delta := \deg(P)$.

Output: $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ annihilating $F_0 = F(t,0)$, i.e. $R(t, F_0) = 0$.

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geometry

(1) Functional equation

↓

(2) Polynomial system

↓

(3) Bounds

- $\deg_t(R) \leq b_t$,
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(5) Compute $R \in \mathbb{K}[t, z_0]$ s.t. $R(t, F_0) = O(t^{\sim b_t b_z})$

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↓
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tools

- Newton iteration

↓

- Algebraic approximants "seriestoalgeq"

↓

- Multiplicity lemma: $R(t, F_0) = O(t^{\sim 2 b_t b_z})$ implies $R(t, F_0) = 0$

Solving our toy example using the Hybrid Guess-and-Prove strategy

Counting walks in $\mathbb{N}$ with steps in $\{+1, -2\}$

$$F(t,u) = 1 + t \cdot u \cdot F(t,u) + t \cdot \frac{F(t,u) - F(t,0) - u \cdot \partial_u F(t,0)}{u^2}$$

- Draw a random $c = 1341$, and a prime number $p = 19541$,
- Using the new algorithm based on elimination theory, we obtain:
  - $R(t,c) \mod p = t^3 + 15794$,
  - $R(c,z_0) \mod p = z_0^3 + 18182z_0 + 1319$.
- Set $b_t = 3$, $b_{z_0} = 3$,
- Compute
  $F(t,0) = 1 + t^3 + 3t^6 + 12t^9 + 55t^{12} + 273t^{15} + 1428t^{18} + O(t^{2 \cdot b_t \cdot b_{z_0} + 1})$
- Guess $A := t^3 z_0^3 - z0 + 1$ such that $A(t, F(t,0)) = O(t^{(b_t+1)\cdot(b_{z_0}+1)-1})$,
- Check that $A(t, F(t,0)) = O(t^{2 \cdot b_t \cdot b_{z_0} + 1})$

The output A is certified.