[ddesolver] A Maple package for Discrete Differential Equations

GT Combinatoire et Interactions, 15 January 2024

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Based on a joint work with:

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Mohab Safey El Din (Sorbonne Université)

Content of the talk

- Motivation through **combinatorial examples**,
- **Examples of algorithms** implemented in **ddesolver**,
- **Presentation** of **ddesolver**,
- **Timings**.

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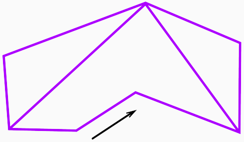
rooted planar maps



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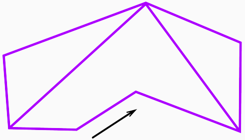

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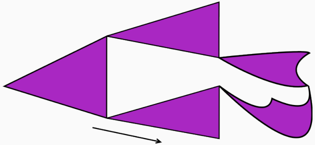

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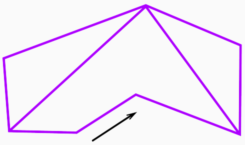

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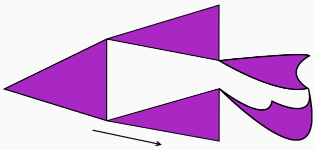
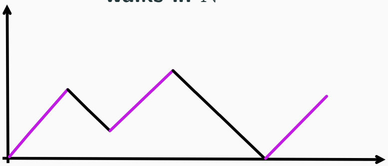


walks in $\mathbb{N}$



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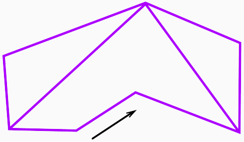

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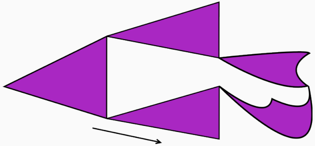

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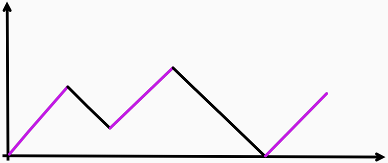


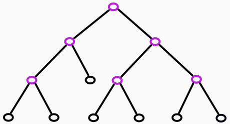
$\mathcal{A}$ set of **discrete** objects, $s : \mathcal{A} \to \mathbb{N}$ **size** function s.t.

$$\#\{a \in \mathcal{A} \mid s(a) = n\} < +\infty, \text{ for all } n \in \mathbb{N}.$$

Define the generating function

$$F(t) := \sum_{a \in \mathcal{A}} t^{s(a)} \qquad \in \mathbb{Q}[[t]]$$

A first toy example: binary trees



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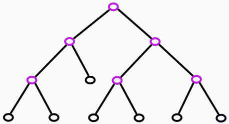

$F_{BT} \in \mathbb{Q}[[t]]$ **generating function** of binary trees, counted by the number of **internal nodes**

$$F_{BT}(t) = 1 + t \cdot F_{BT}(t)^2.$$

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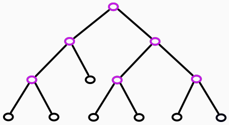

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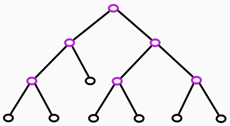

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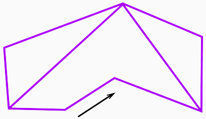
$$F_{BT}(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n$$

Catalan number

First nontrivial example: planar maps enumeration



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$

[Tutte '68]

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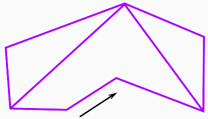

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↓ refinement

$a_{n,d} := \#${planar maps with n edges, d of them on the external face}

$\sum_{n=0}^{\infty} a_n t^n$ generating function

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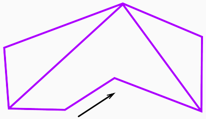

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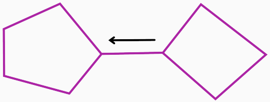

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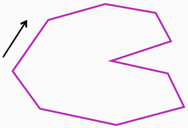

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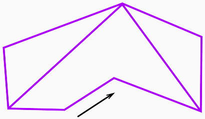

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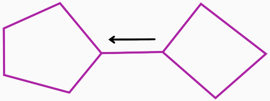

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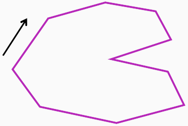

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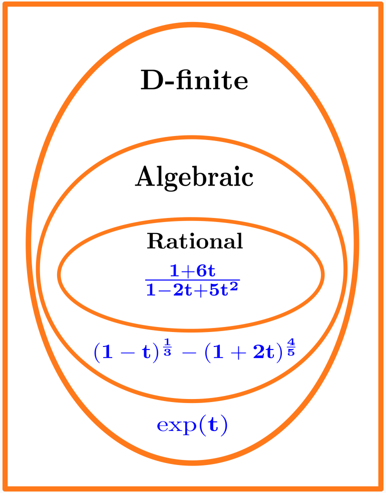
$$F(t,1) = \sum_{n=0}^{\infty} a_n t^n$$

Solving functional equations



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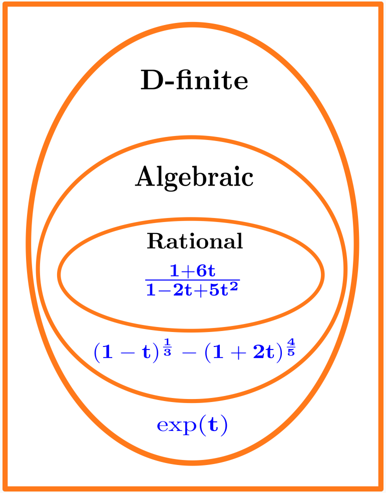

In this talk

Solving $=$ Classifying the initial series $F(t,1)$

$+$ Computing a witness of this classification

(e.g. $R \in \mathbb{Q}[z,t]$ s.t. $R(F(t,1),t) = 0$)

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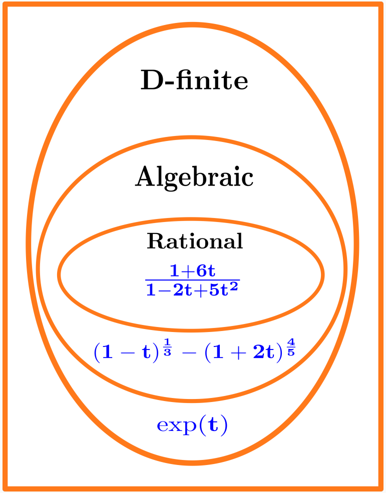

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Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \qquad \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

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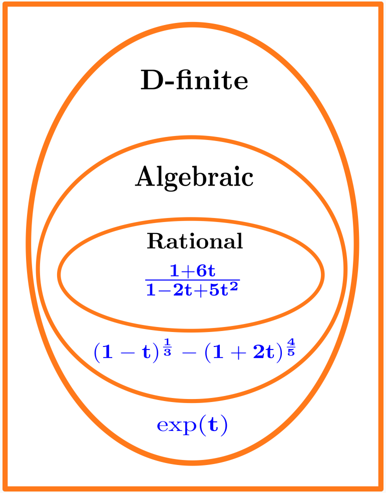

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From R:

- **(Recurrence)** $a_0 = 1$ and $(n+3)a_{n+1} - 6(2n+1)a_n = 0$,
- **(Closed-form)** $a_n = 2\frac{3^n(2n)!}{n(n+2)!}$,
- **(Asymptotics)** $a_n \sim 2\frac{12^n}{\sqrt{\pi n^5}}$, when $n \to +\infty$.

Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \dots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \dots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a **Discrete Differential Equation**, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^\ell \circ \Delta$.

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Are 3-constellations of this shape? YES!

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Theorem [Bousquet-Mélou, Jehanne '06]

The unique solution in $\mathbb{Q}[u][[t]]$ *of* **(DDE)** *is* **algebraic** *over* $\mathbb{Q}(t,u)$.

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- **Gather** the 7 **equations** in 7 **unknowns** and 1 **parameter**

$$\text{For } 1 \leq i \leq 2, \begin{cases} P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_u P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ m \cdot (U_1 - U_2) - 1 = 0. \end{cases}$$

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Elimination theory

- Eliminate all series but $F(t,1)$

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- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0$,
- **Gather** the 7 **equations** in 7 **unknowns** and 1 **parameter**

$$\text{For } 1 \leq i \leq 2, \begin{cases} P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_u P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ m \cdot (U_1 - U_2) - 1 = 0. \end{cases}$$

Elimination theory

- Eliminate all series but $F(t,1)$

$\rightarrow$ **Resultants**

Bousquet-Mélou and Jehanne's duplication algorithm

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right)$,

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2 - 66t + 1)F(t,1) - 3t^2 + 47t - 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x, u, z_0, z_1]$ such that $P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0$,
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Elimination theory

- Eliminate all series but $F(t,1)$

→ **Resultants**

→ **Gröbner bases**

Maple worksheet: Solving the DDE of 3-constellations

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right)$,

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- $R := \text{remove}(\text{has}, \text{Groebner}[\text{Basis}](S, \text{plex}(m, x_1, x_2, u_1, u_2, z_1, z_0, t)), \{m, x_1, x_2, u_1, u_2, z_1\});$

 # eliminates all variables except z_0 and t from S

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  # eliminates all variables except z_0 and t from S

$$R := (16tz_0^2 - 8tz_0 + t - 16)(\mathbf{81t^2z_0^3 - 9t(9t-2)z_0^2 + (27t^2 - 66t + 1)z_0 - 3t^2 + 47t - 1})$$

Example computationally out of reach for this duplication algorithm

3-greedy Tamari intervals $(k = 3)$

- On Dyck paths: *"Swap a down step and the longest Dyck path that follows"*
- Explicit formula for the complete generating function [Bousquet-Mélou, Chapoton '23]
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5-constellations $(k = 4)$

- Colored planar maps with additional constraints (e.g. degree of the faces)
- Solved via a bijective proof in [Bousquet-Mélou, Schaeffer '00]
- Duplication method fails ($<$ 5 days)

Some preliminaries on Gröbner bases

$\mathcal{A} := \mathbb{Q}[x, \boldsymbol{y}]$ polynomial ring, where $\boldsymbol{y} = y_1, \ldots, y_s$.

Monomial orders

- $x^4 y_1^3 y_2^2 \succ_{plex} x^3 y_1^4 y_2^2$ for a **lexicographic order**,
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Fix a monomial order $\succ$ on $\mathcal{A}$. A generating set $G = \{g_1, \ldots, g_t\}$ of an ideal $\mathcal{I} \subset \mathcal{A}$ different from 0 is said to be a **Gröbner basis** if $\langle \mathrm{LT}_{\succ}(g_1), \ldots, \mathrm{LT}_{\succ}(g_t)\rangle = \langle \mathrm{LT}_{\succ}(\mathcal{I})\rangle$.

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Properties

- Such bases always **exist**,
- Computing Gröbner bases is **NP-hard**,
- Gröbner bases are a **powerful tool** in elimination theory.

Geometric modelling of the problem (with Bostan and Safey El Din)

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{0}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}. \end{cases}$$

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$\pi_x : (x, \mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^4 \mapsto (\mathbf{u}, z_0, z_1) \in \overline{\mathbb{Q}(t)}^3,$

$\mathbf{W} := \pi_x(V(\mathbf{P}, \partial_{\mathbf{x}} \mathbf{P}, \partial_{\mathbf{u}} \mathbf{P}) \setminus V(\mathbf{u}))$

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Characterize with polynomial constraints

$$\mathcal{F}_2 := \{\alpha_{\underline{z}} \in \overline{\mathbb{Q}(t)}^2 \mid \# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq 2\}$$

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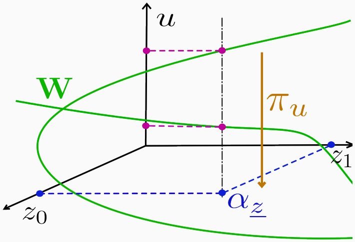

$\# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$

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Solving a toy example... (with Bostan and Safey El Din)

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

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$$\mathbf{B_0}: \quad \gamma_0$$

$$\mathbf{B_1}: \begin{cases} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{cases}, \gamma_i, \beta_j \in \mathbb{Q}(t)[z_0, z_1]$$

$$\mathbf{B_2}: \quad \mathbf{g_2} := u^2 + \beta_{r+1} \cdot u + \gamma_{r+1}$$

"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two distinct solutions in u"

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"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two distinct solutions in u"

At $\alpha \in V(G_u \cap \mathbb{K}[t, z_0, z_1])$ fixed, there exist two solutions in u
$\Longrightarrow \beta_i, \gamma_j = 0$ (**equations**)

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"At $\alpha \in \pi_u(V(G_u)) \subset \overline{\mathbb{Q}(t)}^2$, there exist two **distinct** solutions in u"

At $\alpha \in V(G_u \cap \mathbb{K}[t, z_0, z_1])$ fixed, there exist two solutions in u
$\implies \beta_i, \gamma_j = 0$ $\qquad$ (**equations**)

[Extension theorem]

$\alpha \in \pi_u(V(G_u)) \implies \mathrm{LeadingCoeff}_u(\mathbf{g_2}) \neq 0$

Distinct solutions in $u \implies \mathrm{disc}_u(\mathbf{g_2}) \neq 0$ $\qquad$ (**inequations**)

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- $G_m :=$ Groebner[Basis](S, lexdeg([m], [x, u, z_1, z_0, t])): **# Eliminate m from S**
- $G_x :=$ Groebner[Basis](remove(has, G_m, m), lexdeg([x], [u, z_1, z_0, t])): **# Eliminate x from G_m**
- $G_u :=$ Groebner[Basis](remove(has, G_x, x), lexdeg([u], [z_1, z_0, t])): **# Sort G_u for $\{u\} \succ \{z_1, z_0, t\}$**

Maple worksheet: Solving the DDE of walks in $\mathbb{N}$ with steps in $\{+1, -2\}$

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right),$ **k = 2**

Output: $t^3 F(t,0)^3 - F(t,0) + 1 = 0.$

- $P := (1-x)u^2 + t(u^3x + (x - z_0 - uz_1))$: $S := [P, \mathrm{diff}(P,x), \mathrm{diff}(P,u), mu - 1]$:
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- $L := [\,]$:
- for i to nops(G_u) do: **# The below lines add to L the new relevant polynomial equations**
  - if degree($G_u[i], u$) < 2 then: $L := [\mathrm{op}(L), \mathrm{coeffs}(G_u[i], u)]$: fi: od:

Maple worksheet: Solving the DDE of walks in $\mathbb{N}$ with steps in $\{+1,-2\}$

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u)-F(t,0)-u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

Output: $t^3F(t,0)^3 - F(t,0) + 1 = 0.$

- $P := (1-x)u^2 + t(u^3x + (x - z_0 - uz_1))$: $S := [P, \mathrm{diff}(P,x), \mathrm{diff}(P,u), mu-1]$:
- $G_m :=$ Groebner[Basis](S, lexdeg([m], [x, u, z_1, z_0, t])): **# Eliminate m from S**
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- for i to nops(G_u) do: **# The below lines add to L the new relevant polynomial equations**
  - if degree($G_u[i]$, u) < 2 then: $L :=$ [op(L), coeffs($G_u[i]$, u)]: fi: od:
- $H :=$ Groebner[Basis]([op(G_u), op(L)], plex(u, z_1, z_0, t))[1];

$$t(\mathbf{t^3 z_0^3 - z_0 + 1})$$

Maple worksheet: Solving the DDE of 5-constellations

Input: (The rather big DDE associated with the enumeration of 5-constellations)
Output: $15625t^2F(t,1)^5 - 31250t^2F(t,1)^4 + (25000t^2 - 1000t)F(t,1)^3 - (10000t^2 - 8700t)F(t,1)^2 + (2000t^2 - 15855t + 16)F(t,1) - 160t^2 + 8139t - 16 = 0$

- $P :=$ (a rather big polynomial...): $S := [P, \mathrm{diff}(P,x), \mathrm{diff}(P,u), m(u-1)-1]$:

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- $P :=$ (a rather big polynomial...): $S := [P, \mathrm{diff}(P,x), \mathrm{diff}(P,u), m(u-1)-1]$:

$$P = x^5tu^5 - 4x^5tu^4 + 6x^5tu^3 + 4x^4tu^4 - x^3z_0tu^4 - x^2z_0^2tu^4 - xz_0^3tu^4 - z_0^4tu^4 - 4x^5tu^2 - 12x^4tu^3 + 3x^3z_0tu^3 + 3x^2z_0^2tu^3 + 3xz_0^3tu^3 + 3z_0^4tu^3 - x^2z_1tu^4$$
$$- 3xz_0z_1tu^4 - 6z_0^2z_1tu^4 + x^5tu + 12x^4tu^2 - 3x^3z_0tu^2 - 3x^2z_0^2tu^2 - 3xz_0^3tu^2 - 3z_0^4tu^2 + 6x^3tu^3 - 3x^2z_0tu^3 - 2xz_0^2tu^3 - z_0^3tu^3 + 3x^2z_1tu^3 + 9xz_0z_1tu^3$$
$$+ 18z_0^2z_1tu^3 - 2z_1^2tu^4 - x\frac{z_2}{2}tu^4 - 2z_0z_2tu^4 - 4x^4tu + x^3z_0tu + x^2z_0^2tu + xz_0^3tu + z_0^4tu - 12x^3tu^2 + 6x^2z_0tu^2 + 4xz_0^2tu^2 + 2z_0^3tu^2 - 3x^2z_1tu^2$$
$$- 9xz_0z_1tu^2 - 18z_0^2z_1tu^2 - 2xz_1tu^3 - 3z_0z_1tu^3 + 6z_1^2tu^3 + 3x\frac{z_2}{2}tu^3 + 6z_0z_2tu^3 - \frac{z_3}{6}tu^4 + 6x^3tu - 3x^2z_0tu - 2xz_0^2tu - z_0^3tu + x^2z_1tu$$
$$+ 3xz_0z_1tu + 6z_0^2z_1tu + 4x^2tu^2 - 3xz_0tu^2 - z_0^2tu^2 + 4xz_1tu^2 + 6z_0z_1tu^2 - 6z_1^2tu^2 - 3x\frac{z_2}{2}tu^2 - 6z_0z_2tu^2 - \frac{z_2}{2}tu^3 + \frac{z_3}{2}tu^3 - 4x^2tu + 3xz_0tu$$
$$+ z_0^2tu - 2xz_1tu - 3z_0z_1tu + 2z_1^2tu + x\frac{z_2}{2}tu + 2z_0z_2tu - z_1tu^2 + z_2tu^2 - \frac{z_3}{2}tu^2 + u^4 + xtu - z_0tu + z_1tu - \frac{z_2}{2}tu + \frac{z_3}{6}tu - 4u^3$$
$$- (u^4 - 4u^3 + 6u^2 - 4u + 1)x + 6u^2 - 4u + 1;$$

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Input: (The rather big DDE associated with the enumeration of 5-constellations)
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- $P :=$ (a rather big polynomial...): $S := [P, \mathrm{diff}(P,x), \mathrm{diff}(P,u), m(u-1)-1]$:
- $(\ldots)$ **# Compute G_u, stock in L the coefficients of the elements of** $\{g \in G_u \mid \deg_u(g) < 4\}$

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- $(\dots)$ **# Compute G_u, stock in L the coefficients of the elements of $\{g \in G_u \mid \deg_u(g) < 4\}$**
- $(\dots)$ **# Identify the first polynomial in G_u of degree 4 in u, call it g_4**

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Output: $15625t^2F(t,1)^5 - 31250t^2F(t,1)^4 + (25000t^2 - 1000t)F(t,1)^3 - (10000t^2 - 8700t)F(t,1)^2 + (2000t^2 - 15855t + 16)F(t,1) - 160t^2 + 8139t - 16 = 0$

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- $(\ldots)$ **# Compute G_u, stock in L the coefficients of the elements of $\{g \in G_u \mid \deg_u(g) < 4\}$**
- $(\ldots)$ **# Identify the first polynomial in G_u of degree 4 in u, call it g_4**
- $R :=$ Groebner[Basis]([op(G_u), op(L), $m \cdot \mathrm{coeff}(g_4, u, 4) \cdot \mathrm{disc}(g_u, u) - 1$], plex($m, u, z_3, z_2, z_1, z_0, t$))[1];
  # Here it is mandatory to consider the inequation $\mathrm{coeff}(g_4, u, 4) \cdot \mathrm{disc}(g_u, u) \neq 0$

$$R := (\mathbf{15625t^2z_0^5 - 31250t^2z_0^4 + 25000t^2z_0^3 - 10000t^2z_0^2 - 1000tz_0^3 + 2000t^2z_0 + 8700tz_0^2 - 160t^2 - 15855tz_0 + 8139t + 16z_0 - 16}) \cdot (4096tz_0^4 - 6144tz_0^3 + 3456tz_0^2 - 864tz_0 + 81t - 4096)$$

New proofs in a click of the two computationally challenging DDEs

3-greedy Tamari intervals ($k = 3$)

- On Dyck paths: *"Swap a down step and the longest Dyck path that follows"*
- Explicit formula for the complete generating function [Bousquet-Mélou, Chapoton '23]
- Duplication method fails (< 5 days)

5-constellations ($k = 4$)

- Colored planar maps with additional constraints (e.g. degree of the faces)
- Solved via a bijective proof in [Bousquet-Mélou, Schaeffer '00]
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[ddesolver] **finishes in 1 minute!**

[ddesolver] **finishes in 2 days!**

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[ddesolver] finishes in 2 days!

annihilating_polynomial(P, k)

Multiplicity lemma

Let $F \in \mathbb{Q}[[t]]$ be annihilated by some $P \in \mathbb{Q}[t, z] \setminus \{0\}$. Assume there exists $Q \in \mathbb{Q}[t, z] \setminus \{0\}$ s.t.

$$Q(t, F) = O(t^{(\deg_t(P)+1)\cdot(\deg_z(P)+1)}).$$

If in addition $Q(t, F) = O(t^{2\cdot\deg_t(P)\cdot\deg_z(P)+1})$, then $Q(t, F) = 0$.

Hybrid guess-and-prove: third algorithm (with Bostan and Safey El Din)

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Algorithm:

- **Compute** upper bounds $b_t, b_z \in \mathbb{Z}_{>0}$ on the partial degrees of an annihilating polynomial of F,
- **Compute** $F \bmod t^{2\cdot b_t \cdot b_z + 2}$,
- **Guess** $Q \in \mathbb{Q}[t,z]$ such that $Q(t,F) = O(t^{(b_t+1)\cdot(b_z+1)})$,
- **Check** that $Q(t,F) = O(t^{2\cdot b_t\cdot b_z+1})$ and apply the multiplicity lemma.

Solving 5-constellations using a Hybrid Guess-and-Prove strategy

Input: (The rather big DDE associated with the enumeration of 5-constellations)
Output: $15625t^2F(t,1)^5 - 31250t^2F(t,1)^4 + (25000t^2 - 1000t)F(t,1)^3 - (10000t^2 - 8700t)F(t,1)^2 + (2000t^2 - 15855t + 16)F(t,1) - 160t^2 + 8139t - 16 = 0$

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- **Pick at random** a prime number,
- **Compute** upper bounds $(9,3)$ on the bidegree of $M \in \mathbb{F}_p[z,t]$ annihilating $F(t,1)$ modulo p,
- **Expand** the truncated series $F(t,1)$ mod t^{55}, $55 = 2 \cdot 9 \cdot 3 + 1$
- **Guess** $R \in \mathbb{Q}[z,t]$ such that $R(F(t,1),t) = O(t^{(9+1)\cdot(3+1)-1})$,
- **Check** that $R(t, F(t,1)) = O(t^{55})$.

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[Bostan, N., Safey El Din '23]

$\rightsquigarrow$ **elimination strategy**,

$\rightsquigarrow$ **Newton iteration**,

$\rightsquigarrow$ **Hermite Padé approximants**,

$\rightsquigarrow$ **multiplicity lemma**.

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[Bostan, N., Safey El Din '23]

$\rightsquigarrow$ **elimination strategy**,

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Strategy	Timing	(d_z, d_t)
Elimination	2d	$(\mathbf{9},\mathbf{3})$
Hybrid G-P	**2h41min**	$(\mathbf{5},\mathbf{2})$

Content of the talk

- Motivation through **combinatorial examples**,
- **Examples of algorithms** implemented in **ddesolver**,
- **Presentation** of **ddesolver**,
- **Timings**.

ddesolver: package presentation

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ddesolver: package presentation

ddesolver Public

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- **4 implemented algorithms** in a single function "annihilating_polynomial",
- **One of the algorithms** uses the Maple package **gfun** for guessing annihilating polynomials,
- **Can be coupled with libraries** for efficient Gröbner bases computations like the C library **msolve**.

Maple worksheet: default procedure of ddesolver

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u]$ the polynomial associated to the "numerator DDE",
- k the order of the DDE,
- $[x, z_0, \ldots, z_{k-1}, t, u]$ the variables in P (this order matters!).

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

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- *with*(*ddesolver*);

$$[\text{annihilating\_polynomial}]$$

- $P := (u(u-1)^2x^3+2u(u-1)x^2-u(uz_0-z_0-1)x-u(uz_0^2+uz_1-z_0^2+z_0-z_1))t-(u-1)^2x+(u-1)^2 :$

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annihilating_polynomial

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- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

- $with(ddesolver);$

$$[\text{annihilating\_polynomial}]$$

- $P := (u(u-1)^2x^3 + 2u(u-1)x^2 - u(uz_0 - z_0 - 1)x - u(uz_0^2 + uz_1 - z_0^2 + z_0 - z_1))t - (u-1)^2x + (u-1)^2 :$
- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u]);$

$$(16tz_0^2 - 8tz_0 + t - 16) \cdot (81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$$

Maple worksheet: options of annihilating_polynomial

annihilating_polynomial

Input:

- $P \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u]$: polynomial associated to the "numerator DDE",
- k: order of the DDE,
- $[x, z_0, \ldots, z_{k-1}, t, u]$ the variables in P (this order matters!),
- algorithm: to choose between {"duplication", "elimination", "geometry", "hybrid"},
- variable: variable among t and z_0 on which to perform evaluation–interpolation.

Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

Maple worksheet: options of annihilating_polynomial

annihilating_polynomial

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- $P \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u]$: polynomial associated to the "numerator DDE",
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Output:

- $R \in \mathbb{Q}[t, z_0]$ annihilating the univariate series associated to z_0 in the DDE.

- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u], \text{"}elimination'', t)$;

$$(16tz_0^2 - 8tz_0 + t - 16) \cdot (81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$$

- annihilating_polynomial$(P, 2, [x, z_0, z_1, t, u], \text{"}hybrid'', t)$;

$$81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1$$

Content of the talk

ddesolver Public

- Motivation through **combinatorial examples**,
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Timings with ddesolver

Data	[3]		[6]		[7]		[8]	
k	2		3		3		4	
variable	z_0	t	z_0	t	z_0	t	z_0	t
"duplication"	41m*	10m*	13h	27h	∞	∞	∞	∞
"elimination"	1h20m*	4m*	**2m20s**	35m	1m	7m30s	2d19h	2d
"geometry"	54m*	**2m***	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
"hybrid"	∞		1h42m		**34s**		**2h41m**	
$(\deg_t(R), \deg_{z_0}(R))$	$(132, 6)$		$(5, 16)$		$(2, 4)$		$(3, 9)$	

- [3]: Enumeration of non-separable near-triangulations in which all intern vertices have degree at least 5,
- [6]: Enumeration of 3-Tamari lattices,
- [7]: Enumeration of 3-greedy Tamari intervals,
- [8]: Enumeration of 5-constellations.

- ∞: computations did not finish within 5 days,
- $\times$: algorithm not implemented for $k > 2$,
- $\cdot^*$: added the constraint $(1+t^3)(1-t^3)t \neq 0$.

Intel® Xeon® Gold CPU 6246R v4 @ 3.40GHz, 1.5TB of RAM with 1 thread, Gröbner bases computations run with msolve.

Summary and perspectives

- **ddesolver**: first Maple package dedicated to solving DDEs,
- Can be downloaded from the **git repository**,
- **Play with it** and feel free to report any bug!
- Having a hard computation even with this package? **Email me!**

- A **tutorial paper** will be submitted in the coming weeks.

https://github.com/HNotarantonio/ddesolver

Popescu's theorem yielding algebraicity of the solutions

[Popescu '86, Swan '98]

(1.4) THEOREM. *Let k be a field, $k\langle X\rangle$ the algebraic power series ring in $X=(X_1,\cdots,X_r)$ over k, f a finite system of polynomial equations over $k\langle X\rangle$ and $\hat{y}=(\hat{y}_1,\cdots,\hat{y}_n)\in k[\![X]\!]^n$ a formal solution of f such that $\hat{y}_i\in k[\![X_1,\cdots,X_{s_i}]\!]$, $1\leqslant i\leqslant n$ for some positive integers $s_i\leqslant r$. Then there exists a solution $y=(y_1,\cdots,y_n)$ of f in $k\langle X\rangle$ such that $y_i\in k\langle X_1,\cdots,X_{s_i}\rangle$, $1\leqslant i\leqslant n$.*

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- Solutions of systems of DDEs are **unique** with components in $\mathbb{Q}[u][[t]] \implies$ they are **algebraic**!

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[planar maps]

$$H(t,u) = 1 + t\left(u^2 H(t,u)^2 + u\frac{uH(t,u)-G(t,u)}{u-1}\right)$$

- There exists a solution $(H, G) = (F, F(t,1))$, where $F \in \mathbb{Q}[u][[t]]$,
- The involved series are $F(t,1)$ and $F(t,u)$, and $\{\boldsymbol{t}\} \subset \{\boldsymbol{t}, \boldsymbol{u}\}$.