The 22 periods of a quartic surface
and their integer relations

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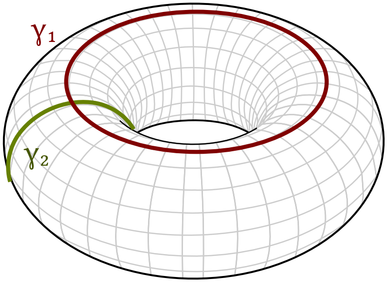
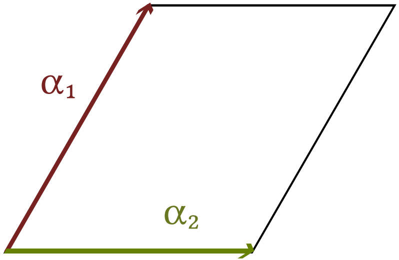
Outline

Periods of an elliptic curve

Let $f \in \mathbb{C}[x,y,z]_3$ (homogeneous polynomial of degree 3) such that $X = V(f) \subseteq \mathbb{P}^2$ is smooth.





- $X \simeq \mathbb{C}/(\alpha_1\mathbb{Z} + \alpha_2\mathbb{Z})$
- $\int_{\gamma_i} \omega_X = \alpha_i$, where ω_X is the unique holomorphic 1-form on X (up to scaling), and γ_1, γ_2 a suitable basis of $H_1(X, \mathbb{Z})$.

Periods of a quartic surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

Let $\gamma_1, \ldots, \gamma_{22}$ be a basis of $H_2(X, \mathbb{Z})$,
and let $\omega_X \in \Omega^2(X)$ be the unique holomorphic 2-form on X.

The *periods* of X are the complex numbers $\alpha_1, \ldots, \alpha_{22}$ defined – up to scaling and choice of basis – by

$$\alpha_i \overset{\text{def}}{=} \int_{\gamma_i} \omega_X.$$

A Torelli theorem for K3 surfaces

Theorem (Pjateckĭı-Šapiro & Šafarevič, 1971)

The periods of a smooth quartic surface determine its isomorphy class.

More precisely, two smooth quartic surfaces X and X' are isomorphic if and only if there is a an isomorphism $\phi : H_2(X, \mathbb{Z}) \longrightarrow H_2(X', \mathbb{Z})$ such that

- $\forall \alpha, \beta \in H_2(X, \mathbb{Z}), \phi(\alpha) \cdot \phi(\beta) = \alpha \cdot \beta$
- $\exists \lambda \in \mathbb{C}^\times \ \forall \gamma \in H_2(X, \mathbb{Z}), \int_\gamma \omega_X = \lambda \int_{\phi(\gamma)} \omega_{X'}$.

Periods determine the isomorphism group

Theorem (Pjateckĭı-Šapiro & Šafarevič, 1971)

The periods of a smooth quartic surface determine its automorphism group.

More precisely, the automorphism group of X is isomorphic to the group of automorphisms of $H_2(X, \mathbb{Z})$

- preserving intersection product
- preserving periods (up to scaling)
- preserving the *ample cone*

Periods determine the Néron-Severi group

The Néron-Severi group of X (a smooth quartic surface) is the sublattice of $H_2(X, \mathbb{Z})$ generated by the classes of algebraic curves on X.

Theorem (Lefschetz, 1924)

$$\mathrm{NS}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \;\middle|\; \int_\gamma \omega_X = 0 \right\}$$

In coordinates, $\mathrm{NS}(X) \simeq \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}$.
This is the lattice of *integer relations between the periods*.

The NS group determine the possible degree and genus of all the algebraic curves lying on X.

The very generic case

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w,x,y,z]_4 \setminus$ (countable union of algebraic hypersurfaces).
Then $\mathrm{NS}(X_f) = \mathbb{Z} \cdot$ (hyperplane section).

Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w,x,y,z]_4$ such that $\mathrm{NS}(X_f) = \mathbb{Z} \cdot h$.

Theorem (van Luijk, 2007)

Let $f = 2w^4 + w^3z + w^2x^2 + 2w^2xy + 2w^2xz - w^2y^2 + w^2z^2 + \cdots$
Then $\mathrm{NS}(X_f) = \mathbb{Z} \cdot h$.

Theorem (Lairez & Sertöz, 2019)

Let $f = wx^3 + w^3y + xz^3 + y^4 + z^4$. Then $\mathrm{NS}(X_f) = \mathbb{Z} \cdot h$.

The Fermat hypersurface

Let $f = w^4 + x^4 + y^4 + z^4$.
The vector of periods is

$$\begin{pmatrix} 1 & i & i & i & i & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -i & -i & -i & -i & -i & -i & 0 \end{pmatrix}$$

$$\operatorname{rank} \mathrm{NS}(X_f) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \{\text{periods}\} = 20.$$

Indeed there are 48 lines on X_f spanning a sublattice of $H_2(X, \mathbb{Z})$ of rank 20.

The Dwork family

Let $f_t = w^4 + x^4 + y^4 + z^4 - 4twxyz$.
The periods of X_t satisfy the differential equation

$$(t^4 - 1)y''' + 6t^3y'' + 7t^2y' + ty = 0,$$

with basis of solutions

$$f_0(t) = {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} & \end{matrix}; t^4\right], \quad f_1(t) = t\,{}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} & \end{matrix}; t^4\right], \quad f_2(t) = t^2\,{}_3F_2\left[\begin{matrix}\frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{5}{4} & \frac{3}{2} & \end{matrix}; t^4\right].$$

If $t^4 \neq 1$, then

$$\operatorname{rank} \mathrm{NS}(X_{f_t}) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \left\{8\pi^4 f_0(t) + \Gamma\left(\tfrac{3}{4}\right)^8 f_2(t), 8\pi^4 i f_0(t) - \Gamma\left(\tfrac{3}{4}\right)^8 i f_2(t), \pi^2\Gamma\left(\tfrac{3}{4}\right)^2 f_1(t)\right\} \in \{19, 20\}$$

Outline

Numerical computation of periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

1. The periods of X_t satisfy a Picard–Fuchs linear differential equation (Picard, 1902).
2. The initial conditions are (generalized) periods of the Fermat quartic, studied by Pham (1965).
3. Numerical analytic continuation provide quasilinear-time algorithms for computing the periods (Mezzarobba, 2016).

⚠ Afflicted by the size of the PF equation (generically order 21 and degree $\sim$ 600), the algorithm does not always terminate in reasonnable time.

Faster computation (with Pichon-Pharabod and Vanhove)

Let $f \in \mathbb{C}[w, x, y, z]_4$ (generic coordinates)
and let $g_t = f(tx, x, y, z) \in \mathbb{C}(t)[x, y, z]_4$.
$\rightsquigarrow$ consider the surface as a family of curves

Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \oint_{\text{cycle in } V(g_t)} \frac{\omega_X}{\mathrm{d}t}.$$

⚠ To be implemented, requires a concrete description of $H_2(X, \mathbb{Z})$
$\rightsquigarrow$ effective Picard–Lefschetz theory

With Eric's implementation, huge improvement of performance compared to previous methods.
(Terminate in reasonnable time for any quartic.)

Computation of the lattice of integer relations

We have the periods $\alpha_1, \dots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

It is an application of the Lenstra–Lenstra–Lovász algorithm:

1. For $1 \leq i \leq 22$, compute the Gaussian integer $[10^{1000}\alpha_i]$.
2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \;\middle|\; \sum_i u_i [10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$, this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.
3. Compute a LLL-reduced basis of L
4. Output the *short* vectors

What is a short vector?

Let $f = 3x^3z - 2x^2y^2 + xz^3 - 8y^4 - 8w^4$.
With 100 digits of precision on the periods, here is a LLL-reduced basis of the lattice L (last 5 columns omitted).

$$
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1669083212117905913652734&0&1937019641160560221317687&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1669083212117905913652734&1937019641160560221317687&\dots\\
1&0&0&-1&0&0&0&1&1&0&0&0&0&0&-1465118299011954436717 89&844784290445878224678 23&-3659802286906301049192 96&\dots\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&-3371677202526783102581 77&2241101519734039462214 21&-7431169559364872799105 52&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&3570314792535223114836 50&7680663376663510994327 48&9405259947193910799984 35&\dots\\
0&0&0&0&0&1&0&0&1&0&1&0&0&0&-5527566718288541531149 05&-1260182482795835854860 71&5350958119531659172108 63&\dots\\
0&-1&1&0&0&0&0&0&1&0&0&-1&0&0&1043354311299086458251 33&-2316162845853183635708 49&5027304085859624110253 06&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-6491595864302031736926 32&7707848679670711009456 65&-2152014469737999315531 272&\dots\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&2777479839347976908352 05&-2862573987306137296638 4&-6387321794083584799900 97&\dots\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&1465118299011954436717 90&-844784290445878224678 23&3659802286906301049192 96&\dots\\
0&0&0&0&0&0&0&0&0&0&0&-1&1&1&2508991467754066459367 61&5756150300112560313950 07&-1148300124261040782472 91&\dots\\
0&1&0&0&0&0&0&1&0&0&-1&0&0&0&1043354311299086458251 33&-2316162845853183635708 49&5027304085859624110253 07&\dots\\
0&0&0&0&0&0&-1&0&0&0&0&0&1&-1&-1406449504434545869194 39&-3930582062123501406142 35&4299330808339302082915 57&\dots\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&5949330706001409509615 61&2731561038203141265890 96&-6718459918484982233168 74&\dots\\
0&0&0&0&1&0&0&-1&0&0&0&0&0&0&3371677202526783102581 77&-2241101519734039462214 21&7431169559364872799105 52&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-8243171548389966819846 21&1771197631974658877549 38&-2367923009246437407024 32&\dots\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0&3793441190239651081048 33&-7697229643267340511839 5&6063667760411549738045 41&\dots\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&5527566718288541531149 05&1260182482795835854860 70&-5350958119531659172108 64&\dots\\
0&0&0&0&0&0&1&0&0&0&0&0&0&-1&-1406449504434545869194 40&-3930582062123501406142 34&4299330808339302082915 57&\dots\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1043354311299086458251 33&2316162845853183635708 49&-5027304085859624110253 07&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&-4672856755854743705009 71&-9506231614652569902135 20&-1255629063127217210042 702&\dots\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&-1465118299011954436717 90&844784290445878224678 23&-3659802286906301049192 96&\dots\\
0&0&0&0&0&0&0&0&0&1&0&-1&0&0&-2777479839347976908352 06&2862573987306137296638 4&6387321794083584799900 97&\dots\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&-690252359306778427451 00&4571029143435868632583 66&6606523468775867078488 17&\dots
\end{bmatrix}
$$

What is short enough? (heuristically)

Let B (maybe 10^{1000}) such that we know the periods up to precision $1/B$.

The first vector of the LLL-reduced basis of L that does not come from a genuine integer relation *should* have norm $\sim B^{\frac{2}{22-\operatorname{rank}\Lambda}}$,
by comparing with random and algebraic numbers (Schmidt, 1971).

def *HeuristicIntegerRelationLattice*(22 periods, B)**:**
  compute a LLL-reduced basis $b_1, \ldots, b_{22}$ of $L \subseteq \mathbb{Z}^{24}$
  search for ρ **such that** $\|b_\rho\| \leq 2^{-22}\|b_{\rho+1}\|$ **and** $B^{\frac{2}{22-\rho}} \approx \|b_{\rho+1}\|$
  if there is none**:**
    fail
  else:
    return the **u**-coordinates of $b_1, \ldots, b_\rho$

A triple alternative

Lemma

If the heuristic algorithm succeeds then one of the following holds:

1. The lattice computed is correct.
2. The IR lattice is not generated by elements of norm $\sim \beta^{\frac{2}{22-\rho}}$.

2' The NS group is not generated by curves of degree $\sim \beta^{\frac{2}{22-\rho}}$.

3. There is a rare numerical coincidence.

Many examples on https://mathexp.eu/lairez/quarticdb

I do not know how to deal with 2/2', there are quartic surfaces with NS group minimaly generated by arbitrary large elements (Mori, 1984).

But we can do something about 3.

Outline

Main result

Let $f \in \mathbb{Q}[w, x, y, z]_4$
and let $\alpha_1, \ldots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

A deformation argument

Idea

If $|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}|$ is *small*,
then we can *slightly* pertub f so that $u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0$.

Lemma

There are $C > 0$ and $\epsilon > 0$ such that for any $\gamma \in H_2(X_f, \mathbb{Z})$
if $\left|\int_\gamma \omega_f\right| < \epsilon$ then there is a monomial m and some $t \in \mathbb{C}$
such that $|t| \leq C\left|\int_\gamma \omega_f\right|$ and $\gamma \in \mathrm{NS}(X_{f+tm})$.

- Smale's α-theory
- The period map is a submersion

The Noether-Lefschetz locus

Let U_4 be the set of quartic polynomials defining a smooth surface.
For $\gamma \in H_2(X, \mathbb{Z})$, let $\Delta(\gamma) = (h \cdot \gamma)^2 - 4(\gamma \cdot \gamma)$.
Let $\Delta > 0$ and let $d, g > 0$ such that $\Delta = d^2 - 8g + 8$.

$$\begin{aligned} \mathcal{NL}_\Delta &\stackrel{\text{def}}{=} \left\{ f \in U_4 \,\middle|\, \exists \gamma \in \mathrm{NS}(X_f), \Delta(\gamma) = \Delta \right\} \\ &= \left\{ f \in U_4 \,\middle|\, X_f \text{ contains a curve of degree } d \text{ and genus } g \right\}. \end{aligned}$$

For example:

- $\mathcal{NL}_9$ is the set of smooth quartic surfaces containing a line;
- $\mathcal{NL}_{12}$ is for conic curves;
- $\mathcal{NL}_{17}$ is for twisted cubic;
- $\mathcal{NL}_9$ is for planar cubic curves

Theorem (???)

For any $\Delta > 0$, the Noether–Lefschetz locus $\mathcal{NL}_\Delta$ is either empty or an algebraic hypersurface in U_4

$\mathcal{NL}_9$ (lines)

$$\begin{aligned}\mathcal{NL}_9 &= \{f \in U_4 \mid V(f) \text{ contains a line}\} \\ &= \mathrm{pr}_1 \operatorname{clo} \left\{(f, u, v) \in U_4 \times (\mathbb{C}^4)^2 \,\middle|\, u \wedge v \neq 0 \text{ and } \forall t, f(u + tv) = 0\right\}\end{aligned}$$

- This is an algebraic hypersurface in U_4 of degree 320
- Don’t try to compute it at home, we may expect up to 30542211864974685251184629048475023725040838973 5 monomials.

$\mathcal{NL}_{12}$ (conic curves)

$$
\begin{aligned}
\mathcal{NL}_{12} &= \{f \in U_4 \mid V(f) \text{ contains a conic curve}\} \\
&= \operatorname{pr}_1 \operatorname{clo} \Big\{ (f, u, v, w) \in U_4 \times (\mathbb{C}^4)^3 \;\Big|\; \\
&\qquad\qquad u \wedge v \wedge w \neq 0 \text{ and } \forall t, f(u + tv + t^2 w) = 0 \Big\}
\end{aligned}
$$

- This is an algebraic hypersurface in U_4 of degree at most 5016

$\mathcal{NL}_{17}$ (twisted cubic)

$$\begin{aligned}\mathcal{NL}_{17} &= \{f \in U_4 \mid V(f) \text{ contains a twisted cubic}\} \\ &= \mathrm{pr}_1 \operatorname{clo} \Big\{(f, u_0, \dots, u_3) \in U_4 \times (\mathbb{C}^4)^4 \;\Big| \\ &\qquad u_0 \wedge \cdots \wedge u_3 \neq 0 \text{ and } \forall t, f\left(\sum_{i=0}^{3} u_i t^i\right) = 0\Big\}\end{aligned}$$

- This is an algebraic hypersurface in U_4 of degree at most 136512

$\mathcal{NL}_{16}$ (intersection of two quadric surfaces)

$$\begin{aligned}\mathcal{NL}_{16} &= \{f \in U_4 \mid V(f) \text{ contains a degree 4 genus 1 curve}\} \\ &= \mathrm{pr}_1\, \mathrm{clo}\Big\{(f, g_1, g_2, h_1, h_2) \in U_4 \times (R_2)^4 \;\Big|\; \\ &\qquad\qquad g_1 \text{ and } g_2 \text{ generic and } f = h_1 g_1 + h_2 g_2\Big\}\end{aligned}$$

- This is an algebraic hypersurface in U_4 of degree at most 76950

$\mathcal{NL}_\Delta$ (general case)

$$\mathcal{NL}_{d^2-8g+8} = \{f \in U_4 \mid V(f) \text{ contains a degree } d \text{ genus } g \text{ curve}\}$$
$$= \mathrm{pr}_1 \operatorname{clo} \Big\{(f, I) \in U_4 \times (\text{ideals of } R) \ \Big| \ \mathrm{Hilb}_I(t) = dt + 1 - g,\ V(I) \text{ reduced and equidim. and } f \in I\Big\}$$

Theorem

There is some absolute constant $A > 0$ such that $\mathcal{NL}_\Delta$ is defined by an integer polynomial NL_Δ such that

$$\deg(\mathrm{NL}_\Delta) \leq A \uparrow \Delta \uparrow \frac{9}{2} \text{ and } \| \mathrm{NL}_\Delta \|_1 \leq 2 \uparrow A \uparrow \Delta \uparrow \frac{9}{2}.$$

- Equations for the Hilbert scheme (Gotzmann, 1978)
- The extended Chow ring, to compute with height and degree of multiprojective varieties (D'Andrea, Krick, & Sombra, 2013)

Exact formula for the degree of $\mathcal{NL}_\Delta$

Theorem (Maulik & Pandharipande, 2013)

Let $A = \sum_{n\in\mathbb{Z}} q^{n^2}$, $B = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$, $\Psi = 108 \sum_{n>0} q^{8n^2}$ and

$$\begin{aligned}\Theta = 2^{-22}\big(&-81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 - 20007A^{16}B^5 - 169092A^{15}B^6 \\ &+ 3A^{21} - 120636A^{14}B^7 - 621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ &- 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} - 361908A^7B^{14} \\ &- 56364A^6B^{15} - 60021A^5B^{16} - 4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}\big) \\ &= -1 + \mathbf{320}\, q^9 + \mathbf{5016}\, q^{12} + \mathbf{76950}\, q^{16} + \mathbf{136512}\, q^{17} + \cdots\end{aligned}$$

Then $\deg \mathrm{NL}_\Delta \leq [q^\Delta](\Theta - \Psi) = O\left(\Delta^{\frac{19}{2}}\right)$.

(There is equality with an appropriate choice of multiplicity for the components of the NL locus.)

Proof of the separation bound

Let $f \in \mathbb{Z}[w,x,y,z]_4$ smooth.
Let $\gamma \in H_2(X,\mathbb{Z})$ such that $\left|\int_\gamma \omega_f\right| \ll 1$.
There is some monomial m such that $\int_\gamma \omega_{f+tm} = 0$, with $|t| \leq C_f \left|\int_\gamma \omega_f\right|$.
The polynomial $\mathrm{NL}_{\Delta(\gamma)}(f+Tm) \in \mathbb{Z}[T]$ has a zero at $T = t$.

So t cannot be *too* small
(at least the inverse of the height of $\mathrm{NL}_{\Delta(\gamma)}(f+Tm)$).

(In case $\mathrm{NL}_{\Delta(\gamma)}(f+Tm)$ vanishes identically, consider a derivative of $\mathrm{NL}_{\Delta(\gamma)}$.)

A supra-Liouville number

Theorem

For any $f \in \mathbb{Q}[w,x,y,z]_4$ *smooth, and any* $\gamma, \eta \in H_2(X_f, \mathbb{Z})$,

$$\sum_{n\geq 0} (2 \uparrow\uparrow 3n)^{-1} \neq \frac{\int_\gamma \omega_f}{\int_\eta \omega_f}.$$

(2 ⇈ b is a tower of exponentiation with b twos.)

Thank you!

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