Periods in action

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What is a period?

A **period** is the integral on a closed path of a rational function in one or several variables with *rational* coefficients.

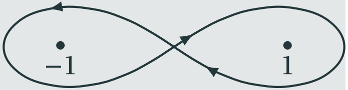
"Rational coefficients" may mean

- coefficients in $\mathbb{Q}$
- coefficients in $\mathbb{C}(t)$, **the period is a function of** t.

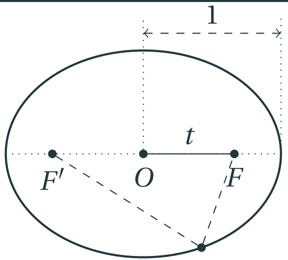
  This is what we will be interested in.

Etymology

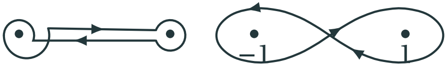
- 2π is a **period** of the sine.
- $\arcsin(z) = \int_0^z \frac{\mathrm{d}x}{\sqrt{1-x^2}}$
- $2\pi = \oint_\infty \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{\pi i} \oint \frac{\mathrm{d}x\mathrm{d}y}{y^2-(1-x^2)}$



An ellipse	
eccentricity	t
major radius	1
perimeter	$E(t)$



$$E(t) = \oint_\infty \sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x$$



Euler (1733) $(t-t^3)E'' + (1-t^2)E' + tE = 0$

Liouville (1834) Not expressible in terms of elementary functions

since then Many applications in algebraic geometry (Gauß-Manin connection)

geometry of the cycles ↔ analytic properties of the periods

Content

1. Computing periods
2. Multiple binomial sums
3. Volume of semialgebraic sets

Computing periods

Representation of algebraic numbers

explicit $\sqrt{5+2\sqrt{6}}$ (also $\sqrt{2}+\sqrt{3}$)

implicit $x^4-10x^2+1=0$ (+ root location)

Representation of D-finite functions

An example by Bostan, Chyzak, van Hoeij, and Pech (2011)

explicit $$1+6\cdot\int_0^t \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\,\frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\,\mathrm{d}w$$

implicit $$t(t-1)(64t-1)(3t-2)(6t+1)y''' + (4608t^4-6372t^3+813t^2+514t-4)y''$$
$$+4(576t^3-801t^2-108t+74)y' = 0$$ (+ init. cond.)

What can we compute?

- addition, multiplication, composition with algebraic functions
- power series expansion
- **equality testing**, given differential equations and initial condtions
- **numerical analytic continuation** with certified precision

  (D. V. Chudnovsky and G. V. Chudnovsky 1990; van der Hoeven 1999; Mezzarobba 2010)

```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
                  [0.7853981633974483 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1], path=[0,i+1,2*i,i-1,0,1])
        [3.926990816987242 +/- 4.81e-17] + [+/- 4.63e-21]*I
```

$R(t, x_1, \ldots, x_n)$ a rational function

$\gamma \subset \mathbb{C}^n$ a n-cycle (n-dim. compact submanifold) which avoids the poles of R, for $t \in U \subset \mathbb{C}$

define $y(t) \triangleq \oint_\gamma R(t, x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$, for $t \in U$

wanted a differential equation $a_r(t) y^{(r)} + \cdots + a_1(t) y' + a_0(t) y = 0$, with polynomial coefficients

One equation fits all cycles, the **Picard-Fuchs equation**.

recall $E(t) = \oint \sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x = \frac{1}{2\pi i}\oint \overbrace{\frac{1}{1-\frac{1-t^2x^2}{(1-x^2)y^2}}}^{R(t,x,y)}\mathrm{d}x\mathrm{d}y$

Picard-Fuchs equation $(t-t^3)E'' + (1-t^2)E' + tE = 0$

Computational proof

$$(t-t^3)\frac{\partial^2 R}{\partial t^2} + (1-t^2)\frac{\partial R}{\partial t} + tR = \frac{\partial}{\partial x}\left(-\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2}\right) + \frac{\partial}{\partial y}\left(\frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2}\right)$$

given $R(t, x_1, \dots, x_n)$, a rational function

find $a_0, \dots, a_r \in \mathbb{Q}[t]$, with $a_r \neq 0$ and r minimal
$C_1, \dots, C_n \in \mathbb{Q}(t, x_1, \dots, x_n)$ with $\operatorname{poles}(C_i) \subseteq \operatorname{poles}(R)$,

such that $$a_r(t)\frac{\partial^r R}{\partial t^r} + \dots + a_1(t)\frac{\partial R}{\partial t} + a_0(t)R = \sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i}.$$

existence Grothendieck (1966), Monsky (1972), etc.
see also Picard (1902) for $n \leqslant 3$

algorithms Almkvist, Apagodu, Bostan, Chen, Christol, Chyzak, van Hoeij, Kauers, Koutschan, Lairez, Lipshitz, Movasati, Nakayama, Nishiyama, Oaku, Salvy, Singer, Takayama, Wilf, Zeilberger, etc.
(People who wrote a paper that solves the problem.)

Problem (mostly) solved!

Multiple binomial sums

joint work with Alin Bostan and Bruno Salvy

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3}$$ (Dixon)

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ (Strehl)

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$

$$\sum_{r\geqslant 0}\sum_{s\geqslant 0} (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n} = \sum_{k\geqslant 0}\binom{n}{k}^4$$

$$\sum_i \sum_j \binom{2n}{n+i}\binom{2n}{n+j}\left|i^3 j^3 (i^2 - j^2)\right| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-3)(2n-1)}\binom{2n}{n}^2$$

Conjectured by Brent, Ohtsuka, Osborn, and Prodinger (2014)

$$1 + F_n^{-1,-1} + 2F_n^{0,0} - F_n^{0,1} + F_n^{1,0} - 3F_n^{1,1} + F_n^{1,2} - F_n^{3,1} + 3F_n^{3,2}$$
$$- F_n^{3,3} - 2F_n^{4,2} + F_n^{4,3} - F_n^{5,2} = \sum_{m=0}^{n} \frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}},$$

where $F_n^{a,b} = \sum_{d=0}^{n-1}\sum_{c=0}^{d-a} \binom{d-a-c}{c}\binom{n}{d-a-c}\left(\binom{n+d+1-2a-2c+2b}{n-a-c+b} - \binom{n+d+1-2a-2c+2b}{n+1-a-c+b}\right)$

Conjectured by Le Borgne

Both proved using periods!

The not so formal grammar of binomial sums

$\bigcirc \to$ integer linear combination of the variables

$$\square \to \binom{\bigcirc}{\bigcirc} \qquad \square \to \mathrm{Cst}^{\bigcirc}$$

$$\square \to \square + \square \qquad \square \to \square \cdot \square \qquad \square \to \sum_{n=\bigcirc}^{\bigcirc} \square$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = ?(-1)^n \frac{(3n)!}{n!^3}$$

basic block $\binom{n}{k} = \frac{1}{2\pi i} \oint \frac{(1+x)^n}{x^k} \frac{dx}{x}$

product $\binom{2n}{k}^3 = \frac{1}{(2\pi i)^3} \oint \frac{(1+x_1)^{2n}}{x_1^k} \frac{(1+x_2)^{2n}}{x_2^k} \frac{(1+x_3)^{2n}}{x_3^k} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$

summation $y(t) = \frac{1}{(2i\pi)^3} \oint \frac{\left(x_1 x_2 x_3 - t \prod_{i=1}^3 (1+x_i)^2\right) dx_1 dx_2 dx_3}{\left(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2\right) \left(1 - t \prod_{i=1}^3 (1+x_i)^2\right)}$

where $y(t)$ is the generating function of the l.h.s.

simplification $y(t) = \frac{1}{(2i\pi)^2} \oint \frac{x_1 x_2 dx_1 dx_2}{x_1^2 x_2^2 - t(1+x_1)^2(1+x_2)^2(1-x_1x_2)^2}$

integration $t(27t+1)y'' + (54t+1)y' + 6y = 0$, i.e. $3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0$

conclusion *Generating functions of binomial sums are periods!*

Computing binomial sums with periods

- Many related works on multiple sums (Chyzak, Egorychev, Garoufalidis, Koutschan, Sun, Wegshaider, Wilf, Zeilberger, etc)
- Subtelties in the translation *recurrence operators* → *actual sequences*, not handled algorithmically

Theorem + Algorithm (Bostan, Lairez, and Salvy 2016)

One can decide the equality between binomal sums.

- ~~"This approach, while it is explicit in principle, in fact yields an infeasible algorithm."~~
  —Wilf and Zeilberger, 1992
- Excellent running times, thanks to **simplification** and better algorithms for **integration**

Binomial sums are diagonals of rational functions

Theorem (Bostan, Lairez, and Salvy 2016)

$(u_n)_{n\geqslant 0}$ is a binomial sum **if and only if** $u_n = a_{n,\dots,n}$, for some rational power series $\sum_I a_I \mathbf{x}^I$.

$$\textbf{NB.}\quad \operatorname{diag} R(x_1,\dots,x_n) = \frac{1}{(2\pi i)^{n-1}} \oint R\left(\frac{t}{x_2\cdots x_n}, x_2,\dots,x_n\right) \frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}.$$

Corollaries of Furstenberg's theorems (Furstenberg 1967)

- If $\sum_n u_n t^n$ is algebraic, then $(u_n)_{n\geqslant 0}$ is a binomial sum.

  The converse does not hold, but...

- If $(u_n)_{n\geqslant 0}$ is a binomial sum, then $\sum_n u_n t^n$ is algebraic modulo p for all prime p (but finitely many).

$$y(t) \triangleq \sum_n \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 t^n$$

$$= \operatorname{diag} \frac{1}{(1-x-y)(1-z-w)-wxyz}$$ (Straub 2014)

$y(t)$ is transcendental,

however $y(t) \equiv (t^2-1)^{-\frac{1}{2}} \mod 5$

$y(t) \equiv \left((t-1)(t^2-1)\right)^{-\frac{1}{3}} \mod 7$

$y(t) \equiv ((t+1)(t+5)(t+7)(t+8)(t+9))^{-\frac{1}{5}} \mod 11$

$y(t) \equiv \left((t^2+8t+1)(t^2+6t+1)(t-1)\right)^{-\frac{1}{6}} (t^2+5t+1)^{-\frac{1}{12}} \mod 13$

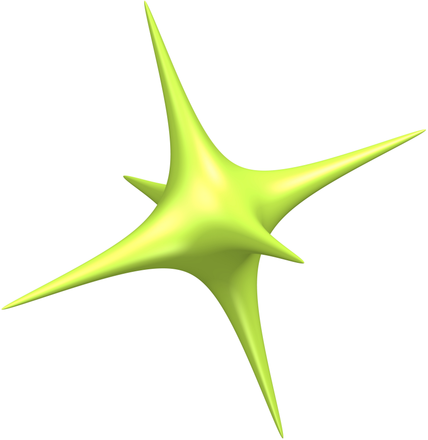
and of course $t^2(t^2-34t+1)y''' + 3t(2t^2-51t+1)y'' + (7t^2-112t+1)y' + (t-5)y = 0.$

Volume of semialgebraic sets

joint work with Mohab Safey El Din

A numeric integral

$$\{x^2 + y^2 + z^2 \leqslant 1 - 2^{10}\left(x^2y^2 + y^2z^2 + z^2x^2\right)\}$$



What is the volume of this shape?

- Basic question
- Few algorihms
  - Monte-Carlo
  - Henrion, Lasserre, and Savorgnan (2009)
- Exponential complexity with respect to precision
- No certification on precision

Volumes are periods

Proposition

For any generic $f \in \mathbb{R}[x_1, \dots, x_n]$,

$$\operatorname{vol}\{f \leqslant 0\} \triangleq \int_{\{f \leqslant 0\}} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{2\pi i} \oint_{\mathrm{Tube}\{f=0\}} \frac{x_1}{f} \frac{\partial f}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

proof Stokes formula + Leray tube map

not so useful There is no parameter.

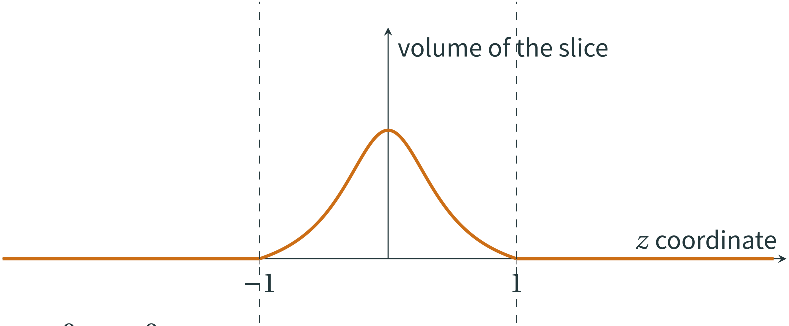
better say For a generic t,

$$\operatorname{vol}\{f \leqslant 0\} \cap \{x_n = t\} = \underbrace{\frac{1}{2\pi i} \oint \frac{x_1}{f|_{x_n=t}} \frac{\partial f|_{x_n=t}}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}}_{\text{satisfies a Picard-Fuchs equation!}}$$

NB. $\operatorname{vol}\{f \leqslant 0\} = \int_{-\infty}^{\infty} \operatorname{vol}\{f \leqslant 0\} \cap \{x_n = t\} \mathrm{d}t$

The "volume of a slice" function

$\{y_1, y_2\}$, basis of the solution space of the Picard-Fuchs equation



$$0 \cdot y_1 + 0 \cdot y_2$$

$$1.0792353\ldots \cdot y_1 - 40.100605\ldots \cdot y_2$$

$$0 \cdot y_1 + 0 \cdot y_2$$

An algorithm for computing volumes

input $f \in \mathbb{R}[x_1, \ldots, x_n]$ generic

symbolic integration Compute a differential equation for $y(t) \triangleq \operatorname{vol}\{f \leqslant 0\} \cap \{x_n = t\}$.

bifurcations Spot singular points where $y(t)$ may not be analytic.

numerical integration On each maximal interval $I \subset \mathbb{R}$ where $y(t)$ is analytic,

- identify $y|_I$ in the solution space of the PF equation,
- compute $\int_I y(t)$.

return $\operatorname{vol}\{f \leqslant 0\} = \sum_I \int_I y(t)$.

The complexity is quasi-linear with respect to the precision!

(To get twice as many digits, you need only twice as much time.)

A hundred digits (within a minute)

vol () $= 0.10857542146036093773950339599420761981091787444660747544447582299328536067303292819494347441406406613662423462795980877810349323467815 68\ldots$

Questions?

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