Algebraic consequences of the fundamental theorem of calculus in differential rings

Georg Regensburger joint work with Clemens G. Raab

U N I K A S S E L V E R S I T A T



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Algebraic consequences of the Leibniz rule and

$$\frac{d}{dx} \int_{a}^{x} f(t) dt = f(x) \quad \text{and} \quad \int_{a}^{x} f'(t) dt = f(x) - f(a)$$

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 instead of $\partial(f)$

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Definition

Let (\mathcal{R}, ∂) be a differential ring with constants \mathcal{C} . We call a \mathcal{C} -linear map $f: \mathcal{R} \to \mathcal{R}$ an **integration** on \mathcal{R} , if

$$\partial \int f = f$$

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Let (\mathcal{R}, ∂) be a differential ring with constants C. We call a C-linear map $\int : \mathcal{R} \to \mathcal{R}$ an **integration** on \mathcal{R} , if

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holds for all $f \in \mathcal{R}$.

A C-linear functional

$$e: \mathcal{R} \to \mathcal{C}$$

acting on C as the identity is called an **evaluation** on \mathcal{R} .

Integro-differential rings

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Let (\mathcal{R}, ∂) be a differential ring and let $\int : \mathcal{R} \to \mathcal{R}$ be an integration on \mathcal{R} .

We call $(\mathcal{R},\partial,\int)$ a (generalized) integro-differential ring and we define the (induced) evaluation E on \mathcal{R} by

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Lemma

Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring with constants C. Then,

$$Ef \in C$$
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for all $f \in \mathcal{R}$ and $c \in \mathcal{C}$. Moreover,

$$\mathcal{R} = C \oplus \int \mathcal{R}$$

as direct sum of C-modules.

(Multiplicative) integro-differential rings and algebras

multiplicative evaluation

$$Efg = (Ef)Eg$$

(Rosenkranz '03 '05, Rosenkranz-R '08, Guo-R-Rosenkranz '14, Hossein Poor-Raab-R '18)

$$\int_{a}^{x} f(t) dt$$
 and evaluation $\mathbf{E}f = f(a)$ of **continuous** functions

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Ex: (Matrices of) polynomials, smooth/analytic functions, formal power series

Motivation and application:

algebraic setting for **boundary value problems** for linear ODEs

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Differential Rota-Baxter algebras

$$(\int f) \int g = \int f \int g + \int (\int f)g$$

(Guo-Keigher '08)

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Laurent series:

$$K((x))[\ln(x)]$$

contain rational functions K(x) and hyperlogarithms

D-finite functions

D-finite functions are closed under antiderivatives

(Abramov-van Hoeij '97)

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Define integration in terms of an evaluation and antiderivates

Lemma

Let (\mathcal{R}, ∂) be a differential ring such that $\partial \mathcal{R} = \mathcal{R}$ and e be an evaluation on \mathcal{R} . Define $\int_e : \mathcal{R} \to \mathcal{R}$ by

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where $g \in \mathcal{R}$ is such that $\partial g = f$.

Then $(\mathcal{R}, \partial, \int_{e})$ is an integro-differential ring with induced evaluation E = e.

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Define an evaluation in terms of formal series solutions

(van Hoeij '97)

Linear operators

- ullet differential operator ∂
- integral ∫
- evaluation $E = id \int \partial$
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R commutative: all operators are C-linear

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Linear operators with composition form a ring

C commutative: a C-algebra C noncommutative: a C-ring

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Let $(\mathcal{R}, \partial, f)$ be an integro-differential ring with constants C.

We define the **ring of integro-differential operators** (IDO)

$$\mathcal{R}\langle\partial,\int,\mathsf{E}\rangle$$

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$$\partial \cdot f = f \cdot \partial + \partial f, \quad \partial \cdot \int = 1, \quad \int \cdot \partial = 1 - E$$

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Identities as rules

$$\begin{array}{ccc}
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\swarrow & \searrow \\
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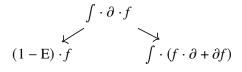
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$$\begin{array}{ccc}
\partial \cdot \int \cdot \partial \\
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$$\int \cdot \partial = 1 - E$$
 and $\partial \cdot f = f \cdot \partial + \partial f$

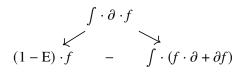
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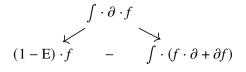
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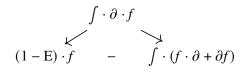
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$$\int \cdot f \cdot \partial = f - \mathbf{E} \cdot f - \int \cdot \partial f$$

A known consequence

$$\int \cdot \partial = 1 - \mathbf{E} \quad \text{and} \quad \partial \cdot f = f \cdot \partial + \partial f$$

Ambiguity



S-polynomial

$$\int \cdot f \cdot \partial = f - \mathbf{E} \cdot f - \int \cdot \partial f$$

integration by parts holds in ${\cal R}$

$$\int f \partial g = fg - Efg - \int (\partial f)g$$

A new consequence

From

$$\int \cdot f \cdot \partial \cdot \int$$

integration by parts and $\partial \cdot \int$,

we obtain

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Rota-Baxter identity with evaluation in ${\mathcal R}$

$$(\int f) \int g = \int f \int g + \int (\int f)g + \mathbf{E}(\int f) \int g$$

products of integrals = nested integrals plus evaluation

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E is multiplicative: Rota-Baxter identity (shuffle identities)

$$(\int f) \int g = \int f \int g + \int (\int f)g$$

All consequences

Table: Rewrite rules for operator expressions

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Theorem

Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring. Then, by repeatedly applying the rewrite rules above in any order, every element of the ring $\mathcal{R}\langle\partial, \int, E\rangle$ can be written as a sum of expressions of the form

$$f \cdot \partial^j, \quad f \cdot \int \cdot g, \quad f \cdot \mathbf{E} \cdot g \cdot \partial^j, \quad \text{or} \quad f \cdot \mathbf{E} \cdot h \cdot \int \cdot g$$

where $j \in \mathbb{N}_0$, $f, g \in \mathcal{R}$, and $h \in \int \mathcal{R}$.

Constructive and algorithmic approach to rings of linear operators via **tensor reduction systems** for tensor algebras and rings

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Mathematica package TenRes

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$$x'(t) + A(t)x(t) = f(t)$$
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Rewrite rules and normal forms: equational prover in calculus, discover identities by ansatz, basics of linear ODEs with initial conditions

Generalized Taylor formula

Integro-differential subring generated by $\boldsymbol{1}$ and constants contains "monomials"

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Let $(\mathcal{R}, \partial, \int)$ be an integro-differential ring such that E is multiplicative on the integro-differential subring generated by 1. Then, for all $n \in \mathbb{N}$, we have

$$1 = \sum_{k=0}^{n} x_k \cdot \mathbf{E} \cdot \partial^k + \sum_{k=0}^{n} (-1)^{n-k} x_k \cdot \int \cdot x_{n-k} \cdot \partial^{n+1}$$
$$- \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^{n-k-j} x_k \cdot \mathbf{E} \cdot x_j \cdot \int \cdot x_{n-k-j} \cdot \partial^{n+1}$$

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$$\mathbb{Q} \subseteq \mathcal{R}$$
:

$$f = \sum_{k=0}^{n} \frac{x_1^k}{k!} E \partial^k f + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} x_1^k \int x_1^{n-k} \partial^{n+1} - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \frac{(-1)^{n-k-j}}{k! j!(n-k-j)!} x_1^k E x_1^j \int x_1^{n-k-j} \partial^{n+1} f x_1^{n-k-j} \partial^{n+1$$

Generalized shuffle relations

 ${\cal R}$ commutative integro-differential ring

$$C\langle\mathcal{R}\rangle=igoplus_{n=0}^\infty\mathcal{R}^{\otimes n}$$
 with **shuffle product** \sqcup , homomorphism

$$\varphi(a_1 \otimes \ldots \otimes a_n) = \int a_1 \int a_2 \ldots \int a_n \in \mathcal{R}$$
 and $a_i^j = a_i \otimes a_{i+1} \otimes \ldots \otimes a_j$

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Theorem

Let $(\mathcal{R}, \partial, \int)$ be a commutative integro-differential ring with constants C. Let $f, g \in C\langle \mathcal{R} \rangle$ be pure tensors of length m and n. Then, the product of $\varphi(f) = \int f_1 \int f_2 \dots \int f_m$ and $\varphi(g) = \int g_1 \int g_2 \dots \int g_n$ is given by

$$\varphi(f)\varphi(g) = \varphi(f \sqcup g) + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} e(f_{i+1}^m, g_{j+1}^n) \varphi(f_1^i \sqcup g_1^j) \in \mathcal{R}$$

with constants $e(f_{i+1}^m, g_{i+1}^n) = \mathbb{E}\varphi(f_{i+1}^m)\varphi(g_{i+1}^n) \in C$.

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E multiplicative:

$$(\int f_1 \int f_2 \cdots \int f_m) (\int g_1 \int g_2 \cdots \int g_n) = \int f_1 \int f_2 \cdots \int f_m \sqcup \int g_1 \int g_2 \cdots \int g_n$$

Outlook

- Integro-differential rings over integral domains
- Tensor reduction systems
- Other operator rings (with linear substitutions, discrete analogs, . . .)
- Free integro-differential rings (integro-differential polynomials)

Clemens G. Raab, R., The fundamental theorem of calculus in differential rings. arXiv:2301.13134 [math.RA] (2023)

Clemens G. Raab, R., The free commutative generalized integro-differential ring. (2023) In preparation.