Symbolic Manipulation, Summation, and Integration of Special Functions and Combinatorial Sequences

The Holonomic Approach

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Symbolic Manipulation, Summation, and Integration of Special Functions and Combinatorial Sequences

1. about works initiated by Zeilberger in the 90's
2. from combinatorics and combinatorial interests to computer algebra manipulations and algorithms
3. imports theory of holonomy from $\mathcal{D}$-module theory in pure/traditional maths

Outline

0. Motivation;
1. Implicit Representations of Special Functions and Sequences;
2. Gröbner Bases in Algebras of Linear Operators;
3. Holonomy;
4. Holonomic Summation and Integration.

Part 0

Motivation

Motivation

1. Motivation of the whole series through examples of algebraic manipulation of ∂-finite functions.

Automatic Calculation of Identities

$$\sum_{k=1}^{n}\binom{k}{m}H_k=\binom{n+1}{m+1}\left(H_{n+1}-\frac{1}{m+1}\right),\qquad J_0(z)^2+2\sum_{k=1}^{\infty}J_k(z)^2=1,$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3,$$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=n2^{3n-1}+2^{3n}-3n2^{n-2}\binom{2n}{n},$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-(r+s)}{n}=\sum_{k=0}^{\infty}\binom{n}{k}^4,$$

$$\sum_{n=0}^{+\infty}P_n(x)y^n=\frac{1}{\sqrt{1-2xy+y^2}},\qquad\sum_{n=0}^{\infty}H_n(x)H_n(y)\frac{u^n}{n!}=\frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-u^2}},$$

$$\frac{\pi}{2}J_0(z)=\int_0^1\frac{\cos(zu)}{\sqrt{1-u^2}}\,du=\int_1^{+\infty}\frac{\sin(zu)}{\sqrt{u^2-1}}\,du,\qquad\int_{-1}^{+1}\frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx=(-1)^n\pi I_n(p),$$

$$\int_0^{+\infty}xe^{-px^2}J_n(bx)I_n(cx)\,dx=\frac{1}{2p}\exp\left(\frac{c^2-b^2}{4p}\right)J_n\left(\frac{bc}{2p}\right),$$

$$\int_0^{+\infty}xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{n!H_n(x)}{\lfloor n/2\rfloor!},$$

$$\int_0^{\infty}\int_0^{\infty}J_1(x)J_1(y)J_2(c\sqrt{xy})\,\frac{dxdy}{e^{x+y}}=\frac{c^2+8}{8\pi c^2}\left((2-k^2)\boldsymbol{K}\left(\frac{4c}{c^2+8}\right)-2\boldsymbol{E}\left(\frac{4c}{c^2+8}\right)\right),$$

$$\sum_{-\infty}^{+\infty}\frac{(a;q)_n\,x^n}{(b;q)_n}=\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty(b;q)_\infty\left(\frac{q}{a};q\right)_\infty},$$

$$\sum_{n=0}^{\infty}\frac{r_n(a,b;q)z^n}{(q;q)_n}=\frac{1}{(az;q)_\infty(bz;q)_\infty},$$

$$\sum_{n=0}^{\infty}\frac{r_n(a,b;q)r_n(c,d;q)z^n}{(q;q)_n}=\frac{(acz;q)_\infty(adz;q)_\infty(bcz;q)_\infty(bdz;q)_\infty}{(abcdz^2;q)_\infty},$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

Based on an effective theory of holonomy.

Automatic Calculation of Identities

1. Holonomy
   - large variety of objects, computable class
   - goal achieved during the 90's: cover the class by theory and the corresponding programs
2. Decidability of the identities
   - simplifications of expressions in terms of special functions by computer algebra
   - parameterized integrations and summations
3. Sources for holonomic problems
   - combinatorics (problems of recursive nature and with ordering constraints)
   - analysis of algorithms
   - physical problems with cylindrical/spherical symmetry
   - statistical physics, q-calculus, divide and conquer
4. Thanks to implementation
   - each identity in a matter of minutes of calculation
   - recover recent research results

Holonomy for Special Functions and Sequences in Computer Algebra

Holonomic functions: derivatives live in a finite dimensional vector space.
Holonomic sequences: Taylor coefficients of holonomic functions.
$\Longrightarrow$ cancelled by linear PDE's and recurrences.

Class of ubiquitous functions and sequences:

Algebraic functions, exp, ln, sin, cos, sinh, cosh, arcsin, arccos, arcsinh, arccosh; $n!$, $\binom{n}{m}$, $\binom{n}{m_1,\ldots,m_r}$, H_n, $H_n^{(p)}$, hypergeometric sequences; erf, erfc, Ci, Si; P_ν^μ, J_ν, Y_ν, I_ν, K_ν, $\boldsymbol{H}_\nu$, Ki; ${}_2F_1(a,b,c;z)$, ${}_pF_q((a_i),(b_i);z)$, $F_4(a,b,c,d;x,y)$, $M(a,b,z)$, $W_{\kappa,\mu}(z)$; $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\alpha)}(x)$, $T_n(x)$, $U_n(x)$, $P_n(x)$, $L_n^{(\alpha)}(x)$, $L_n(x)$, $H_n(x)$, $r_n(q;a,b)$.

Algorithms to compute with **systems** of linear functional equations:

$+$, $\times$, ∂ ... ; simplification; definite and indefinite $\int$ and $\sum$.

New type of computation with implicit forms.

Unified theoretic framework, unified algorithms, unified implementation.

First full implementation (MGFUN in MAPLE).

Holonomy for Special Functions and Sequences in Computer Algebra

1. Holonomy = finitely specifiable $\Rightarrow$ computer algebra
   [early informal definition]
   - description by
     - finite number of linear equations
     - finite numer of initial conditions
   - computations in finite dimensional vector spaces
2. Ubiquitous nature
   - 70% of sequences and functions from the handbook by Abramovitz and Stegun
   - half from Erdélyi's *Bateman Project*
   - 1/4 of sequences from the *Encyclopedia of Integer Sequences* by Sloane and Plouffe
3. Computation without closed forms (at least during intermediate calculations)
   - explicit forms (= closed f.)/implicits ones (systems)
   - CAS: failure of the methods that restrict to closed forms
   - my thesis: general method to compute even in the absence of closed forms
   - representation by systems
4. Unifying viewpoint
   - differential + recurrences
   - use of polynomial elimination
   - full implementation

Two Typical Basic Problems

PROBLEM: "Simplify"

$$s_n = F_{n+2}F_n - F_{n+1}^2,$$
$$y_{m,n}(x) = 2T_m(x)T_n(x) - T_{n+m}(x) - T_{n-m}(x),$$
$$z_n(a,b) = p_{n+1} - p_{n-1} + \frac{2n}{a}\frac{\partial p_n}{\partial b} + \frac{2n}{b}\frac{\partial p_n}{\partial a},$$

where F_n are the Fibonacci numbers, $T_n(x)$ is the n-th Chebychev orthogonal polynomial and

$$p_n(a,b) = J_n(a)Y_n(b) - J_n(b)Y_n(a)$$

in terms of the Bessel functions of 1st and 2nd kinds and order n.
SOLUTION: $s_n = (-1)^n$, $y_{m,n}(x) = z_n(a,b) = 0$.

PROBLEM: Compute a simple particular solution of the LODE in F

$$F''(x) + \frac{F'(x)}{x} - q^2F(x) = (1-x^2)J_0(qx) - 2q^2xJ_1(qx).$$

SOLUTION:

$$F(x) = \frac{2q-1+x^2}{2q^2}J_0(qx) + \frac{(q^3-1)x}{q^3}J_1(qx).$$

REMARK: Generalizes indefinite integration problem

$$F'(x) = \text{given closed form.}$$

Two Typical Basic Problems

(= side-products of the method)

1. Maple: code for Wronskians in this particular case;
   algorithm for normal forms in a general class;
2. Maple: ugly result that can no longer be manipulated;
   algorithm for searching for particular solutions in a general class.
   Application: study of a liquid flow with molecular diffusion.

Computations of Integrals and Sums

Wave equation: model that describes the vibration of a drum membrane:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx \quad \longrightarrow \quad \pi(-1)^n I_n(p) \qquad (5\ sec).$$

Theory of special functions: Bessel functions integrals [M. L. Glasser and E. Montaldi (1994)]:

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx \quad \longrightarrow \quad -\frac{\ln(1-a^4)}{2\pi a^2} \qquad (75\ sec).$$

$$a^3(a^4-1)\phi^{(4)}(a) + 4a^2(4a^4-1)\phi^{(3)}(a) + a(73a^4+3)\phi''(a) + (103a^4-3)\phi'(a) + 32a^3\phi(a) = 0.$$

Denumeration problems, combinatorial sums: generating functions of a product of Hermite functions:

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} \quad \longrightarrow \quad \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}} \qquad (30\ sec).$$

Probability computation: the expectance of the maximum of three Gaussian independent random variables is the limit of [N. J. Calkin (1994)]:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 \quad \longrightarrow \quad \frac{n}{2} 8^n + 8^n - \frac{3n}{4} 2^n \binom{2n}{n} \qquad (195\ sec).$$

$$(n+1)u_{n+2} - (7n+12)u_{n+1} - 4(2n+1)u_n = (18n-20)8^n.$$

Computations of Integrals and Sums

1. Pre- and postprocessing pass from closed form to implicit form and conversely.
2. Variety of applications (maths, mathematical physics, engineering sciences)
3. Computation of parameterized integrals and sums
   - ≥ 2 variables $\Rightarrow$ 1 linear functional equation $\Rightarrow$ 1 variable
   - my contribution: unified algorithmization and a palet of algorithms to get the linear functional equation
   - when no closed form available: possible following up of the calculations
4. Guaranteed general algorithms
   - recent research results due to specialists, experts $\Rightarrow$ short computer computations
   - first automatic calculations

Automatic Proof of Identities

Theory of special functions:

$$\int_0^\infty \int_0^\infty e^{-x}e^{-y}J_1(x)J_1(y)J_2(c\sqrt{xy})\,dxdy$$
$$= \frac{1}{2\pi ck}((2-k^2)\boldsymbol{K}(k) - 2\boldsymbol{E}(k)) \qquad (50\ sec),$$

where $k = 4c/(c^2+8)$ and $\boldsymbol{K}$ and $\boldsymbol{E}$ are elliptic functions.

Partition theory, hard hexagon models, statistical physics:

[Rogers–Ramanujan] $$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad (80\ sec),$$

[Gordon] $$\sum_{j=0}^{n}\sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad (120\ sec),$$

where $(a;q)_n = (1-a)\cdots(1-aq^{n-1})$.

Number theory: Th. [Apéry (1978)]: $\zeta(3) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \notin \mathbb{Q}$.

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad (60\ sec).$$

$$(n+2)^3 u_{n+2} - \left((n+2)^3 + (n+1)^3 + 4(2n+3)^3\right) u_{n+1} + (n+1)^3 u_n = 0.$$

Automatic Proof of Identites

1. Pre- and postprocessing pass from closed form to implicit form and conversely.
2. Variety of applications (maths, mathematical physics, engineering sciences)
3. Proof of identities
   - absence of closed forms or resolution not possible
   - comparing/matching descriptions ⇒ zero test
4. Guaranteed general algorithms
   - Gordon: first automatic proof
     ⇒ solution in 2 min to a problem suggested by Knuth
   - Apéry: 1 min compared to several months of failure by Cohen + Van der Poorten

The Holonomic Approach

The Mehler formula for Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)u^n}{n!} = \frac{e^{\frac{4u(u(x^2+y^2)-xy)}{4u^2-1}}}{\sqrt{1-4u^2}}.$$

Maple fails to sum it:

```
sum(orthopoly[H](n,x)*orthopoly[H](n,y)*u^n/n!,
    n=0..infinity);
```

$$\sum_{n=0}^{\infty} \frac{orthopoly_H(n,x)orthopoly_H(n,y)u^n}{n!}$$

Load experimental version of Mgfun:

```
with(Mgfun);
```

$$[dfinite_expr_to_sys, diag_of_sys, int_of_sys, pol_to_sys, sum_of_sys, sys * sys, sys + sys]$$

The Holonomic Approach (I)

1. Mehler formula: evaluation of the generating function of a product of the same special function
2. Computer algebra fails on closed (=explicit) forms

Follow the holonomic approach: introduce the summand.

```
h:=HermiteH(n,x)*HermiteH(n,y)*u^n/n!:
sys:=dfinite_expr_to_sys(h,f(n::shift,u::diff)):
collect(sys,{f,diff},factor);
```

$$
\begin{aligned}
[&-n\mathrm{f}(n,u)+u\frac{\partial}{\partial u}\mathrm{f}(n,u),\\
&(n+4)\mathrm{f}(n+4,u)-4xyu\mathrm{f}(n+3,u)\\
&-4\left(2n+5-2x^2-2y^2\right)u^2\mathrm{f}(n+2,u)-16xyu^3\mathrm{f}(n+1,u)\\
&+16(n+1)u^4\mathrm{f}(n,u)]
\end{aligned}
$$

Justification: $H_n = 0$ for negative n, u^n goes to 0 when n goes to ∞.

```
sys:=sum_of_sys(sys,n=0..infinity,takayama_algo):
collect(sys,{f,diff},factor);
```

$$
\begin{aligned}
\{&(2u-1)^2(2u+1)^2(\frac{\partial}{\partial u}\mathrm{f}(u))\\
&+\left(-4xy+8x^2u+8y^2u-4u-16xyu^2+16u^3\right)\mathrm{f}(u)\}
\end{aligned}
$$

Solve differential equation.

```
dsolve({op(sys),f(0)=1},f(u)):
simplify(expand(%),symbolic);
```

$$
\mathrm{f}(u)=-\frac{Ie^{\left(4\frac{u(ux^2+uy^2-xy)}{(2u-1)(2u+1)}\right)}}{\sqrt{4u^2-1}}
$$

The Holonomic Approach (II)

1. Closed form to system: (easy) preprocessing based on a "small" database of atomic functions and on algorithmic closure properties
2. Computation on systems: main topic of this talk
3. Resolution: (generally easy) postprocessing
   - dedicated algorithms to search for hypergeometric function solutions
   - not necessary when less than a result under closed form is wanted

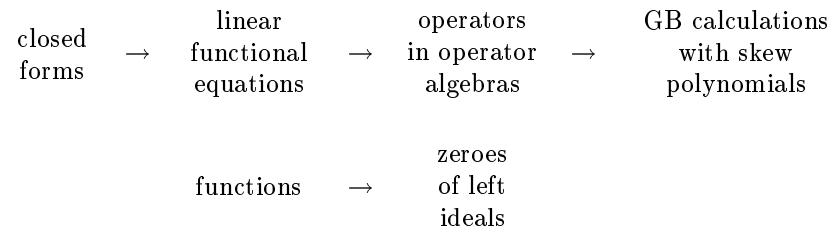
Part I

Implicit Representations of Special Functions and Sequences

Implicit Representations of Special Functions and Sequences

1. Operator algebras; skew polynomial rings; "commutation algebras"; Ore algebras;
2. Annihilating ideals of a function with respect to an Ore algebra;
3. ∂-Finiteness;
4. ∂-Finite closures in a non-algorithmic way.

Summary:



Linear Differential Operators

Functional notation: (Leibniz rule)

$$(xf)'(x) = xf'(x) + f(x),$$
$$(Pf)'(x) = P(x)f'(x) + P'(x)f(x).$$

Operator notation: (commutation rule)

$$D = \text{differentiation operator with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$DX = XD + 1,$$
$$DP = PD + P'.$$

Action on functions:

$$1 \cdot f = f, \qquad D \cdot f = f', \qquad D^2 \cdot f = f'', \qquad \dots$$
$$P(X)D^r \cdot f = P(x)f^{(r)}.$$

Multivariate case: here for $f(x_1, x_2)$,

$$1 \cdot f = f,$$
$$D_1 \cdot f = \frac{\partial f}{\partial x_1}, \qquad D_2 \cdot f = \frac{\partial f}{\partial x_2},$$
$$D_1^2 \cdot f = \frac{\partial^2 f}{\partial x_1^2}, \qquad D_1 D_2 \cdot f = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \qquad D_2^2 \cdot f = \frac{\partial^2 f}{\partial x_2^2},$$
$$\vdots$$
$$P(X_1, X_2)D_1^r D_2^s \cdot f = P(x_1, x_2)\frac{\partial^{r+s} f}{\partial x_1^r \partial x_2^s}.$$

Linear Differential Operators

1. simple other description: $\mathbb{K}\langle X, D; DX = XD + 1\rangle$
2. possible extension
   - rational functions
   - formal power series
   - convergent series
   - Laurent series
   - analytic functions
   - meromorphic functions
   - any differential ring

Linear Difference Operators

Functional notation: (ring morphism)

$$(xf)(x+1) = (x+1)f(x+1),$$
$$(Pf)(x+1) = P(x+1)f(x+1).$$

Operator notation: (commutation rule)

$$S = \text{shift operator with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$SX = (X+1)S,$$
$$SP(X) = P(X+1)S.$$

Action on functions:

$$(1 \cdot f)(x) = f(x), \quad (S \cdot f)(x) = f(x+1), \quad (S^2 \cdot f)(x) = f(x+2), \quad \ldots$$
$$(P(X)S^r \cdot f)(x) = P(x)f(x+r).$$

Multivariate case: here for $f(x_1, x_2)$,

$$(1 \cdot f)(x_1, x_2) = f(x_1, x_2),$$
$$(S_1 \cdot f)(x_1, x_2) = f(x_1+1, x_2), \quad (S_2 \cdot f)(x_1, x_2) = f(x_1, x_2+1),$$
$$(S_1^2 \cdot f)(x_1, x_2) = f(x_1+2, x_2), \quad (S_1S_2 \cdot f)(x_1, x_2) = f(x_1+1, x_2+1),$$
$$(S_2^2 \cdot f)(x_1, x_2) = f(x_1, x_2+2), \quad \ldots$$
$$(P(X_1, X_2)S_1^r S_2^s \cdot f)(x_1, x_2) = P(x_1, x_2)f(x_1+r, x_2+s).$$

Linear Difference Operators

1. difference vs shift
2. simple other description: $\mathbb{K}\langle X, S; SX = XS + S\rangle$
3. possible extension
   - rational functions
   - formal power series
   - Laurent series
   - analytic functions
   - meromorphic functions

Linear q-Difference Operators

Functional notation: (ring morphism)

$$(xf)(qx) = qxf(qx),$$
$$(Pf)(qx) = P(qx)f(qx).$$

Operator notation: (commutation rule)

$$H = \text{operator of } q\text{-dilation with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$HX = qXH,$$
$$HP(X) = P(qX)H.$$

Action on functions:

$$(1 \cdot f)(x) = f(x), \quad (H \cdot f)(x) = f(qx), \quad (H^2 \cdot f)(x) = f(q^2x), \quad \ldots$$
$$(P(X)H^r \cdot f)(x) = P(x)f(q^r x).$$

Multivariate case: here for $f(x_1, x_2)$,

$$(1 \cdot f)(x_1, x_2) = f(x_1, x_2),$$
$$(H_1 \cdot f)(x_1, x_2) = f(qx_1, x_2), \quad (H_2 \cdot f)(x_1, x_2) = f(x_1, qx_2),$$
$$(H_1^2 \cdot f)(x_1, x_2) = f(q^2x_1, x_2), \quad (H_1H_2 \cdot f)(x_1, x_2) = f(qx_1, qx_2),$$
$$(H_2^2 \cdot f)(x_1, x_2) = f(x_1, q^2x_2), \quad \ldots$$
$$(P(X_1, X_2)H_1^r H_2^s \cdot f)(x_1, x_2) = P(x_1, x_2)f(q^r x_1, q^s x_2).$$

Linear q-Difference Operators

1. q-difference vs q-dilation
2. simple other description: $\mathbb{K}\langle X, H; HX = qXH\rangle$
3. possible extension
   - rational functions
   - formal power series
   - Laurent series
   - analytic functions
   - meromorphic functions

Mixed Cases as Well!

The Chebychev polynomials

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$$

satisfy the equations

$$(1-x^2){T_n}''(x) - x{T_n}'(x) + n^2 T_n(x) = 0,$$
$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0,$$
$$(1-x^2){T_{n+1}}'(x) - (n+1)xT_{n+1}(x) - (n+1)T_n(x) = 0.$$

Equivalently, they are cancelled by the following operators:

$$(1-x^2)D_x^2 - xD_x + n^2,$$
$$S_n^2 - 2xS_n + 1,$$
$$(1-x^2)D_x S_n - (n+1)xS_n - (n+1).$$

Mixed differential difference framework!

$$(1 \cdot f)(n,x) = f(n,x),$$
$$(S_n \cdot f)(n,x) = f(n+1,x), \quad (D_x \cdot f)(n,x) = \frac{\partial f}{\partial x}(n,x),$$
$$(S_n^2 \cdot f)(n,x) = f(n+2,x), \quad (S_n D_x \cdot f)(n,x) = \frac{\partial f}{\partial x}(n+1,x),$$
$$(D_x^2 \cdot f)(n,x) = \frac{\partial^2 f}{\partial x^2} f(n,x), \qquad \ldots$$
$$(P(X_1,X_2) S_n^r D_x^s \cdot f)(n,x) = P(n,x) \frac{\partial^s f}{\partial x^s}(n+r,x).$$

Mixed Cases as Well!

1. Similar in the case of any other family of orthogonal polynomials
2. Abandon closed forms
   - replaced by linear operators
   - possibility of mixed framework
3. Polynomial notation
   - in view of calculations by Gröbner bases
   - non-commutativity

Skew Polynomial Rings

General framework of skew polynomial rings $\mathbb{A}[\partial; \sigma, \delta]$:

$$\partial a = a^\sigma \partial + a^\delta \qquad \textit{[Ore]}.$$

$a^\sigma = \sigma(a) =$ conjugate of a; $\qquad a^\delta = \delta(a) = \sigma$-derivative of a.

σ: ring endomorphism of $\mathbb{A}$; $\qquad \delta$: σ-derivation of $\mathbb{A}$.

$$(fg)^\delta = f^\sigma g^\delta + f^\delta g \qquad \text{(skew Leibniz rule)}.$$

Special cases:

1. $\sigma = 1$ and $\delta = 0$: commutative polynomial rings;
2. $\sigma = 1$ and $\delta = '$: usual derivation rings; (usual) Leibniz rule;
3. $\delta = 0$ or $\delta = \sigma - 1$: usual difference rings; ring morphism.

Properties: ($\mathbb{A}$ a field)

- injective $\sigma \Rightarrow$ degree, right euclidean, left gcd;
- bijective $\sigma \Rightarrow$ degree, left and right euclidean, left and right gcd;
- left Ore condition, left fraction field.

Skew Polynomial Rings

1. indeterminate does not commute with coefficients
2. Ore's goals:
   - linear operators
   - fraction field with application to linear algebra
3. coefficients in a general ring of coefficients
4. general commutations
5. great deal of linear operators
6. example of euclidean algorithm: Chebychev polynomials from previous slide

Ore Algebras

Multivariate framework for several indeterminates ∂_i's $\mathbb{A}[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$:

$$\partial_j\partial_i = \partial_i\partial_j, \qquad \sigma_j\delta_i = \delta_i\sigma_j \qquad (i \neq j).$$

derivation	$f(x) \mapsto f'(x)$	$D_x x = xD_x + 1$
shift	$f(x) \mapsto f(x+1)$	$S_x x = (x+1)S_x$
difference	$f(x) \mapsto f(x+1) - f(x)$	$\Delta_x x = (x+1)\Delta_x + 1$
q-dilation	$f(x) \mapsto f(qx)$	$H_x^{(q)} x = qxH_x^{(q)}$
q-difference	$f(x) \mapsto f(qx) - f(x)$	$\Delta_x^{(q)} x = qx\Delta_x^{(q)} + (q-1)x$
q-derivation	$f(x) \mapsto \frac{f(qx)-f(x)}{x(q-1)}$	$D_x^{(q)} x = qxD_x^{(q)} + 1$
q-shift	$f(x) \mapsto f(x+1)$	$S_x q^x = qq^x S_x$
q-difference $(*)$	$f(x) \mapsto f(x+1) - f(x)$	$\Delta_x^{(q)} x = qq^x\Delta_x^{(q)} + (q-1)q^x$
Eulerian operator	$f(x) \mapsto xf'(x)$	$\Theta_x x = x\Theta_x + x$
derivation $(*)$	$f(x) \mapsto f'(x)$	$D_x e^x = e^x D_x + e^x$
Mahlerian operators	$f(x) \mapsto f(x^p)$	$M_x^{(p)} x = x^p M_x^{(p)}$
Frobenius operators	$f(x) \mapsto f(x)^p$	$F_x^{(p)} x = x^p F_x^{(p)}$
divided differences	$f(x) \mapsto \frac{f(x)-f(a)}{x-a}$	$\Delta_x^{(a)} x = a\Delta_x^{(a)} + 1$

q-shift $(*)$	$S_n^{(q)} n^\alpha \left(q^n\right)^\beta = q^\beta (n+1)^\alpha \left(q^n\right)^\beta S_n^{(q)}$
derivation $(**)$	$\mathbb{K}(x,\, e^x, e^{x^2}, e^{e^x}, \ln x, \tan x)[D_x; 1, D_x]$

Effective framework of Ore algebras $\mathbb{K}(\mathbf{x})[\mathbf{y}][\partial;\sigma,\delta]$:

$$\partial x = (qx+p)\partial + (sx+r) \qquad (q \neq 0).$$

Same type of commutation for y.

Ore Algebras

1. Effective framework of algebras of linear operators
   - in view of calculations by Gröbner bases
   - description by commutations
   - independence from $\boldsymbol{\partial}$
   - the $\mathbf{x}$ and $\mathbf{y}$ and in fact the same variables, but viewed differently in different contexts
2. $\mathbb{K}$ commutative: algebras described by generators and relations

$$\mathbb{K}\langle x, \partial; \partial x = (qx+p)\partial + (sx+r)\rangle$$

Annihilating Ideals

"Function space": a $\mathbb{K}$-algebra $\mathcal{F}$ which is a left $\mathbb{O}$-module for the Ore algebra $\mathbb{O} = \mathbb{A}[\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}]$ over the $\mathbb{K}$-algebra $\mathbb{A}$:

$$\mathcal{F} + \mathcal{F} \subseteq \mathcal{F} \qquad \mathbb{O} \cdot \mathcal{F} \subseteq \mathcal{F}.$$

In applications, $\mathbb{K}$ is commutative and $\mathbb{A} = \mathbb{K}(\mathbf{x})[\mathbf{y}]$.

The annihilating set of $f \in \mathcal{F}$,

$$\operatorname{Ann} f = \big\{ P(\mathbf{x}, \mathbf{y}, \boldsymbol{\partial}) \in \mathbb{O} \mid P \cdot f = 0 \big\},$$

is a left ideal of $\mathbb{O}$:

$$\begin{cases} P \cdot f = 0 \\ Q \cdot f = 0 \end{cases} \Rightarrow (P + Q) \cdot f = 0,$$

$$P \cdot f = 0 \Rightarrow \begin{cases} x \cdot (P \cdot f) = x \times 0 = 0, \\ \partial \cdot (P \cdot f) = \partial \cdot 0 = 0. \end{cases}$$

Computations in the left module over $\mathbb{O}$

$$\mathbb{O} \cdot f \simeq \mathbb{O} / \operatorname{Ann} f.$$

Annihilating Ideals

1. Annihilator $\longleftrightarrow$ set of all linear functional equations
2. Difference between isomorphism of vector spaces, resp. modules (give product law in each module or quotient ring)

∂-Finite Terms

t is ∂-finite w.r.t. the operator algebra $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$

$\Updownarrow$

the $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot t$'s span a finite dimensional $\mathbb{K}(\mathbf{x})$-vector space:

$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}] \cdot t < +\infty.$

- $\exp(z^2)$ for D_z over $\mathbb{Q}(z)$; $d = 1$
- $\exp\exp z$ for D_z over $\mathbb{Q}(\exp z)$; $d = 1$
- $n!$ for S_n over $\mathbb{Q}(n)$; $d = 1$
- $\binom{n}{k}$ for S_n and S_k over $\mathbb{Q}(n,k)$; $d = 1$
- $(x;q)_n$ for $S_{q,n}$ and $H_{q,x}$ over $\mathbb{Q}(q,q^n,x)$; $d = 1$
- F_n, H_n for S_n over $\mathbb{Q}(n)$; $d = 2$
- $J_n(z), T_n(z), P_n(z), \ldots$ for S_n and D_z over $\mathbb{Q}(n,z)$; $d = 2$
- ${}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix}\middle| z\right)$ for S_a, S_b, S_c and D_z over $\mathbb{K}(a,b,c,z)$; $d = 2$
- $r_n(a,b;q), \ldots$ for $S_{q,n}$ over $\mathbb{Q}(q,a,b,q^n)$; $d = 2$
- $\mathrm{Ci}(z)$ for D_z over $\mathbb{Q}(z)$; $d = 3$
- $J_\mu(x)J_\nu(y)$ for D_x, D_y, S_μ and S_ν over $\mathbb{Q}(\mu,\nu,x,y)$; $d = 4$
- $J_1(ax)I_1(ax)Y_0(x)K_0(x)$ for D_a and D_x over $\mathbb{Q}(a,x)$. $d = 16$

Special cases: (q)-hypergeometric terms, hyperexponential functions, Zeilberger's holonomic functions.

Algorithmic closure under $+$, $\times$ and the ∂_i's

$\Longrightarrow$ simplification and zero test of ∂-finite expressions.

∂-Finite Terms

1. ∂-finiteness vs (rational) holonomy; vs: D-finite f., P-finite f. and s., P-recursive s.
2. ∂ in terminology = mere symbol (no relation to indeterminates)
3. Term vs function
   - univariate case: meromorphic functions
   - multivariate case: ?
4. Describe:
   - algebra $\mathbb{O}$,
   - annihilator $\mathrm{Ann}\, f$,
   - vector space $\mathbb{O} \cdot f$ (choice of basis is irrelevant),
   - corresponding module structure

∂-Finite Closure Properties under $+$, $\times$, ∂

$\mathcal{F}$: function space over $\mathbb{O}$, $\quad f, g$: ∂-finite functions.

f and g are ∂-finite w.r.t. $\mathbb{O} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}]$

$\Downarrow$

$f+g$, fg, and $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot f$ are ∂-finite w.r.t. $\mathbb{O}$.

$$\boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot (f+g) \longrightarrow \boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot f, \boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot g, \qquad \boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot (fg) \longrightarrow (\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot f)(\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot g).$$

Computational setting for $+$: direct sum $\mathbb{O} \cdot f \oplus \mathbb{O} \cdot g \subset \mathcal{F} \oplus \mathcal{F}$:

$$\begin{aligned} \partial_i \cdot (f \oplus g) &= \partial_i \cdot f \oplus \partial_i \cdot g && \text{(action);} \\ NF(f \oplus g) &= NF(f) \oplus NF(g) && \text{(normal forms).} \end{aligned}$$

Computational setting for $\times$: tensor product $\mathbb{O} \cdot f \otimes \mathbb{O} \cdot g \subset \mathcal{F} \otimes \mathcal{F}$:

$$\begin{aligned} \partial_i \cdot (f \otimes g) &= \begin{cases} \partial_i \cdot f \otimes g + f \otimes \partial_i \cdot g, \\ \text{or} \\ \partial_i \cdot f \otimes \partial_i \cdot g \end{cases} && \text{(action);} \\ NF(f \otimes g) &= NF(f) \otimes NF(g) && \text{(normal forms).} \end{aligned}$$

"Algorithm" = Normal Forms + Gaussian Elimination

$\rightarrow$ computes $\mathrm{Ann}(f \oplus g)$, $\mathrm{Ann}(f \otimes g)$, or $\mathrm{Ann}(\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot f)$.

∂-Finite Closure Properties under $+$, $\times$, ∂

1. Suitable finite-dimensional vector space: (homomorphic image of) direct sum or tensor product
2. Evaluation by addition, resp. multiplication map
3. Product requires technical constraints
   - for iterated ∂'s, σ's and δ's are polynomials in ∂
   - for product, well-definedness of $\otimes$
4. ∂-finiteness:

$$\mathbb{O} \cdot f = \bigoplus_{\boldsymbol{\alpha} \in I} \mathbb{K}(\mathbf{x}) \boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot f \subset \mathcal{F}, \qquad I \text{ a finite set;}$$
$$\mathbb{O} \cdot g = \bigoplus_{\boldsymbol{\beta} \in J} \mathbb{K}(\mathbf{x}) \boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot g \subset \mathcal{F}, \qquad J \text{ a finite set.}$$

5. Tensor products over $\mathbb{K}(\mathbf{x})$
6. Complete algorithm:
   - normal forms by Gröbner bases
   - efficiency by (an adaptation of) FGLM algorithm

Single Variable Case

Functions f and g defined by:

$$\begin{aligned} f &= \cos(x), \quad & D^2 \cdot f + f &= 0, \quad & f(0) &= 1, \quad & f'(0) &= 0; \\ g &= \sin(x), \quad & D^2 \cdot g + g &= 0, \quad & g(0) &= 0, \quad & g'(0) &= 1. \end{aligned}$$

Rewrite successive derivatives of $h = f^2 + g^2$:

$$\begin{aligned} h &= f^2 + g^2, & h(0) &= 1; \\ D \cdot h &= 2(fD \cdot f + gD \cdot g), & h'(0) &= 0; \\ D^2 \cdot h &= 2((D \cdot f)^2 - f^2 + (D \cdot g)^2 - g^2), & h''(0) &= 0; \\ D^3 \cdot h &= -8(fD \cdot f + gD \cdot g), & h'''(0) &= 0. \end{aligned}$$

Conclusion:

- $h = f^2 + g^2$ satisfies $D^3 \cdot h + 4D \cdot h = 0$;
- initial conditions $\implies h(x) = \boxed{\cos(x)^2 + \sin(x)^2 = 1}$.

Same algorithm applied to Fibonacci numbers F_n yields:

$$\boxed{F_{n+2}F_n - F_{n+1}^2 = (-1)^n}.$$

Single Variable Case

1. Direct calculation for a polynomial expression (also show calculation on $(f \otimes f) \oplus (g \otimes g)$).
2. Compare to calculation where $\sin = -D \cdot \cos$ (and $h' = 0$).
3. Compare to calculation with three different F's (and $S^3 - 2S^2 - 2S + 1$).
4. Allude to rectangular calculations, and announce example of the Mehler formula for Hermite polynomials.

The Mehler Formula for Hermite Polynomials

The sum

$$F(u) = \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!}$$

equals the closed form

$$\frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}.$$

1. Computing product $H_n(x) \times H_n(y) \times \frac{u^n}{n!}$ yields:

$$\begin{aligned}(n+4)S_n^4 - 4xyuS_n^3 - 4(2n+5-2x^2-2y^2)u^2S_n^2 \\ -16xyu^3S_n + 16(n+1)u^4, \\ uD_u - n;\end{aligned}$$

2. Derive the following annihilator of the summand:

$$\begin{aligned}16u^3 + 16u^4D_u - 16u^2xyS_n + D_uS_n^4 - 8u^2S_n^2D_u \\ +4u(2x^2-1+2y^2)S_n^2 - 4xyS_n^3;\end{aligned}$$

3. Creative telescoping yields

$$((1-4u^2)^2D_u + 16u^3 + 4(2x^2+2y^2-1)u - 4xy(4u^2+1)) \cdot F = 0;$$

4. Solve to get the announced closed form.

The Mehler Formula for Hermite Polynomials

1. Multivariate product
   - perform first steps of the reduction by

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

   - describe lacks in the "algorithm":
     * procedure for normal forms
     * derivatives have to be ordered for more efficiency
2. Conclusion:
   - need of Gröbner bases:
     * normal forms for ∂-finite closures
     * elimination for holonomic closures
   - need of holonomy:
     * existence of non-trivial eliminated polynomial

Part II

Gröbner Bases in Algebras of Linear Operators

Gröbner Bases in Algebras of Linear Operators

1. Gröbner bases for the skew case;
2. Other possible variations on Gröbner bases:
   [not described here]
   - FGLM algorithm;
   - Gröbner bases for modules;
   - computation of syzygies and their evaluation;
   - Gröbner walk.
3. ∂-Finite closures $(+, \times, \partial^{-1})$ in an algorithmic way.

Skew Gröbner Bases

Distinguished bases of an ideal
satisfying a divisibility property.

Generalize the Euclidean division algorithm and Euclidean gcd calculations to multivariate (skew) polynomial rings.

Provide us with:

1. normal forms for ideals:

$$\{p_i\} \to \{g_i\} \text{ such that } \sum_i \mathcal{A}p_i = \sum_i \mathcal{A}g_i;$$

2. divisions with uniqueness of the remainder:

$$p \to \sum_i a_i g_i + r;$$

2'. normal forms in the quotient ring:

$$\mathcal{A}/\sum_i \mathcal{A}g_i;$$

3. polynomial elimination: $(\mathcal{A} = \mathcal{B}[x; \sigma, \delta])$

$$\{p_i\} \to \{g_i\} \text{ such that } \sum_i \mathcal{B}g_i = \sum_i \mathcal{A}p_i \cap \mathcal{B}.$$

Algorithmic procedures based on Buchberger's algorithm.

Skew Gröbner Bases

1. Effective calculations with multivariate ideals
   - originally developped in the commutative case
   - main difference to the Euclidean division: dependency in a term order
2. Same algorithms and program in MAPLE

Gröbner-Based Algorithms

Extension of Buchberger's algorithm to the skew case:
focus on left ideals; Gröbner bases still finite (noetherianity).

[Galligo, Takayama, Mora, Robbiano, Kandri-Rody and Weispfenning, Kredel]

Direct extension of:

- FGLM algorithm;
- Gröbner walk;
- Gröbner bases for (left-) modules;
- Syzygy calculations.

"Skew Varieties"

Hilbert dimension

For an Ore algebra $\mathbb{O}_{\text{rat}} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}]$:

Dimension 0 $\longleftrightarrow$ ∂-finite function.

For an Ore algebra $\mathbb{O}_{\text{poly}} = \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}]$:

"Small" dimension $\longleftrightarrow$ holonomic modules and functions.

[Bernstein, Kashiwara, Sabbah, Zeilberger]

Gröbner-Based Algorithms and "Skew Varieties"

1. Generalization to Ore algebras:
   - as a subcase of polynomial rings of solvable type
   - possible restriction on term orders allowed
2. Skew Gröbner bases: different from
   - non-commutative Gröbner bases
   - differential Gröbner bases
3. Detail/define each application
4. Dimension theory
   - invariant = Hilbert dimension for ideals
     $\neq$ dimension as vector spaces
   - same as Gel'fand Kirillov dimension (and other ones)
   - link to (commutative) algebraic varieties
   - multiplicity
   - definition of holonomy originates
     in $\mathcal{D}$-module theory
5. Two frameworks
   - $+$, $\times$, simplifications: calculation with "non-commutative algebraic numbers"
   - $\int$, $\sum$: eliminations

Term Orderings

Set of terms = totally ordered monoid T (neutral e)
with compatibility between $\preceq$ and product:

$$\forall t \in T \qquad e \preceq t,$$
$$\forall (t,u,v) \in T^3 \qquad u \preceq v \Rightarrow tu \preceq tv.$$

Usually: $T = [x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ = free commutative monoid.

Examples:

- lexicographical ordering $x \succ \partial$:

$$e \prec \partial \prec \partial^2 \prec \cdots \prec x \prec x\partial \prec x^2\partial \prec \cdots \prec x^3 \prec \ldots$$

- total degree ordering $x \succ \partial$:

$$e \prec \partial \prec x \prec \partial^2 \prec x\partial \prec x^2 \prec \partial^3 \prec x\partial^2 \prec x^2\partial \prec x^3 \prec \ldots$$

- intermediate orderings for elimination.

(Skew) polynomials = linear combinations of terms:

- leading term of a polynomial p;
- leading coefficient of a polynomial p;
- leading monomial of a polynomial p.

Term Orderings

1. Admissibility: positivity + continuity
   (positivity mainly for termination)
2. Compatibility for correctness
3. Term vs monomial

The FGLM Algorithm

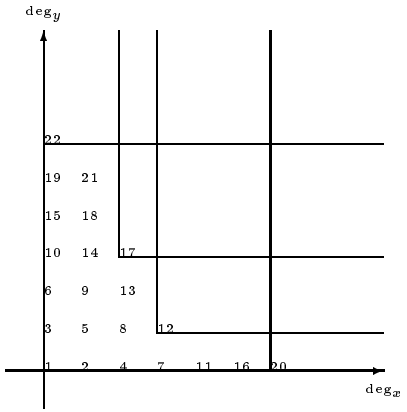
INPUT: $\begin{cases} \mathcal{A} = \mathbb{K}\langle x_1, \ldots, x_n; \ldots \rangle, \ \mathfrak{I} \text{ a left ideal;} \\ \text{an algorithm } NF \text{ for normal forms in } \mathcal{A}/\mathfrak{I}; \\ \text{a term order } T \text{ on } \mathcal{A}; \end{cases}$

OUTPUT: a Gröbner basis of $\mathfrak{I}$ with respect to T.

ALGORITHM: consider terms in increasing order and find linear dependencies between their normal forms.

Termination for zero-dimensional ideals only.

Example: $\mathcal{A} = \mathbb{K}[x, y]$, T is the total degree order with $x < y$.



Terms reduced: 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^2y^3, xy^4, y^5, x^6, xy^5, y^6.

Leading terms of the computed Gröbner basis: x^3y, x^2y^3, x^6, y^6.

The FGLM algorithm

1. Original problem (commutative case):
   - compute a Gröbner basis by a change of term ordering
   - reduction by original Gröbner basis = taking a normal form
   - linear dependencies obtained by Gaussian elimination
2. Non zero-dimensional case: Gröbner walk

Algorithmic ∂-Finite Closures

Compute an annihilating system for $f+g$, $f \times g$, $\partial \cdot f$
for ∂-finite functions f and g.

Example: the product $h = T_n(x) \times \dfrac{e^{-ux}}{\sqrt{1-x^2}}$.

Ore algebra: $\mathbb{O} = \mathbb{Q}(x,n,u)\langle D_x, S_n, D_u\rangle$.

Term order: total degree in $D_x \succ S_n \succ D_u$.

Gröbner basis for $f = T_n(x)$:

$$D_u, \qquad (x^2-1)D_x - nS_n + nx, \qquad S_n^2 - 2xS_n + 1.$$

Gröbner basis for $g = \frac{e^{-ux}}{\sqrt{1-x^2}}$:

$$D_u + x, \qquad S_n - 1, \qquad (x^2-1)D_x + ux^2 + x - u.$$

$\longrightarrow$ Normal forms in

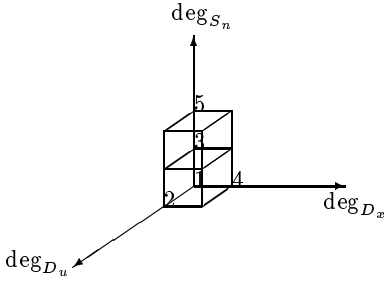
$$\mathbb{O}\cdot f = \mathbb{Q}(x,n,u)f \oplus \mathbb{Q}(x,n,u)(S_n \cdot f) \quad \text{and} \quad \mathbb{O}\cdot g = \mathbb{Q}(x,n,u)g.$$

Algorithmic ∂-Finite Closures

1. Goal: $(\mathrm{sys}_i)_{i\in I} \longrightarrow \mathrm{sys}$
2. Algorithm:
   - normalize the action of the successive $\boldsymbol{\partial}$ at the input expression
   - look for linear combinations
   - order of calculations: à la FGLM
   - termination due to 0-dimensionality
3. Example of $\times$:
   - action on $\otimes$
   - normal forms

In $\mathbb{O}$:

$$1 \prec \boxed{D_u} \prec S_n \prec \boxed{D_x} \prec D_u^2 \prec D_u S_n \prec D_u D_x \prec \boxed{S_n^2} \prec S_n D_x \prec D_x^2 \dots$$



Normal forms in $\mathbb{O} \cdot f \otimes \mathbb{O} \cdot g$:

$$\begin{aligned}
1 \cdot (f \otimes g) &= f \otimes g, \\
\boxed{D_u} \cdot (f \otimes g) &= (D_u \cdot f) \otimes g + f \otimes (D_u \cdot g) = -x(f \otimes g), \\
S_n \cdot (f \otimes g) &= (S_n \cdot f) \otimes (S_n \cdot g) = (S_n \cdot f) \otimes g, \\
\boxed{D_x} \cdot (f \otimes g) &= (D_x \cdot f) \otimes g + f \otimes (D_x \cdot g) \\
&= (x^2 - 1)^{-1}[n(S_n \cdot f) \otimes g - (ux^2 + (n+1)x - u)(f \otimes g)], \\
\boxed{S_n^2} \cdot (f \otimes g) &= (S_n^2 \cdot f) \otimes (S_n^2 \cdot g) = 2x(S_n \cdot f) \otimes g - f \otimes g.
\end{aligned}$$

Gröbner basis computed for $f \otimes g$:

$$\boxed{D_u} + x, \quad (x^2 - 1)\boxed{D_x} - nS_n + ux^2 + (n+1)x - u, \quad \boxed{S_n^2} - 2xS_n + 1.$$

Example of $\otimes$

1. Emphasize
   - colored terms: leading terms in input GB's and rewritten to obtain NF
   - boxed terms: leading terms in output GB that allow to avoid considering their multiples
2. Simpler action on $\oplus$

Part III

Holonomy and Holonomic Functions

Holonomy and Holonomic Functions

1. Three fundamental types of operator algebras; their ideal and module structures;
2. Hilbert dimensions;
3. Bernstein-like inequalities;
4. Complete definitions of holonomy in each of the three fundamental cases;
5. Dimension and elimination; fundamental lemma;
6. Holonomic closures;
7. ∂-Finiteness vs holonomy.

Analytic Continuation Problem

Gel'fand's Problem (ICM, 1954):

For $P \in \mathbb{R}[x_1, \ldots, x_r]$ and $f(\mathbf{x}) \in C_0^\infty(\mathbb{R}^r)$, the integral

$$\Gamma_f(\lambda) = \int_{\mathbb{R}^r} P^\lambda(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$$

is analytic in $\{\lambda \mid \operatorname{Re}\lambda > 0\}$.

Is Γ_f continuable as a meromorphic function of λ?

Two Difficult Proofs by Atiyah (1968), and by Bernstein and Gel'fand (1968):

Hironaka's resolution of singularities.

Elementary Solution by Bernstein (1972):

There exist a polynomial $b \in \mathbb{C}[\lambda]$ and a linear differential operator $L \in \mathbb{C}[\lambda]\langle x_1, \ldots, x_r, \partial_1, \ldots, \partial_r\rangle$ such that

$$b(\lambda)\Gamma_f(\lambda) = \int_{\mathbb{R}^r} P^{\lambda+1}(\mathbf{x})(L \cdot f)(\mathbf{x})\, d\mathbf{x} = \Gamma_{L\cdot f}(\lambda + 1).$$

Using the theory of holonomy (Bernstein, 1971).

Analytic Continuation Problem

1. Holonomy: studied started by Joseph Bernstein in the 1970's, further developed by Masaki Kashiwara (and others)
2. Solution to problem:
   - meromorphic continuation by recurrence
   - poles are on arithmetical progressions
   - further developments of the theory: functional equations obtained for formal series or analytical function instead of polynomial P
3. Bernstein-Sato polynomial $b(\lambda)$:
   - existence by holonomy (Bernstein)
     * $\mathcal{W}_r(\mathbb{K}(s))$-module $\mathbb{K}(s)[\mathbf{x}, p^{-1}] \cdot p^s$ is holonomic
     * so is its $\mathcal{W}_r(\mathbb{K}(s))$-submodule generated by p^s
     * finite length: for sufficiently large k,
     $$p^k p^s \in \mathcal{W}_r(\mathbb{K}(s)) p^{k+1} \cdot p^s$$
     * clearing denominators, shifting by S_λ^{-k},
     $$b(\lambda) p^s \in \mathcal{W}_r(\mathbb{K}[s]) p \cdot p^s$$
   - zeroes are negative rational numbers (Kashiwara)
4. Example: $r = 1$, $P = x^2$,
$$(2\lambda + 1)(2\lambda + 2) \int_{\mathbb{R}} x^{2\lambda} f(x)\, dx = \int_{\mathbb{R}} x^{2(\lambda+1)} f(x)\, dx.$$

Holonomic Identities

Zeilberger's Observation (1990):

Many special functions and sequences are holonomic.
The holonomic class is closed under:

$$+,\ \times,\ \partial,\ \int,\ \sum,\ \int_a^b,\ \sum_a^b,\ \ldots$$

$$2T_m(x)\times T_n(x) - T_{n+m}(x) - T_{n-m}(x) = 0.$$

$T_n(x)$ = Chebychev orthogonal polynomials.

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}.$$

$$a^3(a^4-1)\phi^{(4)}(a) + 4a^2(4a^4-1)\phi^{(3)}(a) + a(73a^4+3)\phi''(a) + (103a^4-3)\phi'(a) + 32a^3\phi(a) = 0.$$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = \frac{n}{2}8^n + 8^n - \frac{3n}{4}2^n\binom{2n}{n}.$$

$$(n+1)u_{n+2} - (7n+12)u_{n+1} - 4(2n+1)u_n = (18n-20)8^n.$$

Holonomic Identities

1. Holonomic class
   - large variety of objects, computable class
   - ubiquitous nature
   - finitely specifiable $\Rightarrow$ computer algebra
2. Decidability of the identities
   - simplifications of expressions in terms of special functions by computer algebra
   - parameterized integrations and summations

Experimental Facts

$$f = \exp(x^2).$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i \dfrac{d^j f}{dx^j}$
for $i \geq 0$, $j \geq 0$, and $i + j \leq n$.

n	0	1	2	3	4	5	6	7	8	9	10
h_n	1	2	3	4	5	6	7	8	9	10	11
$\Delta_n h_n$		1	1	1	1	1	1	1	1	1	1

$h_n = n^1 + 1$, the exponent 1 is the number of variables!

$$f = \frac{1}{x^2 - x + y}.$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i y^j \dfrac{d^{k+l} f}{dx^k dy^l}$
for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	13	25	41	61
$\Delta_n h_n$		4	8	12	16	20
$\Delta_n^2 h_n$			4	4	4	4

$h_n = 2n^2 + 2n + 1$, the exponent 2 is the number of variables!

Experimental Facts (I)

Experimental Facts (II)

$$f = \frac{1}{\sqrt{y^3 - y^2 + xy}}.$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i y^j \dfrac{d^{k+l} f}{dx^k dy^l}$

for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	14	29	50	77
$\Delta_n h_n$		4	9	15	21	27
$\Delta_n^2 h_n$			5	6	6	6

$h_n = 3n^2 + 2$ for large enough n, 2 is the number of variables!

$$f = J_\mu(x) J_\nu(y).$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i y^j \dfrac{d^{k+l} f}{dx^k dy^l}$

for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	15	35	68	116
$\Delta_n h_n$		4	10	20	33	48
$\Delta_n^2 h_n$			6	10	13	15

6	7	8	9	10	11	12
180	260	356	468	596	740	900
64	80	96	112	128	144	160
16	16	16	16	16	16	16

$h_n = 8n^2 - 24n + 36$ for large enough n, 2 is the number of variables!

Experimental Facts (III)

$$f = \cos(xy - z).$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

the $x^i y^j z^k \dfrac{d^{t+u+v} f}{dx^t dy^u dz^v}$ for $\begin{cases} i \geq 0,\ j \geq 0,\ k \geq 0,\ t \geq 0,\ u \geq 0,\ v \geq 0, \\ \text{and } i + j + k + t + u + b \leq n. \end{cases}$

n	0	1	2	3	4	5
h_n	1	7	19	39	69	111
$\Delta_n h_n$		6	12	20	30	42
$\Delta_n^2 h_n$			6	8	10	12
$\Delta_n^3 h_n$				2	2	2

$h_n = \dfrac{n^3}{3} + 2n^2 + \dfrac{11n}{3} + 1,$ the exponent 3 is the number of variables!

$$f = \binom{x}{y}.$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i y^j \binom{x+k}{y+l}$

for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5	6
h_n	1	5	12	22	35	51	70
$\Delta_n h_n$		4	7	10	13	16	19
$\Delta_n^2 h_n$			3	3	3	3	3

$h_n = \dfrac{3n^2}{2} + \dfrac{5n}{2} + 1,$ the exponent 2 is the number of variables!

Experimental Facts (IV)

$$f = 1.$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

the $x_1^{\alpha_1} \dots x_r^{\alpha_r} \dfrac{d^{\beta_1+\dots+\beta_r} f}{dx_1^{\beta_1} \dots dx_r^{\beta_r}}$ for $\alpha_k \geq 0$, $\beta_k \geq 0$, and $\displaystyle\sum_{k=1}^{r} \alpha_k + \sum_{k=1}^{r} \beta_k \leq n$.

$h_n = \binom{n+r}{r} = \Theta(n^r)$, the exponent r is the number of variables!

$$f = \exp(\sin(x)).$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i \dfrac{d^j f}{dx^j}$

for $i \geq 0$, $j \geq 0$, and $i + j \leq n$.

n	0	1	2	3	4	5	6
h_n	1	3	6	10	15	21	28
$\Delta_n h_n$		2	3	4	5	6	7
$\Delta_n^2 h_n$			1	1	1	1	1

$\rightarrow$ $h_n = \dfrac{n^2}{2} + \dfrac{3n}{2} + 1$, the exponent 2 is **not** the number of variables!? (But $2 \geq 1$.)

$$f = \frac{1}{x^2 + y^2}.$$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

the $\dfrac{x^i y^j}{(x+k)^2 + (y+l)^2}$ for $\begin{cases} i \geq 0,\ j \geq 0,\ k \geq 0,\ l \geq 0, \\ \text{and } i + j + k + l \leq n. \end{cases}$

$h_n = \binom{n+4}{4} = \Theta(n^4)$, the exponent 4 is **not** the number of variables!?

(But $4 \geq 2$.)

Fundamental Types of Operator Algebras

Three types of associative algebras over the ring $\mathbb{A}$:

$$\begin{aligned}
\mathcal{W}_1(\mathbb{A}) &= \mathbb{A}\langle x, \partial; \partial x - (x\partial + 1)\rangle, \\
\mathcal{R}_1(\mathbb{A}) &= \mathbb{A}\langle x, \sigma; \sigma x - (x+1)\sigma\rangle, \\
\mathcal{Q}_1(\mathbb{A}) &= \mathbb{A}\langle x, \tau; \tau x - qx\tau\rangle \qquad\qquad (q \in \mathbb{A} \setminus \{0\});
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \ldots, x_r, \partial_1, \ldots, \partial_r; \partial_i x_j - (x_j \partial_i + \delta_{i,j})\rangle, \\
\mathcal{R}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \ldots, x_r, \sigma_1, \ldots, \sigma_r; \sigma_i x_j - (x_j + \delta_{i,j})\sigma_i\rangle, \\
\mathcal{Q}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \ldots, x_r, \tau_1, \ldots, \tau_r; \tau_i x_j - q^{\delta_{i,j}} x_j \tau_i\rangle.
\end{aligned}$$

The $\mathcal{W}_r(\mathbb{A})$'s are known as the Weyl algebras.

Two well-defined notions of degree:

- total degree in all the $2r$ generators;
- total degree in the r generators ∂_i's, σ_i's, or τ_i's only.

Fundamental Types of Operator Algebras

1. Multivariate case $\mathcal{A}_r$ = external tensor product of r copies of $\mathcal{A}_1$ (where $\mathcal{A} = \mathcal{W}, \mathcal{R}, \mathcal{Q}$)
2. Mixed case = external tensor product $\mathcal{W}_r \otimes \mathcal{R}_s \otimes \mathcal{Q}_t$ (over the same ring $\mathbb{A}$)
3. Several possible extensions to algebras with inverse indeterminates

   $$x^{-1}, \partial^{-1}, \sigma^{-1}, \tau^{-1};$$

   stick to these in view of Gröbner bases
4. As an abuse of notation, single notation for the associative algebras and any of their concrete representations
5. Total degree: two notions
   - in the $2r$ generators
   - in the r non-x generators
   - well-defined in view of the commutations
   - usual properties with respect to $+$ and $\times$

Hilbert Dimension

For $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$, consider the filtration by the finite-dimensional vector spaces

$$\mathcal{F}_n = \{P \in \mathcal{A}_r(\mathbb{K}) \text{ with total degree less than or equal to } n\}.$$

For a module $\mathfrak{m}$ over $\mathcal{A}_r(\mathbb{K})$ with generators μ_i's for a finite set I, consider the filtration by the finite-dimensional vector spaces

$$\Phi_n = \sum_{i \in I} \mathcal{F}_{n-d_i} \cdot \mu_i \qquad \text{for fixed } d_i.$$

$$\mathcal{F}_n \mathcal{F}_m \subseteq \mathcal{F}_{n+m}, \qquad \mathcal{F}_n \Phi_m \subseteq \Phi_{n+m}.$$

Th. [Bernstein, Sabbah, ...]: let $h(n) = \dim_{\mathbb{K}} \Phi_n$ (Hilbert function). There exist positive integers d (dimension) and e, and a polynomial

$$p(n) = \frac{en^d}{d!} + \text{lower order terms} \qquad \text{(Hilbert polynomial)}$$

such that $h(n) = p(n)$ for all sufficiently large n.

$$0 \leq d(\mathfrak{m}) \leq 2r \quad (\mathfrak{m} \neq 0),$$
$$d(0) = -\infty, \qquad d(\mathbb{K}[x_1, \ldots, x_r]) = r, \qquad d(\mathcal{A}_r(\mathbb{K})) = 2r.$$

Hilbert Dimension

1. Total degree: in the (r or) $2r$ generators
2. Product of two "homogeneous" polynomials need not be "homogeneous"
3. Filtration of a module defined with respect to a given filtration of the algebra
4. Generator μ_i is assigned a formal degree d_i
5. Dimension for ideals and modules: to refine between infinite-dimensional vector spaces
6. Dimensions: equivalent/related definitions by
   - analytic dimension of an algebraic variety (two concepts for two different filtrations) in the commutations case
   - Hilbert dimension in the commutative case
   - extends to q-case by standard bases (= "combinatorial" dimension)
   - Krull dimension of a quotient ring in the commutative case
   - Gel'fand-Kirillov dimension (all contexts)
   - homological dimension (all contexts)

   $\longrightarrow$ coincide if $\mathbb{K}$ is algebraically closed in the commutative case
   (link to algebraic variety detailed in next slide)
7. Hilbert dimension and multiplicity: independent from the graduation
8. Proof that $d(\mathbb{K}[\mathbf{x}]) = r$ as an $\mathcal{A}_r(\mathbb{K})$-module: consider

$$\mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \partial_i, \quad \mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \sigma_i, \quad \text{and} \quad \mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \tau_i$$

Bernstein's Inequality

For any finitely generated left module $\mathfrak{m}$ over $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$,

$$d(\mathfrak{m}) \geq r \qquad \text{[Bernstein, Sabbah, ...]},$$

unless it is zero, and, assuming the injectivity of the σ_i's in the case of $\mathcal{R}_r$, or the injectivity of the x_i's and the τ_i's in the case of $\mathcal{Q}_r$.

Holonomy

A finitely generated left module $\mathfrak{m}$ over $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$, is holonomic when

$$\mathfrak{m} = 0 \qquad \text{or} \qquad d(\mathfrak{m}) = r,$$

and when the σ_i's are injective in the case of $\mathcal{R}_r$, or the x_i's and the τ_i's are injective in the case of $\mathcal{Q}_r$. [Bernstein, Sabbah, ...].

$f \in \mathfrak{n}$ is holonomic in the left $\mathcal{A}_r$-module $\mathfrak{n}$

$$\Updownarrow$$

$f \in \mathcal{A}_r \cdot f \subseteq \mathfrak{m} \subseteq \mathfrak{n}$ for a holonomic submodule $\mathfrak{m}$.

Bernstein's Inequality

1. A priori: dimensions are $-\infty$ or between 0 and $2r$
   $\longrightarrow$ reduce to $-\infty$ or between r and $2r$
2. Injectivity assumption:
   - $\mathfrak{m} \to \mathbb{K}\langle x, x^{-1}, \tau, \tau^{-1}\rangle \otimes \mathfrak{m}$ is an injection
   - both modules describe the same information
   - sequences identified when they match for $n \geq n_0$ and $m \geq m_0$ (n_0 and m_0 sufficiently large)

Holonomy

1. Simplified definition: to avoid
   - algebras with inverse indeterminates
   - need of homological algebra
2. Holonomic elements:
   - equivalently, $\mathcal{A}_r \cdot f$ is a holonomic module
   - holonomic module has all its elements holonomic
   - holonomic elements span a holonomic module

Dimension and Elimination

$\mathfrak{I}$ = left ideal of $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$.

$\mathcal{B}$ = subalgebra generated by u generators chosen from the $2r$ ones of $\mathcal{A}_r$.

$$d(\mathcal{A}_r/\mathfrak{I}) < u \implies \mathcal{B} \cap \mathfrak{I} \text{ is not } \{0\}.$$

Any set of at most $2r-d-1$ generators can be eliminated simultaneously.

Fundamental Lemma

f a holonomic element in a left $\mathcal{A}_r$-module

$\Downarrow$

$x_2, \ldots, x_r$ can be eliminated simultaneously from $\operatorname{Ann} f$.

f a holonomic element in a left $\mathcal{A}_r$-module

$\Downarrow$

$\partial_2, \ldots, \partial_r$ (resp. $\sigma_2, \ldots, \sigma_r$, resp. $\tau_2, \ldots, \tau_r$)
can be eliminated simultaneously from $\operatorname{Ann} f$.

Dimension and Elimination, and Fundamental Lemma

1. More strongly: dimension is $d \Leftrightarrow$ any set of $2r-d-1$ indeterminates can be eliminated

2. Proof: consider the linear map

$$\begin{array}{ccc} \mathcal{F}_p\mathcal{B} = \mathcal{F}_p\mathcal{A}_r \cap \mathcal{B} & \to & \Gamma_p(\mathcal{A}_r/\mathfrak{I}) = \mathcal{F}_p\mathcal{A}_r \cdot f \\ P & \mapsto & P \cdot f \end{array}$$

   for $f = 1 + \mathfrak{I} \in \mathcal{A}_r/\mathfrak{I}$, and note that for p going to infinity

$$\dim_{\mathbb{K}} \mathcal{F}_p\mathcal{B} = \Theta\left(p^u\right) > \dim_{\mathbb{K}} \Gamma_p(\mathcal{A}_r/\mathfrak{I}) = \Theta\left(p^d\right).$$

3. Proof: f is holonomic $\implies d = r$.

Holonomy vs ∂-Finiteness

$\mathfrak{m} = \mathcal{A}_r \cdot f$ is a holonomic $\mathcal{A}_r$-module

$\Downarrow$ (simple)

f is ∂-finite with respect to $\mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{A}_r$

(tensoring over $\mathbb{K}[\mathbf{x}]$; for a suitable action of $\mathcal{A}_r$ on the tensor product)

f is ∂-finite with respect to $\mathbb{O} = \mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{W}_r$

$\Updownarrow$ (def.)

$V = \mathbb{O} \cdot f$ is finite-dimensional

$\Downarrow$ [Kashiwara]

any finitely generated $\mathcal{W}_r$-submodule $\mathfrak{m}$ of V is holonomic

Example: rational functions are holonomic (with usual derivations).

f is ∂-finite with respect to $\mathbb{O} = \mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{Q}_r$

$\Updownarrow$ (def.)

$V = \mathbb{O} \cdot f$ is finite-dimensional

$\Downarrow$ [Sabbah]

for any non-zero finitely generated $\mathcal{Q}_r$-submodule $\mathfrak{n}$ of V,

there exists a non-zero holonomic $\mathcal{Q}_r$-submodule $\mathfrak{m}$ of $\mathfrak{n}$

Example: $\dfrac{1}{q^{2n} + q^{2k}}$ is rationally holonomic but not holonomic!

Holonomy vs ∂-Finiteness

1. Tensor product: extended to an $\mathcal{A}_r$-module by

$$\begin{aligned}\partial_i \cdot (r \otimes m) &= (\partial_i \cdot r) \otimes m + r \otimes (\partial_i \cdot m),\\ \sigma_i \cdot (r \otimes m) &= (\sigma_i \cdot r) \otimes (\sigma_i \cdot m),\\ \tau_i \cdot (r \otimes m) &= (\tau_i \cdot r) \otimes (\tau_i \cdot m)\end{aligned}$$

2. Case of recurrences: presumably similar to the case of q-recurrences

3. Differential case [Kashiwara+Takayama]: for

$$\mathfrak{m} = \sum_k \mathcal{W}_r \cdot b_k$$

a finitely generated $\mathcal{W}_r$-submodule of V, let $L \in \mathbb{K}[\mathbf{x}]$ be such that for a fixed d but each i and k

$$\partial_i \cdot b_k \in L^{-1} \sum_k \mathcal{F}_d \mathcal{W}_r \cdot b_k \subseteq L^{-1}\mathfrak{m} \subseteq \mathfrak{n} = \bigcup_{p \geq 0} L^{-p}\mathfrak{m}.$$

Set $m = \max(1 + \deg P, d)$. Then $\mathfrak{n}$, therefore $\mathfrak{m}$ is holonomic

$\rightarrow \dim_{\mathbb{K}} \Phi_p \mathfrak{n} = \Theta(p^r)$ for $\Phi_p \mathfrak{n} = \sum_k P^{-p} \mathcal{F}_{mp} \mathcal{Q}_r \cdot b_k$

4. Counter-example: prove that it is not holonomic by starting from a recurrence free from q^k and obtain a contradiction

Part IV

Holonomic Summation and Integration

Holonomic Summation and Integration

1. Creative telescoping;
2. Naive, but most intuitive, Gröbner basis approach;
3. Takayama's algorithm for the case of natural boundaries;
4. Extended Zeilberger's algorithm.

Creative Telescoping

Let $\mathcal{A} = \mathbb{Q}(n,k)\langle S_n, S_k\rangle$ and $u_{n,k}$; we want to determine

$$U_n = \sum_{k=a}^{b} u_{n,k}.$$

$$P(n, S_n)\cdot u = (S_k - 1)Q(n,k,S_n,S_k)\cdot u$$

$$\Downarrow$$

$$P(n, S_n)\cdot \sum_{k=a}^{b} u_{n,k} = [Q(n,k,S_n,S_k)\cdot u]_a^{b+1}.$$

Let $\mathcal{A} = \mathbb{Q}(z,t)\langle D_z, D_t\rangle$ and $u(z,t)$; we want to determine

$$U(z) = \int_a^b u(z,t)\,dt.$$

$$P(z, D_z)\cdot u = D_t Q(z,t,D_z,D_t)\cdot u$$

$$\Downarrow$$

$$P(z, D_z)\cdot \int_a^b u(z,t)\,dt = [Q(z,t,D_z,D_t)\cdot u]_a^b.$$

P is known $\Rightarrow$ indefinite $\sum$ or $\int$ finds Q;

Existence of a non-trivial (P,Q) guaranteed when u is holonomic.

Creative Telescoping

1. Paradigm:

   linear functional system which defines u_n, resp. $u(z)$, up to initial conditions

   $\longrightarrow$ equation that defines U_n, resp. $U(z)$, up to initial conditions

2. Multivariate case, mixed case and other operators possible

3. Possibility of linear dependency of the bounds of summation in n, and of algebraic dependency of the bounds of integration in z

Hypergeometric Terms

Determined by first-order linear recurrence equations:

$$\frac{t(x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_r)}{t(x_1,\ldots,x_r)} \in \mathbb{K}(x_1,\ldots,x_r).$$

As a consequence:

$$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]\cdot t = 1.$$

∂-Finite Terms

Determined by higher-order linear functional equations:

$$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]\cdot t \ < +\infty.$$

Holonomic Terms

Smallest possible Hilbert dimension (unless $t=0$):

$$d(\mathbb{K}[x_1,\ldots,x_r][\partial_1;\sigma_1,\delta_1]\ldots[\partial_1;\sigma_1,\delta_1]\cdot t) = r.$$

Fundamental property:

There exist non-trivial annihilating operators
free from $r-1$ generators.

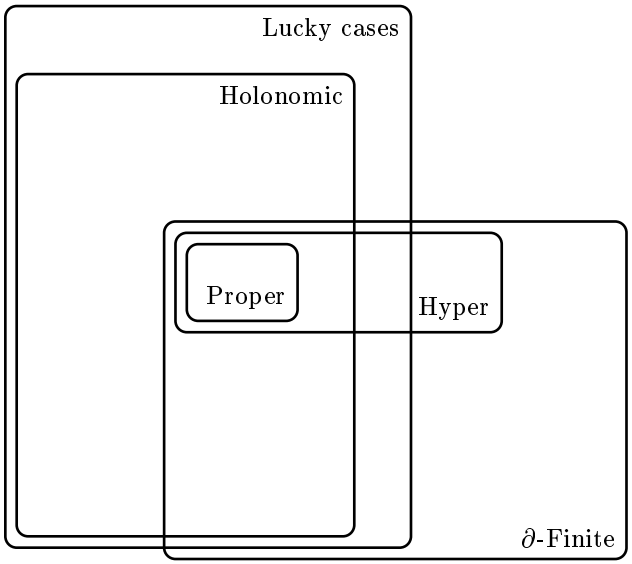
Hypergeometric, ∂-Finite, and Holonomic Terms

1. Hypergeometric terms:
   - closure under product, but no closure under addition ($n!+1$ is not hypergeometric)
   - concept of similarity: $t \simeq u$ iff $t/u \in \mathbb{K}(\mathbf{x})$; $t+u$ is again hypergeometric
   - special case of proper hypergeometric terms

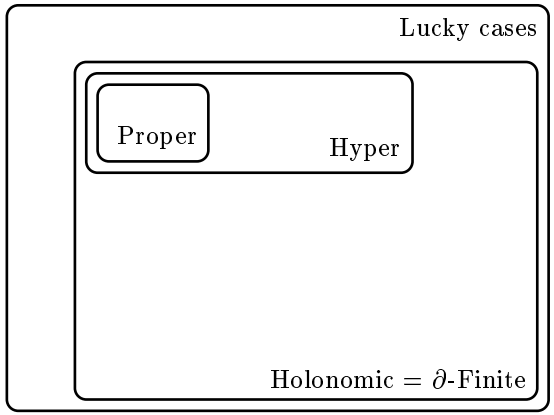
     $$t_{n,k} = P(n,k)\prod_{i=1}^{\ell}\Gamma(a_in+b_ik+c_i)^{d_i}z^k$$

     (for $a_i,b_i,d_i\in\mathbb{Z}$, $c_i,z\in\mathbb{K}$, and $P\in\mathbb{K}[n,k]$)
   - similar definitions for q-hypergeometric terms and hyperexponential terms
2. Holonomic terms
   - class of operator algebras restricted to differential, difference and q-difference operators
   - proper hypergeometric terms are holonomic
   - differentially ∂-finite terms are holonomic
   - no satisfying result in other ∂-finite cases
3. Two frameworks
   - $+$, $\times$, simplifications: calculation with "non-commutative algebraic numbers"
   - $\int$, $\sum$: eliminations

Hierarchy of Classes: General Case



Hierarchy of Classes: Differential Case



Hierarchy of Classes

Definite Summation and Integration: ∂_Ω^{-1}

$\mathcal{A}_r = \mathbb{K}[x_1, \ldots, x_r][\partial_1; \sigma_1, \delta_1] \ldots [\partial_r; \sigma_r, \delta_r]$;

$\mathcal{F}_r =$ left $\mathcal{A}_r$-module = "space of functions in r variables".

$\partial_{r,\Omega}^{-1} =$ "definite anti-∂" from $\mathcal{F}_r$ to $\mathcal{F}_{r-1} =$ generalization of

$$f \mapsto \int_\Omega f(x_1, \ldots, x_r)\, dx_r \qquad \text{and} \qquad f \mapsto \sum_{x_r \in \Omega} f(x_1, \ldots, x_r).$$

Requirements: derivation/shift under $\int / \sum$, and $\int / \sum$ by parts:

$$\text{if } i \neq r, \qquad \partial_i \partial_{r,\Omega}^{-1} = \partial_{r,\Omega}^{-1} \partial_i;$$
$$\partial_{r,\Omega}^{-1} \cdot ((\partial_r \cdot f)g) = (\partial_{r,\Omega}^{-1} \partial_r) \cdot (fg) - \partial_{r,\Omega}^{-1} \cdot (\sigma_r(f)(\partial_r \cdot g)).$$

Natural Boundaries: $\partial_\Omega^{-1} \partial = 0$.

Analytical contexts:

- $\sum$ and $\int$ over compact support;
- $\int$ over closed contours (residues);
- $\sum$ and $\int$ of exponentially decreasing functions.

Definite Summation and Integration and Natural Boundaries

∂_Ω^{-1} of ∂-Finite Functions by Creative Telescoping: Gröbner Bases for Elimination

Compute an annihilating system for the **definite** ∂_Ω^{-1} of a ∂-finite function.

$\mathcal{A}_r = \mathbb{K}[x_1, \ldots, x_r][\partial_1; \sigma_1, \delta_1] \ldots [\partial_r; \sigma_r, \delta_r]$;

$\mathcal{F}_r =$ left $\mathcal{A}_r$-module = "space of functions in r variables".

Idea: $U = \partial_{r,\Omega}^{-1} \cdot u$ w.r.t. $\mathcal{A}_r$: the elements from

$$\mathcal{A}_{r-1}[\partial_r] \cap \operatorname{Ann} u$$

are given by a Gröbner basis for $\operatorname{Ann} u$ for any order such that

$$x_r \succcurlyeq \partial_i, \qquad x_r \succcurlyeq x_j \qquad (j \neq r).$$

Any such Z rewrites into the form $Z = P - \partial_r Q$.

∂_Ω^{-1} of ∂-Finite Functions by Creative Telescoping: Gröbner Bases for Elimination

1. Existence of a non-zero (P, Q) ensured by holonomy:
   - related to the Hilbert dimension of $\operatorname{Ann} h$
   - this dimension can be computed from a Gröbner basis for $\operatorname{Ann} h$
2. More requirements for the success of creative telescoping
   - $P \neq 0$: if $P = 0$, another pair (P', Q') with $P' \neq 0$ exists
     - Zeilberger's proof (requires natural boundaries)
     - Wegschaider's general proof
   - Q non-singular: I know no result, but was said that Schorn has partial results

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(I)}$$

The integrand $h = T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}$ is annihilated by the Gröbner basis

$$\begin{cases} (x^2-1)D_x^2 + (2px^2+3x-2p)D_x + p^2x^2 + 3px - n^2 - p^2 + 1, \\ (x^2-1)D_x S_n + (px^2 - nx - p)S_n + n + 1, \\ S_n^2 - 2xS_n + 1, \qquad D_u + x, \end{cases}$$

in the Ore algebra $\mathbb{O} = \mathbb{Q}[n,u,x]\langle D_x, S_n, D_u\rangle$.

Eliminating x by the term order

$$x \gg D_x \succ S_n \succ D_u \succ n \succ u$$

yields another Gröbner basis:

$$\begin{aligned} &S_n^2 + 2S_nD_u + 1, \\ &D_x(D_u^2-1) - nS_n + uD_u^2 - (n-1)D_u - u, \qquad x + D_u. \end{aligned}$$

After integration, the integral $H = \int_{-1}^{+1} h\,dx$ satisfies

$$\begin{aligned} (S_n^2 + 2S_nD_u + 1)\cdot H &= 0, \\ [nS_n - uD_u^2 + (n-1)D_u + u]\cdot H &= [(D_u^2-1)\cdot h]_{x=-1}^{x=+1} = 0. \end{aligned}$$

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(I)}$$

1. used to model the vibration of a drum membrane

Generating Function of the Jacobi Polynomials

The Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}$$

are determined up to initial conditions by

$$\begin{aligned}
&(1-x^2)P_n^{(\alpha,\beta)\prime\prime}(x)\\
&\quad+(\beta-\alpha-(\alpha+\beta+2)x)P_n^{(\alpha,\beta)\prime}(x)\\
&\quad+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0,\\
&2(n+2)(n+\alpha+\beta+2)(2n+\alpha+\beta+2)P_{n+2}^{(\alpha,\beta)}(x)\\
&\quad-(2n+\alpha+\beta+3)((\alpha^2-\beta^2)\\
&\qquad+(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)x)P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad+2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+4)P_n^{(\alpha,\beta)}(x)=0,\\
&(2n+\alpha+\beta+2)(1-x^2)P_{n+1}^{(\alpha,\beta)\prime}(x)\\
&\quad+(n+1)(\alpha-\beta-(2n+\alpha+\beta+2)x)P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad-2(n+\alpha+1)(n+\beta+1)P_n(x)^{(\alpha,\beta)}=0.
\end{aligned}$$

We **prove** that their generating function is

$$\sum_{n\geq 0}P_n^{(\alpha,\beta)}(x)y^n=\left(R(x,y)\,(1-y+R(x,y))^{\alpha}\,(1+y+R(x,y))^{\beta}\right)^{-1},$$

for $R(x,y)=\sqrt{1-2yx+y^2}$.

Generating Function of the Jacobi Polynomials (I)

Generating Function of the Jacobi Polynomials (II)

Initialization and calculation of equations satisfied by $P_n^{(\alpha,\beta)}(x)y^n$ in $\mathbb{K}(x,y,n)\langle D_x, D_y, S_n\rangle$:

```
G:=[...,...,...]:
with(Ore_algebra):
A:=skew_algebra(diff=[Dx,x],diff=[Dy,y],
    shift=[Sn,n],comm=[alpha,beta],polynom=n):
G:=map(primpart,map(numer,[op(subs(Sn=Sn/y,G)),
    y*Dy-n]),[Sn,Dx,Dy]):
```

Compute the sum by creative telescoping (elimination of n):

```
with(Groebner):
T:=termorder(A,lexdeg([n],[Sn,Dx,Dy])):
GB:=gbasis(G,T):
CT:=map(collect,subs(Sn=1,remove(has,GB,n)),
    [Dx,Dy],distributed,factor);
```

(Maximum degree 3 in D_x and D_y, too large to be displayed.)

Proving the identity:

```
A:=diff_algebra([Dx,x],[Dy,y],comm=[alpha,beta]):
T:=termorder(A,tdeg(Dx,Dy)):
GB:=gbasis(CT,T):
collect(GB,[Dx,Dy],factor);
```

$$(\dots)D_x + (\dots)D_y + (\dots), \qquad (\dots)D_y^2 + (\dots)D_y + (\dots).$$

(Two initial conditions needed.)

Summation and Integration over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

Compute an annihilating system for the **definite** sum or integral of a ∂-finite function in the case of **natural boundaries**.

Goal: compute the P's such that there exists a Q such that

$$[P(n,u,D_x,S_n,D_u) - D_x Q(x,n,u,D_x,S_n,D_u)] \cdot h = 0,$$

or equivalently,

$$P(n,u,S_n,D_u) \cdot H = [Q(x,n,u,D_x,S_n,D_u) \cdot h]_{x=-1}^{x=+1},$$

without computing the Q's.

Idea:

if $$\operatorname{Ann} h = \sum_{i=1}^{\ell} \mathbb{Q}[n,u,x]\langle D_x, S_n, D_u\rangle g_i,$$

then $$\operatorname{Ann} h = \lim_{N\to\infty} \sum_{i=1}^{\ell}\sum_{j=0}^{N} \mathbb{Q}[n,u]\langle D_x, S_n, D_u\rangle x^j g_i.$$

Summation and Integration over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

1. Algorithm:
   - by troncation of the module according the the degree in x
   - using Gröbner bases for modules: multiplication by x forbidden!
   - from finitely-generated $\mathcal{A}_r$-modules (ideals) to non-finitely-generated $\mathcal{A}_{r-1}$-modules
   - calculations modulo D_x on the left

∂_Ω^{-1} over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

Compute an annihilating system for the **definite** ∂_Ω^{-1} of a ∂-finite function in the case of **natural boundaries**.

Recall: the elimination of x_r by a lexicographical Gröbner basis computes

$$\mathcal{A}_{r-1}[\partial_r] \cap \operatorname{Ann} u.$$

Better: determine

$$(\mathcal{A}_{r-1} \oplus \partial_r \mathcal{A}_r) \cap \operatorname{Ann} u = (\mathcal{A}_{r-1} \oplus \partial_r \mathcal{A}_{r-1}[x_r][\partial_r; \sigma_r, \delta_r]) \cap \operatorname{Ann} u.$$

Even better: determine

$$\mathcal{A}_{r-1} \cap (\partial_r \mathcal{A}_r + \operatorname{Ann} u).$$

Idea:

$$\text{if} \qquad \operatorname{Ann} u = \sum_{i=1}^{\ell} \mathcal{A}_{r-1}[x_r][\partial_r; \sigma_r, \delta_r] g_i,$$

$$\text{then} \qquad \operatorname{Ann} u = \lim_{N \to \infty} \sum_{i=1}^{\ell} \sum_{j=0}^{N} \mathcal{A}_{r-1}[\partial_r; \sigma_r, \delta_r] x^j g_i.$$

$\longrightarrow$ straighforward extension to simultaneous $\partial_{i,\Omega}^{-1}$'s.

∂_Ω^{-1} over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

1. Algorithm:
   - by troncation of the module according the degree in x_r
   - from finitely-generated modules over $\mathcal{A}_r$ (= ideals in $\mathcal{A}_r$) to non-finitely-generated modules over $\mathcal{A}_{r-1}$
   - calculations modulo ∂_r on the left, resp. using

$$P(x_r) D_r^d = (-1)^d P^{(d)}(x_r) + D_r Q(x_r, D_r),$$
$$P(x_r) S_r^d = S_r^d P(x_r - d), \quad P(x_r) H_r^d = H_r^d P(q^{-d} x_r).$$

2. Simultaneous multivariate ∂_Ω^{-1}: first automatic proofs of Gordon's identities in a matter of minutes

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(II)}$$

For $N = 3$:

$$\begin{gathered}
1D_u + x, \qquad xD_u + x^2, \qquad x^2D_u + x^3,\\
1(-D_x - nS_n - u) + x(n+1) + x^2(u + D_x),\\
x(-D_x - nS_n - u) + x^2(n+1) + x^3(u + D_x),\\
1(S_n^2+1) - x2S_n, \qquad x(S_n^2+1) - x^2 2S_n, \qquad x^2(S_n^2+1) - x^3 2S_n.
\end{gathered}$$

Optimisation: Taking remainder modulo D_x on the left
by $p(x)D_x = D_x p(x) - p'(x)$ yields

$$\begin{gathered}
D_u 1 + x, \qquad D_u x + x^2, \qquad D_u x^2 + x^3,\\
(-nS_n - u)1 + (n-1)x + ux^2,\\
1 + (-nS_n - u)x + (n-2)x^2 + ux^3,\\
(S_n^2+1)1 - 2S_n x, \qquad (S_n^2+1)x - 2S_n x^2, \qquad (S_n^2+1)x^2 - 2S_n x^3.
\end{gathered}$$

Gröbner basis (in free module) for $x \gg S_n \succ D_u$:

$$\begin{gathered}
\boxed{(-u - nS_n - (n-1)D_u + uD_u^2)1}, \qquad \boxed{(S_n^2 + 1 + 2D_u S_n)1},\\
D_u 1 + x, \qquad (-u - nS_n - (n-1)D_u)1 + ux^2,\\
(unS_nD_u + n(n-2)S_n - u + (u^2 + n^2 - 3n + 2)D_u + nu)1 + u^2x^3.
\end{gathered}$$

$\longrightarrow$ same two eliminated polynomials.

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(II)}$$

1. Reminiscent to the resultant method
2. Contribution: optimisation

Indefinite Hypergeometric Summation: Gosper's Algorithm

INPUT: a hypergeometric term t_k.
OUTPUT: a hypergeometric term T_k from $\mathbb{K}(k)t_k$ such that $T_{k+1} - T_k = t_k$, or $\perp$.

$$\sum_{j=0}^{k} \boxed{\frac{4^j}{\binom{2j}{j}}} = (T_k - T_0) + t_k = \frac{2(k+1)}{3}\boxed{\frac{4^k}{\binom{2k}{k}}} + \frac{1}{3}.$$

Definite Hypergeometric Summation: Zeilberger's Fast Algorithm

ALGORITHM: $T = \textbf{Zeilberger}(h_{n,k})$.
INPUT: a holonomic hypergeometric term $h_{n,k}$.
OUTPUT: a recurrence satisfied by the definite sum over k.

$$f_n = \sum_{k\in\mathbb{Z}} s_{n,k} = \sum_{k\in\mathbb{Z}} \binom{n}{2k}\binom{2k}{k}4^{-k} \qquad \text{is} \qquad \frac{\binom{2n}{n}}{2^n}.$$

$$(n+1)f_{n+1} - (2n+1)f_n = 0.$$

Proof: $\eta_1(k)s_{n+1,k} + \eta_0(k)s_{n,k}$ is Gosper-summable when $\eta_0(k) = -(2n+1)$ and $\eta_1(k) = n+1$.

Indefinite Hypergeometric Summation: Gosper's Algorithm

1. Decision algorithm: $\perp$ means "returns a proof of no answer", and does not mean "does not terminate"
2. Key idea:
   - work in the 1-dimensional vector space of hypergeometric terms similar to t_k
   - each hypergeometric term u_k under consideration can be represented by a single rational function r such that $u_k = r(k)t_k$
3. Generalizations to q-calculus and integration

Indefinite ∂-Finite ∂^{-1}

ALGORITHM: $T = \textbf{Indefinite}(t, (b_i)_{i=1,\dots,N}, \mathcal{A})$.

INPUT:
- a ∂-finite term $t \in V = \bigoplus_{i=1}^{N} \mathbb{F} b_i$,
- an operator algebra $\mathcal{A} = \mathbb{F}\langle \partial_0, \partial_1, \dots, \partial_r \rangle$,
- the action of $\mathcal{A}$ on V.

OUTPUT: $T \in V$ such that $\partial_0 \cdot T = t$; or $\perp$.

1. let $T = \sum_{i=1}^{N} \phi_i b_i$ for undetermined coefficients $\phi_i \in \mathbb{F}$;
2. extract the coefficients of $\partial_0 \cdot T - t$ in the b_i's to obtain a first order functional system in the ϕ_i's;
3. solve for rational solutions [Abramov et al.];
4. if solvable return T; otherwise return $\perp$.

Special case: (a variant of) Gosper's algorithm when $r = 0$, $\partial_0 = S_n$, $\mathcal{A} = \mathbb{Q}(n)\langle S_n \rangle$, $N = 1$.

Indefinite ∂-Finite ∂^{-1}

1. Gosper's algorithm
   - indefinite hypergeometric summation
   - crucial step in Zeilberger's "fast" algorithm for definite hypergeometric summation
   - notion of similarity in the hypergeometric case
2. Algorithm
   - any finite-dimensional vector space
   - use undetermined coefficients
3. Two methods to solve for rational solutions:
   - by uncoupling the first-order functional system to get a higher-order equation in a single ϕ_i
   - recent algorithms to work at the level of a system [Barkatou et al.]
4. Extension to a linearly parametrized input
$$t = \sum_i \eta_i t_i$$
   (for parameters η_i's, $t_i \in V$)
5. Extended algorithms:
   - replacing $\partial_0 \cdot T - t$ by $H \cdot T - t$ yields a method to obtain particular solutions of non-homogeneous functional equations
   - possibility of an extension to algebras where ∂a would be a polynomial in ∂ (not necessarily linear)?

Examples of Indefinite ∂-Finite ∂^{-1}

Example of indefinite ∂-finite summation:

$$\sum_{j=1}^{k} t_j \text{ for } t_k = \binom{k}{p} H_k,$$
$$(\dots)t_{k+2} + (\dots)t_{k+1} + (\dots)t_k = 0,$$
$$\mathcal{A} = \mathbb{Q}(p,k)\langle S_k \rangle, \qquad T = \phi_0 t_k + \phi_1 t_{k+1},$$

$$\text{Output:} \quad \phi_0(k) = \frac{(k+1)^2}{(p+1)^2} - 1, \quad \phi_1(k) = -\frac{(k-p)(k-p+1)}{(p+1)^2},$$
$$T = \sum t_k \, \delta k = \binom{k}{p} \frac{k-p}{(p+1)^2} \left((p+1)H_k - 1\right).$$

Example of indefinite ∂-finite integration:

$$\int \mathrm{Ci}(z)\, dz \text{ for } t(z) = \mathrm{Ci}(z) = \int_0^z \frac{\cos(t)}{t}\, dt,$$
$$(\dots)t'''(z) + (\dots)t''(z) + (\dots)t'(z) = 0,$$
$$\mathcal{A} = \mathbb{Q}(z)\langle D_z \rangle, \qquad T = \phi_0 t(z) + \phi_1 t'(z) + \phi_2 t''(z),$$

$$\text{Output:} \qquad \phi_0(z) = z, \qquad \phi_1(z) = 1, \qquad \phi_2(z) = z,$$
$$T = \int \mathrm{Ci}(z)\, dz = z\,\mathrm{Ci}(z) - \sin(z).$$

Examples of Indefinite ∂-Finite ∂^{-1}

1. Example of the indefinite integral

$$F = \int f\, dx = \int \left(P \cdot T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}} \right) dx,$$

where

$$P = u^2 D_u^4 + 5u D_u^3 + (4 - 2u^2 - n^2) D_u^2 \\ -5u D_u + n^2 + u^2 - 2$$

shows the necessity of Gröbner bases

Definite ∂-Finite ∂_Ω^{-1}

ALGORITHM: $(P, Q) = \textbf{Definite}(u, (b_i)_{i=1,\dots,N}, \mathcal{A})$.

INPUT:
- a ∂-finite term u w.r.t. $\mathcal{A} = \mathbb{F}\langle \partial_0, \partial_1 \rangle$,
- a finite $\mathbb{F}$-basis $(b_i)_{i=1,\dots,N}$ of $\mathcal{A} \cdot u$.

OUTPUT:
- $P \in \mathbb{F}\langle \partial_1 \rangle$ that commutes with ∂_0,
- $Q \in \mathcal{A}$ such that $P \cdot u = \partial_0 Q \cdot u$.

For increasing values of r:

1. let $P = \sum_{i=0}^{r} \eta_i \partial_1^i$ and $t = P \cdot u$ for undetermined coefficients η_i that commute with ∂_0;
2. let $(T, (\eta_i)) = \textbf{ParamIndefinite}(t, (b_i), \mathcal{A}, \{\eta_0, \dots, \eta_r\})$.
3. if $T \neq \perp$ return (P, Q) for $Q \cdot u = T$.

Termination: when u is holonomic.

Special case: Zeilberger's algorithm when $N = 1$.

Definite ∂-Finite ∂_Ω^{-1}

1. Advantages:
   - few calculations with Gröbner bases
   - Gröbner bases for term orders in two times less indeterminates
   - does not require a holonomic representation of the input
2. For a ∂-finite holonomic term $h_{n,k}$:
   $$[P(n, S_n) + (S_k - 1)Q(n, k, S_n, S_k)] \cdot h = 0$$
   If P is known such that such a Q exists, the previous algorithm finds it
3. $t = P \cdot h$ depends linearly on the η_i's:
   ParamIndefinite computes them together with the coefficients of Q

Neumann's Addition Theorem

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1$$

$\mathcal{A} = \mathbb{Q}(z,k)\langle D_z, S_k\rangle$, $J_k(z)$ satisfies :

$$\begin{cases} z^2 D_z^2 + zD_z + z^2 - k^2, \\ zS_k + zD_z - k. \end{cases}$$

$J_k(z)^2$ satisfies :

$$\begin{cases} zD_z^2 + (-2k+1)D_z - 2zS_k + 2z, \\ zD_z S_k + zD_z + (2k+2)S_k - 2k, \\ z^2 S_k^2 - 4(k+1)^2 S_k - 2z(k+1)D_z + 4k(k+1) - z^2. \end{cases}$$

Basis: $\left(J_k(z)^2, S_k \cdot J_k(z)^2, D_z \cdot J_k(z)^2\right)$.

Output: $P = D_z, \qquad Q = \dfrac{k}{z} + \dfrac{1}{2} D_z,$

$$P \cdot \left(\sum_{k=0}^{\infty} J_k(z)^2\right) + \left[Q \cdot J_k(z)^2\right]_{k=0}^{k=\infty} = 0, \qquad \text{i.e.}$$

$$D_z \cdot \left(2\sum_{k=1}^{\infty} J_k(z)^2 + J_0(z)^2 - 1\right) = 0.$$

Initial condition: $2\sum_{k=1}^{\infty} J_k(0)^2 + J_0(0)^2 - 1 = 0.$

Neumann's Addition Theorem

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1$$

$$\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = J_0(z)$$

$$\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = J_0(z)$$

$\mathcal{A} = \mathbb{Q}(z,t)\langle D_z, D_t\rangle$, $\quad \frac{\cos(zt)}{\sqrt{1-t^2}}$ satisfies :

$$\begin{cases} D_z^2 + t^2, \\ t(1-t^2)D_t - (1-t^2)zD_z - t^2. \end{cases}$$

Basis: $\left(\frac{\cos(zt)}{\sqrt{1-t^2}}, D_z \cdot \frac{\cos(zt)}{\sqrt{1-t^2}} = -t\frac{\sin(zt)}{\sqrt{1-t^2}}\right)$.

Output: $P = zD_z^2 + D_z + z, \qquad Q = \frac{1-t^2}{t}D_z,$

$$P \cdot \left(\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\right) + \left[Q \cdot \frac{\cos(zt)}{\sqrt{1-t^2}}\right]_0^1 = 0, \qquad \text{i.e.}$$

$$(zD_z^2 + D_z + z) \cdot \frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = 0.$$

Correct initial conditions.

Calkin's Curious Identity

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = n2^{3n-1} + 2^{3n} - 3n2^{n-2}\binom{2n}{n}$$

Only 3 minutes!

Starting from a system for $\binom{n}{k}$, summation and cubation yield a system of second and third order recurrence equations for

$$u_{n,k} = \left(\sum_{j=0}^{k}\binom{n}{j}\right)^3.$$

Reduce $P + (S_k - 1)Q$ where

$$P = \sum_{i=0}^{2}\eta_i(n)S_n^i,$$
$$Q = \phi_0(n,k) + \phi_1(n,k)S_k + \phi_2(n,k)S_k^2 + \psi(n,k)S_nS_k.$$

Uncoupling yields a 4-th order recurrence in $\phi(k)$.

After non-homogeneous summation, the sum U_n satisfies

$$(n+1)U_{n+2} - (7n+12)U_{n+1} - 4(2n+1)U_n = 2(9n-10)8^n,$$

from which the result follows by Petkovšek's algorithm.

Calkin's Curious Identity

Multivariate Definite ∂-Finite ∂_Ω^{-1}

ALGORITHM: $(P, Q) = \mathbf{MultiDefinite}(u, (b_i)_{i=1,\dots,N}, \mathcal{A})$.

INPUT:
- a ∂-finite term u w.r.t. $\mathcal{A} = \mathbb{F}\langle \partial_0, \dots, \partial_r \rangle$,
- a finite $\mathbb{F}$-basis $(b_i)_{i=1,\dots,N}$ of $\mathcal{A} \cdot u$.

OUTPUT:
- $P \in \mathbb{F}\langle \partial_1, \dots, \partial_r \rangle$ that commutes with ∂_0,
- $Q \in \mathcal{A}$ such that $P \cdot u = \partial_0 Q \cdot u$.

For increasing supports of the polynomials P:

1. let $P = \sum_{\boldsymbol{\alpha}} \eta_{\boldsymbol{\alpha}} \partial_1^{\alpha_1} \dots \partial_r^{\alpha_r}$ and $t = P \cdot u$ for undetermined coefficients $\eta_{\boldsymbol{\alpha}}$ that commute with ∂_0;
2. let $(T, (\eta_{\boldsymbol{\alpha}})) = \mathbf{ParamIndefinite}(t, (b_i), \mathcal{A}, \{\eta_{\boldsymbol{\alpha}}\})$.
3. if $T \neq \perp$ return (P, Q) for $Q \cdot u = T$.

Termination: when u is holonomic.

Possible strategies:

- any sequence of terms;
- filtration by total degree;
- filtration by a term order (FGLM-like).

Multivariate Definite ∂-Finite ∂_Ω^{-1}

1. Idea: replace candidate of a given degree by candidate of a given support (= set of terms with a non-zero coefficient)
2. Extension already possible in the hypergeometric case (Zeilberger's algorithm)
3. Iterated summation: possible if each inner sum is holonomic ∂-finite
4. Merging with the approach suggested by Verbaeten and Wegschaider could provide a 4th strategy in the future

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-(r+s)}{n}=\sum_{k=0}^{\infty}\binom{n}{k}^4$$

The summand u is cancelled by:

$$\begin{cases}(n+1)(n+1-r)(n+1-s)(n+1-r-s)S_n\\ \quad+(n+s+1)(n+r+1)(2n+1-r-s)(2n+2-r-s),\\ (r+1)^2(2n-r-s)S_r+(n+r+1)(n-r)(n-r-s),\\ (s+1)^2(2n-r-s)S_s+(n+s+1)(n-s)(n-r-s),\end{cases}$$

with respect to the algebra $\mathbb{Q}(n,r,s)[S_n;S_n,0][S_r;S_r,0][S_s;S_s,0]$.

1. Inner simple sum over s: for $L=2$, introducing

$$\begin{cases}P=\eta_{0,0}+\eta_{1,0}S_n+\eta_{0,1}S_r+\eta_{2,0}S_n^2+\eta_{1,1}S_nS_r+\eta_{0,2}S_r^2,\\ Q=\phi(s).\end{cases}$$

one obtains:

$$P_1=(\dots)S_n^2+(\dots)S_r^2+(\dots)S_n+(\dots)S_r+(\dots),$$
$$P_2=(\dots)S_r^2+(\dots)S_n+(\dots)S_r+(\dots),$$
$$Q_1=\frac{(\dots)}{(\dots)},\qquad Q_2=\frac{(\dots)}{(\dots)}.$$

Homogeneous summation: $[Q_i\cdot u]_{s=-\infty}^{s=+\infty}=0$.

$\longrightarrow$ the inner sum is ∂-finite w.r.t. $\mathbb{Q}(n,r)[S_n;S_n,0][S_r;S_r,0]$.

2. Simple case of the algorithm for the summation over r yields

$$(n+2)^3S_n^2-2(2n+3)(3n^2+9n+7)S_n-4(4n+5)(4n+3)(n+1).$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-(r+s)}{n}$$

1. Sum to be found in $A=B$; a better example, with non-hypergeometric summand, has to be found
2. Difficult part is left-hand side (Wegschaider gave another derivation)
3. Iteration: because P_1 and P_2 span a ∂-finite ideal
4. End of the proof:
   - same second-order operator for the right-hand side
   - both sides agree at $n=0$ and $n=1$, where they are 1 and 2,
   - leading coefficient $(n+2)^3$ does not vanish for non-negative integers n