Generalized Hermite Reduction, Creative Telescoping, and Definite Integration of D-Finite Functions

Frédéric Chyzak

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Based on joint work with A. Bostan, P. Lairez, and B. Salvy

Parametrized Definite Integrals

$$\int_0^{+\infty} x\, J_1(ax)\, I_1(ax)\, Y_0(x)\, K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ (Glasser & Montaldi, 1994)

$$\int_0^{\infty}\int_0^{\infty} J_1(x)\, J_1(y)\, J_2(c\sqrt{xy})\, \frac{dx\, dy}{e^{x+y}}$$ (has a 2nd-order linear ODE)

$$\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$ (Doetsch, 1930)

$$\int_{-1}^{+1} \frac{e^{-px}\, T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi\, I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx)\, dx = \frac{1}{2p}\exp\left(\frac{c^2-b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

$$\frac{1}{(2i\pi)^2}\oint\oint \frac{f(s,t/s,x/t)}{st}\, ds\, dt = 1 + 6\cdot \int_0^x \frac{{}_2F_1\left(\begin{matrix}1/3, 2/3\\ 2\end{matrix}\middle| \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\, dw$$

$$\text{where} \quad f(s,t,u) = \frac{(1-s)(1-t)(1-u)}{1-2(s+t+u)+3(st+tu+us)-4stu}$$

(Bostan, Chyzak, van Hoeij, Pech, 2011)

Differentiating under the Integral Sign

Zeilberger's derivation (1982) of a classical integral

$$\text{Given } f(b,x) = e^{-x^2}\cos 2bx, \text{ find } F(b) = \int_{-\infty}^{+\infty} f(b,x)\,dx = ?.$$

$$\frac{dF}{db}(b) = \int_{-\infty}^{+\infty} -2xe^{-x^2}\sin 2bx\,dx =$$

$$\left[e^{-x^2}\sin 2bx\right]_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} -2be^{-x^2}\cos 2bx\,dx = -2b\,F(b).$$

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Continuous form of **"Creative Telescoping"**:

$$\frac{df}{db}(b,x) + 2b\,f(b,x) = \frac{dg}{dx}(b,x) \quad \text{for} \quad g(b,x) = -\frac{1}{2x}\frac{df}{db}(b,x).$$

After integration over x from $-\infty$ to $+\infty$:

$$\frac{dF}{db}(b) + 2bF(b) = \left[\frac{dg}{dx}(b,x)\right]_{x=-\infty}^{x=+\infty} = 0 - 0 = 0.$$

Hermite Reduction (1872)

$$EA - mFA' = P \implies \int \frac{P}{A^{m+1}} = \frac{F}{A^m} + \int \frac{E + F'}{A^m}$$

Cela posé, l'intégrale $\int \frac{P\,dx}{A^{\alpha+1}}$ se traitera comme il suit : nous effectuerons sur A et sa dérivée A′ les opérations du plus grand commun diviseur, de manière à obtenir deux polynômes G et H, satisfaisant à la condition

$$AG - A'H = 1.$$

Nous formerons ensuite deux séries de fonctions entieres :

$$V_0,\ V_1, \ldots,\ V_{\alpha-1},$$
$$P_1,\ P_2, \ldots,\ P_\alpha,$$

par ces relations, où les polynômes Q, Q_1, Q_2,... sont entièrement arbitraires, savoir :

$$\begin{aligned} \alpha V_0 &= HP - AQ,\\ (\alpha-1)V_1 &= HP_1 - AQ_1,\\ (\alpha-2)V_2 &= HP_2 - AQ_2,\\ &\ldots\ldots\ldots\ldots,\\ V_{\alpha-1} &= HP_{\alpha-1} - AQ_{\alpha-1},\\ P_1 &= GP - A'Q - V_0',\\ P_2 &= GP_1 - A'Q_1 - V_1',\\ &\ldots\ldots\ldots\ldots,\\ P_\alpha &= GP_{\alpha-1} - A'Q_{\alpha-1} - V_{\alpha-1}'. \end{aligned}$$

Maintenant je prouverai qu'en faisant

$$V = V_0 + AV_1 + A^2V_2 + \ldots + A^{\alpha-1}V_{\alpha-1},$$
$$U = P_\alpha,$$

on a l'égalité

$$\frac{P}{A^{\alpha+1}} = \frac{U}{A} + \left(\frac{V}{A^\alpha}\right)';$$

d'où

$$\int \frac{P\,dx}{A^{\alpha+1}} = \int \frac{U\,dx}{A} + \frac{V}{A^\alpha},$$

de sorte que $\frac{V}{A^\alpha}$ est la partie algébrique de l'intégrale proposée, et $\int \frac{U\,dx}{A}$ la partie transcendante.

A cet effet, j'élimine G et H entre les trois égalités

$$\begin{aligned} AG - A'H &= 1,\\ (\alpha - i)V_i &= HP_i - AQ_i,\\ P_{i+1} &= GP_i - A'Q_i - V_i', \end{aligned}$$

ce qui donne

$$AP_{i+1} = P_i + (\alpha - i)A'V_i - AV_i'.$$

Or on peut écrire cette relation de la manière suivante :

$$\frac{P_i}{A^{\alpha-i+1}} - \frac{P_{i+1}}{A^{\alpha-i}} = \left(\frac{V_i}{A^{\alpha-i}}\right)'.$$

En supposant ensuite $i = 0, 1, 2, \ldots, \alpha - 1$ et ajoutant membre à membre, nous en conclurons

$$\frac{P}{A^{\alpha+1}} - \frac{P_\alpha}{A} = \left(\frac{V_0}{A^\alpha} + \frac{V_1}{A^{\alpha-1}} + \cdots + \frac{V_{\alpha-1}}{A}\right)',$$

ce qui fait bien voir qu'on satisfait à la condition proposée

$$\frac{P}{A^{\alpha+1}} = \frac{U}{A} + \left(\frac{V}{A^\alpha}\right)'$$

par les valeurs

$$V = V_0 + AV_1 + A^2V_2 + \ldots + A^{\alpha-1}V_{\alpha-1},$$
$$U = P_\alpha,$$

comme il s'agissait de le démontrer.

Ostrogradsky–Hermite Reduction

$$EA - mFA' = P \implies \int \frac{P}{A^{m+1}} = \frac{F}{A^m} + \int \frac{E + F'}{A^m}$$

Cela posé, l'intégrale $\int \frac{\mathrm{P}\,dx}{\mathrm{A}^{\alpha+1}}$ se traitera comme il suit : nous effectuerons sur A et sa dérivée A′ les opérations du plus grand commun diviseur, de manière à obtenir deux polynômes G et H, satisfaisant à la condition

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$$\mathrm{P}_1,\ \mathrm{P}_2, \ldots,\ \mathrm{P}_\alpha,$$

par ces relations, où les polynômes Q, Q_1, Q_2,… sont entièrement arbitraires, savoir :

$$\begin{aligned}
\alpha \mathrm{V}_0 &= \mathrm{HP} - \mathrm{AQ},\\
(\alpha-1)\mathrm{V}_1 &= \mathrm{HP}_1 - \mathrm{AQ}_1,\\
(\alpha-2)\mathrm{V}_2 &= \mathrm{HP}_2 - \mathrm{AQ}_2,\\
&\ldots\ldots\ldots\ldots\ldots,\\
\mathrm{V}_{\alpha-1} &= \mathrm{HP}_{\alpha-1} - \mathrm{AQ}_{\alpha-1},\\
\mathrm{P}_1 &= \mathrm{GP} - \mathrm{A'Q} - \mathrm{V}'_0,\\
\mathrm{P}_2 &= \mathrm{GP}_1 - \mathrm{A'Q}_1 - \mathrm{V}'_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
\mathrm{P}_\alpha &= \mathrm{GP}_{\alpha-1} - \mathrm{A'Q}_{\alpha-1} - \mathrm{V}'_{\alpha-1}.
\end{aligned}$$

Maintenant je prouverai qu'en faisant

$$\mathrm{V} = \mathrm{V}_0 + \mathrm{AV}_1 + \mathrm{A}^2\mathrm{V}_2 + \ldots + \mathrm{A}^{\alpha-1}\mathrm{V}_{\alpha-1},$$
$$\mathrm{U} = \mathrm{P}_\alpha,$$

on a l'égalité

$$\frac{\mathrm{P}}{\mathrm{A}^{\alpha+1}} = \frac{\mathrm{U}}{\mathrm{A}} + \left(\frac{\mathrm{V}}{\mathrm{A}^\alpha}\right)';$$

d'où

$$\int \frac{\mathrm{P}\,dx}{\mathrm{A}^{\alpha+1}} = \int \frac{\mathrm{U}\,dx}{\mathrm{A}} + \frac{\mathrm{V}}{\mathrm{A}^\alpha},$$

de sorte que $\frac{\mathrm{V}}{\mathrm{A}^\alpha}$ est la partie algébrique de l'intégrale proposée, et $\int \frac{\mathrm{U}\,dx}{\mathrm{A}}$ la partie transcendante.

A cet effet, j'élimine G et H entre les trois égalités

$$\begin{aligned}
\mathrm{AG} - \mathrm{A'H} &= 1,\\
(\alpha - i)\mathrm{V}_i &= \mathrm{HP}_i - \mathrm{AQ}_i,\\
\mathrm{P}_{i+1} &= \mathrm{GP}_i - \mathrm{A'Q}_i - \mathrm{V}'_i,
\end{aligned}$$

ce qui donne

$$\mathrm{AP}_{i+1} = \mathrm{P}_i + (\alpha - i)\mathrm{A'V}_i - \mathrm{AV}'_i.$$

Or on peut écrire cette relation de la manière suivante :

$$\frac{\mathrm{P}_i}{\mathrm{A}^{\alpha-i+1}} - \frac{\mathrm{P}_{i+1}}{\mathrm{A}^{\alpha-i}} = \left(\frac{\mathrm{V}_i}{\mathrm{A}^{\alpha-i}}\right)'.$$

En supposant ensuite $i = 0, 1, 2, \ldots, \alpha - 1$ et ajoutant membre à membre, nous en conclurons

$$\frac{\mathrm{P}}{\mathrm{A}^{\alpha+1}} - \frac{\mathrm{P}_\alpha}{\mathrm{A}} = \left(\frac{\mathrm{V}_0}{\mathrm{A}^\alpha} + \frac{\mathrm{V}_1}{\mathrm{A}^{\alpha-1}} + \cdots + \frac{\mathrm{V}_{\alpha-1}}{\mathrm{A}}\right)',$$

ce qui fait bien voir qu'on satisfait à la condition proposée

$$\frac{\mathrm{P}}{\mathrm{A}^{\alpha+1}} = \frac{\mathrm{U}}{\mathrm{A}} + \left(\frac{\mathrm{V}}{\mathrm{A}^\alpha}\right)'$$

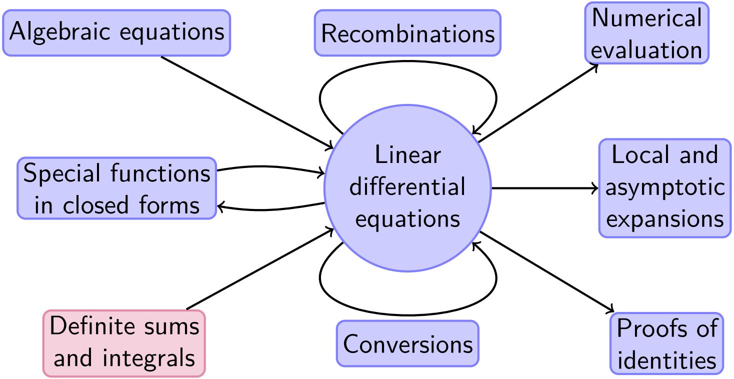
par les valeurs

$$\mathrm{V} = \mathrm{V}_0 + \mathrm{AV}_1 + \mathrm{A}^2\mathrm{V}_2 + \ldots + \mathrm{A}^{\alpha-1}\mathrm{V}_{\alpha-1},$$
$$\mathrm{U} = \mathrm{P}_\alpha,$$

comme il s'agissait de le démontrer.

See also (Ostrogradsky, 1833, 1844/45).

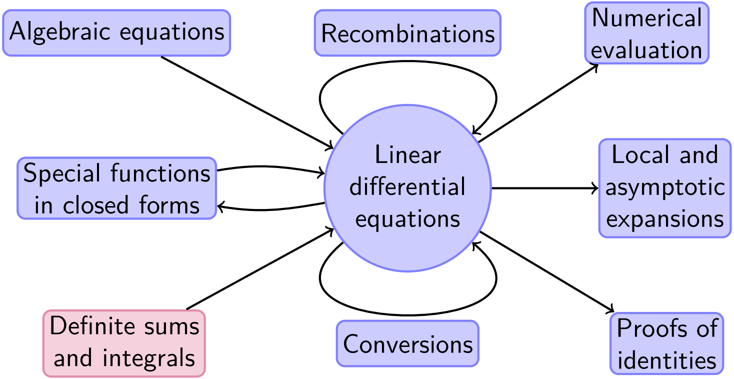
Linear Differential Equations as a Data Structure



Def: differentially finite functions (a.k.a. D-finite)

A function $f(x)$ is D-finite if its derivatives $f(x)$, $f'(x)$, $f''(x)$, ..., span a finite-dimensional vector space over $\mathbb{C}(x)$.

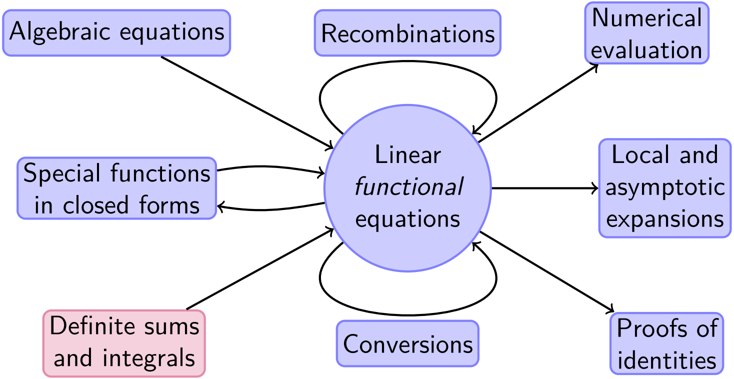
Linear Differential Equations as a Data Structure



Def: multivariate D-finite functions

A function $f(x, y, z)$ is D-finite iff its derivatives $\frac{\partial^{i+j+k} f}{\partial_x^i \partial_y^j \partial_z^k}(x, y, z)$, $i, j, k \geq 0$, span a finite-dimensional vector space over $\mathbb{C}(x, y, z)$.

Linear Differential Equations as a Data Structure



Def: multivariate ∂-finite functions

A function $f_{n,m}(x, y, z)$ is ∂-finite iff a similar confinement holds for derivatives w.r.t. x, y, z, shifts w.r.t. n, m, etc.

Creative Telescoping for Sums and Integrals

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

Given a relation $a_r(n) u_{n+r,k} + \cdots + a_0(n) u_{n,k} = v_{n,k+1} - v_{n,k}$, summation leads by "telescoping" to

$$a_r(n) U_{n+r} + \cdots + a_0(n) U_n = v_{n,b+1} - v_{n,a} \overset{\text{often}}{=} 0.$$

$$U(t) = \int_a^b u(t,x)\, dx = ?$$

Given a relation $a_r(t) \frac{\partial^r u}{\partial t^r} + \cdots + a_0(t) u = \frac{\partial}{\partial x} v(t,x)$, integrating leads by "telescoping" to

$$a_r(t) \frac{\partial^r U}{\partial t^r} + \cdots + a_0(t) U = v(t,b) - v(t,a) \overset{\text{often}}{=} 0.$$

Adapts easily to $U(t) = \sum_{k=a}^{b} u_k(t)$, $U_n = \int_a^b u_n(x)\, dx$, etc.

Creative Telescoping for Sums and Integrals

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$$a_r(t)\frac{\partial^r U}{\partial t^r} + \cdots + a_0(t)U = v(t,b) - v(t,a) \overset{\text{often}}{=} 0.$$

Telescoper Certificate

History of Algorithms for Creative Telescoping

Algorithmic Literature (≤ 2018)

Fasenmyer (1947, 1949); Rainville (1960); Verbaeten (1974); Gosper (1978); Lipshitz (1988); Zeilberger (1982, 1990, 1991); Takayama (1990); Almkvist, Zeilberger (1990); Wilf, Zeilberger (1992); Hornegger (1992); Koornwinder (1993); Paule, Schorn (1995); Majewicz (1996); Riese (1996); Petkovšek, Wilf, Zeilberger (1996); Paule, Riese (1997); Wegschaider (1997); Chyzak, Salvy (1998); Sturmfels, Takayama (1998); Chyzak (2000); Saito, Sturmfels, Takayama (2000); Oaku, Takayama (2001); Le (2001); Riese (2001); Tefera (2000, 2002); Riese (2003); Apagodu, Zeilberger (2006); Kauers (2007); Chen W.Y.C., Sun (2009); Chyzak, Kauers, Salvy (2009); Koutschan (2010); Bostan, Chen S., Chyzak, Li (2010); Chen S., Kauers, Singer (2012); Bostan, Lairez, Salvy (2013); Bostan, Chen S., Chyzak, Li, Xin (2013); Chen S., Huang, Kauers, Li (2015); Lairez (2016); Chen S., Kauers, Koutschan (2016); Huang (2016); Bostan, Dumont, Salvy (2016); Hoeven (2017); Chen S., Hoeij, Kauers, Koutschan (2018); Bostan, Chyzak, Lairez, Salvy (2018).

Applicable to

first-order vs higher-order equations; shift vs differential vs q-analogues vs mixed; ∂-finite vs non-∂-finite; w/ vs wo/ certificate.

Approaches

- bound denominators + bound degrees + linear algebra
- bound denominators + solve functional equations
- elimination by skew Gröbner bases/skew resultants
- reduction of singularity order + linear algebra

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Running Example

Problem

Integrate $f(n, p, x) = \dfrac{\exp(-px)\, T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

$$F(n, p) := \int_{-1}^{+1} f(n, p, x)\, dx = (-1)^n \pi I_n(p).$$

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Generating LFEs by algorithm for closure under product yields:

$$\begin{aligned}
&\frac{\partial f}{\partial p}(n,p,x) + xf(n,p,x) = 0,\\
&nf(n+1,p,x) + (1-x^2)\frac{\partial f}{\partial x}(n,p,x)\\
&\qquad\qquad + \big(p(1-x^2) - (n+1)x\big) f(n,p,x) = 0,\\
&(1-x^2)\frac{\partial^2 f}{\partial x^2}(n,p,x) - (2px^2+3x-2p)\frac{\partial f}{\partial x}(n,p,x)\\
&\qquad\qquad - (p^2x^2 + 3px - n^2 - p^2 + 1) f(n,p,x) = 0.
\end{aligned}$$

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Compact notation using $f_n = f(n+1,p,x)$, $f_x = \frac{\partial f}{\partial x}(n,p,x)$, etc:

$$f_p + xf = 0,$$

$$nf_n + (1-x^2)f_x + \big(p(1-x^2) - (n+1)x\big)f = 0,$$

$$(1-x^2)f_{xx} - (2px^2 + 3x - 2p)f_x - (p^2x^2 + 3px - n^2 - p^2 + 1)f = 0.$$

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$$nf_n + (1-x^2)f_x + \big(p(1-x^2) - (n+1)x\big)f = 0,$$
$$(1-x^2)f_{xx} - (2px^2 + 3x - 2p)f_x - (p^2x^2 + 3px - n^2 - p^2 + 1)f = 0.$$

Observe: any $f_{n^u p^v x^w}$ is a $\mathbb{Q}(n,p,x)$-linear combination of f_x and f.

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$$(1-x^2)f_{xx} - (2px^2+3x-2p)f_x - (p^2x^2+3px-n^2-p^2+1)f = 0.$$

Goal: Find a telescoper such that there is a certificate satisfying

$$\sum_{u,v} c_{u,v}(n,p) f_{n^u p^v} = g_x \quad \text{for} \quad g = b(n,p,x)f_x + a(n,p,x)f.$$

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

$$f_p = (\dots)f_x + (\dots)f,$$
$$f_n = (\dots)f_x + (\dots)f,$$
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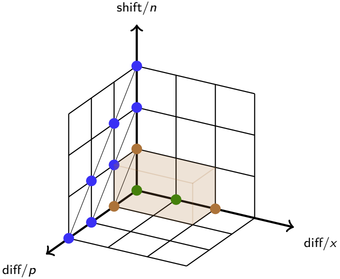

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} c_{u,v}(n,p) f_{n^u p^v} = \frac{\partial}{\partial x}\big(b(n,p,x)f_x + a(n,p,x)f\big).$$

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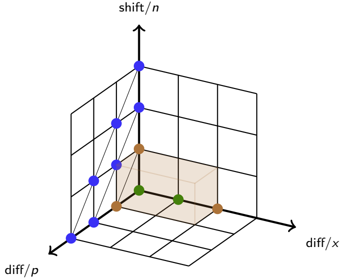

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} (\dots)c_{u,v}(n,p)f_x + (\dots)c_{u,v}(n,p)f = \frac{\partial}{\partial x}\big(b(n,p,x)f_x + a(n,p,x)f\big).$$

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$$f_n = (\dots)f_x + (\dots)f,$$
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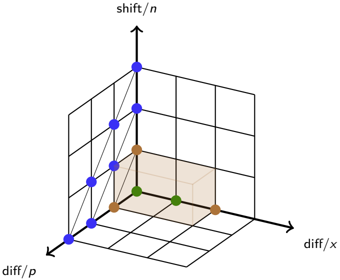

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\leq r} (\dots)c_{u,v}f_x + (\dots)c_{u,v}f = \big((\dots)b + b_x + a\big)f_x + \big((\dots)b + a_x\big)f.$$

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

$$f_p = (\dots)f_x + (\dots)f,$$
$$f_n = (\dots)f_x + (\dots)f,$$
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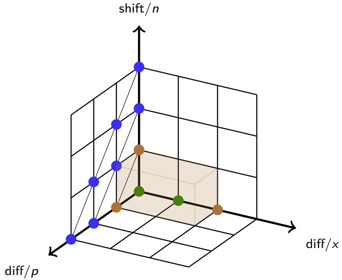

For $r = 1, 2, \dots$ until a nonzero equation can be found, solve:

$$\sum_{u+v\le r} (\dots)c_{u,v} = (\dots)b + b_x + a \quad \text{and} \quad \sum_{u+v\le r} (\dots)c_{u,v} = (\dots)b + a_x.$$

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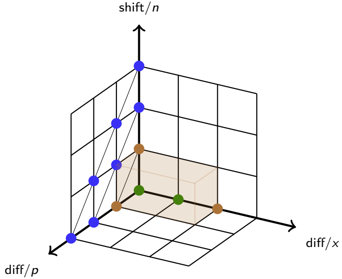

For $r = 1, 2, \dots$ until a nonzero equation can be found:

- eliminating a yields: $b_{xx} + (\dots)b_x + (\dots)b = \sum_{u+v\leq r} (\dots)c_{u,v}$;
- a variant of Abramov's decision algorithm finds $b \in \mathbb{Q}(n,p,x)$ and the $c_{u,v} \in \mathbb{Q}(n,p)$; substituting next gives a.

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

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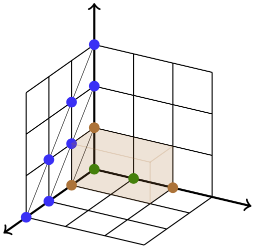

For the running example, the algorithm stops at $r = 2$ and outputs:

$$pf_p + pf_n - nf = g_x \quad \text{for} \quad ng = (1-x^2)f_x + \big(p(1-x^2) - x\big)f,$$
$$pf_{nn} - 2(n+1)f_n - pf = g_x \quad \text{for}$$
$$ng = 2x(1-x^2)f_x + 2\big((px+n)(1-x^2) - x^2\big)f.$$

Chyzak's Algorithm (2000): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = (-1)^n \pi I_n(p)$$

$$f_p = (\dots) f_x + (\dots) f,$$
$$f_n = (\dots) f_x + (\dots) f,$$
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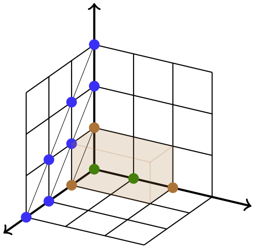

Upon integrating and using properties of $T_n(\pm 1)$:

$$pF_p + pF_n - nF = [g]_{x=-1}^{x=+1} = 0,$$
$$pF_{nn} - 2(n+1)F_n - pF = [g]_{x=-1}^{x=+1} = 0.$$

One recognizes the equations for the right-hand side $(-1)^n \pi I_n(p)$.

```
[chyzak@slowfox (16:08:44) ~]$ maple -b Mgfun.mla -B
    |\^/|     Maple 2018 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2018
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> with(Mgfun);
[MG_Internals, creative_telescoping, dfinite_expr_to_diffeq,

    dfinite_expr_to_rec, dfinite_expr_to_sys, diag_of_sys, int_of_sys,

    pol_to_sys, rational_creative_telescoping, sum_of_sys, sys*sys, sys+sys]

> f := ChebyshevT(n,x)/sqrt(1-x^2)*exp(-p*x);
                             ChebyshevT(n, x) exp(-p x)
                        f := --------------------------
                                       2     1/2
                                   (-x  + 1)

> ct := creative_telescoping(f, {n::shift, p::diff}, x::diff);
memory used=30.3MB, alloc=78.3MB, time=0.37
                           /d            \
ct := [[p _F(n + 1, p) + p |-- _F(n, p)| - n _F(n, p),
                           \dp           /

    x _f(n, p, x) - _f(n + 1, p, x)], [

    -p _F(n, p) + p _F(n + 2, p) + (-2 n - 2) _F(n + 1, p),

    -2 x _f(n + 1, p, x) + 2 _f(n, p, x)]]
```

Chyzak's Algorithm: Three Issues

1. The telescoper (wanted output) is a by-product of the certificate, which is obtained in dense, expanded form (likely to be unneeded in further calculations).
2. In dense, expanded form, the certificate is intrinsically large.
3. The rational-solving step is sensitive to r, allowing for little reuse of intermediate calculations.

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1. The telescoper (wanted output) is a by-product of the certificate, which is obtained in dense, expanded form (likely to be unneeded in further calculations).
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3. The rational-solving step is sensitive to r, allowing for little reuse of intermediate calculations.

Example (walks in $\mathbb{N}^2$ using $\nwarrow, \leftarrow, \downarrow, \rightarrow, \nearrow$, counted by length):

$$\oint\oint \frac{-(1+x)(1+x^2-xy^2)}{(1+x^2)(1-y)(t-xy+ty+tx^2+tx^2y+txy^2)}\,dx\,dy$$

$$(16312320t^{20}+\cdots)f_{t^5}+(407808000t^{19}+\cdots)f_{t^4}+\ldots=\frac{\partial g}{\partial x}+\frac{\partial h}{\partial y}$$

$$\text{LHS}=2\text{ kB},\qquad g=33\text{ kB},\qquad h=896\text{ kB}$$

Rational Integration: the Classics

Hermite reduction (Ostrogradsky, 1833, 1844/45; Hermite, 1872)

Given P/Q, Hermite reduction finds polynomials A and a such that

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{A(x)}{Q^-(x)} + \int \frac{a(x)}{Q^*(x)}\,dx,$$

where $Q^*(x)$ is the squarefree part of $Q(x)$, $Q(x) = Q^-(x)Q^*(x)$, and $\deg a < \deg Q^*$.

Squarefree factorization

Given Q monic, one can in good complexity compute m and 2-by-2 coprime monic Q_i satisfying

$$Q = Q_1^1 Q_2^2 \dots Q_m^m, \quad Q^- = Q_2^1 \dots Q_m^{m-1}, \quad Q^* = Q_1 Q_2 \dots Q_m.$$

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Logarithmic part = obstruction to existence of rational primitive

For $R(w) = \operatorname{res}_x(b(x), a(x) - b'(x)w)$,

$$\int \frac{a(x)}{b(x)}\,dx = \sum_{R(c)=0} c \ln(\gcd(b(x), a(x) - b'(x)c))$$

(Trager, 1976).

Hermite Revisited (Bostan, Chen, Chyzak, Li, 2010)

$$F(t) := \oint \frac{P(t,x)}{Q(t,x)}\,dx = ? \qquad \text{ODE w.r.t. } t?$$

Hermite reduction in $K(x)$

Given P/Q, find polynomials A and a with $\deg a < \deg Q^*$ such that

$$\int \frac{P}{Q}\,dx = \frac{A}{Q^-} + \int \frac{a}{Q^*}\,dx.$$

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Bivariate Hermite reduction for creative telescoping in $K(t,x)$

$$\frac{P}{Q} = \frac{\partial}{\partial x}\left(\frac{A^{(0)}}{Q^-}\right) + \frac{a^{(0)}}{Q^*}.$$

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$$\left(\frac{P}{Q}\right)_t = \frac{\partial}{\partial x}\left(\left(\frac{A^{(0)}}{Q^-}\right)_t\right) + \frac{a_t^{(0)}}{Q^*} - \frac{a^{(0)}\,Q_t^*}{(Q^*)^2}.$$

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$$\left(\frac{P}{Q}\right)_{t^i} = \frac{\partial}{\partial x}\left(E^{(i)}\right) + \frac{a^{(i)}}{Q^*}.$$

- Confinement $\deg_x a^{(i)} < d := \deg_x Q^* \leq \deg_x Q$:

$$\sum_{i=0}^{d} c_i(t)a^{(i)}(t,x) = 0 \implies \sum_{i=0}^{d} c_i F_{t^i} = 0.$$

- Incremental algorithm that does not compute $(P/Q)_{t^i}$.
- Degree bounds in $K(t)$ + eval./interpol. $\implies$ good complexity.

Key Idea: Reduce Coordinates, not Functions

D-finite functions can have complicated singularities.
Rational functions have only poles.

Previous algorithms	New algorithm
$f \to [f]$	$f = R_0\, f \to [R_0]\, f$
$f_t \to [f_t]$	$f_t = R_1\, f \to [R_1]\, f$
$f_{tt} \to [f_{tt}]$	$f_{tt} = R_2\, f \to [R_2]\, f$
$\vdots$	$\vdots$

Operator Notation

Algebra of linear differential operators with rational coefficients

$$\mathbb{A} = K(t,x)\langle \partial_t, \partial_x \rangle, \qquad M_f = \mathbb{A}(f) = \{P(f) : P \in \mathbb{A}\}$$

$$P = \sum p_{i,j}(t,x)\partial_t^i \partial_x^j \in \mathbb{A} \implies P(f) = \sum p_{i,j}(t,x) f_{t^i x^j} \in M_f$$

$$\mathbb{S} = K(t,x)\langle \partial_x \rangle \subset \mathbb{A}$$

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Hypotheses of D-finiteness

- f is D-finite w.r.t. $\mathbb{A} \implies d := \dim_{K(t,x)}(M_f) < \infty$.
- Let $h \in M_f$ be cyclic, that is to say, $M_f = \bigoplus_{i=0}^{d-1} K(t,x) h_{x^i} = \mathbb{S}(h)$.
- For all $g \in M_f$, there is $A_g \in \mathbb{S}$ such that $g = A_g(h)$.

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Interpretation of creative telescoping

Given f, find a telescoper $T \in K(t)\langle\partial_t\rangle$ and a certificate $g \in M_f$ such that $T(f) = \partial_x(g)$. This really computes $\big(K(t)\langle\partial_t\rangle\big)(f) \cap \partial_x(M_f)$.

Lagrange's Identity

Dual of operators

$$P = \sum_{i=0}^{r} p_i(t,x)\partial_x^i \in \mathbb{S} \xleftrightarrow{\ *\ } P^* = \sum_{i=0}^{r} (-\partial_x)^i p_i(t,x) \in \mathbb{S}$$

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Lagrange's identity

There is a map $\mathcal{L}_P$, bilinear w.r.t. $(h, \ldots, h_{x^{(r-1)}})$ and $(u, \ldots, u_{x^{(r-1)}})$, such that

$$\forall u \in K(t,x),\ \forall h \in M_f,\ uP(h) - P^*(u)h = \partial_x(\mathcal{L}_P(h,u)).$$

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Proof: $\mathcal{L}_P(h,u) = \sum_{i=0}^{r}\sum_{j=0}^{i-1}(-1)^j (p_i u)_{x^j} h_{x^{i-j-1}}$.

Consequences of Lagrange's Identity

Lagrange's identity:

$$\forall h \in M_f,\ \forall u \in K(t,x),\ uP(h) - P^*(u)h = \partial_x(\mathcal{L}_P(h,u)).$$

Let h be cyclic and $L \in \mathbb{S}$ be such that $L(h) = 0$. Then, for all $g \in M_f$:

Operator to rational function: $\mathbb{A}(f) = \mathbb{S}(h) \to K(t,x)h$

$g \in A_g^*(1)h + \partial_x(M_f)$. [by $u = 1$, $P = A_g$]

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Equivalent rational factors: $K(t,x)h \to \big(K(t,x) \bmod L^*(K(t,x))\big)h$

$\forall u \in K(t,x),\ g \in \big(A_g^*(1) - L^*(u)\big)h + \partial_x(M_f)$. [by $P = L$]

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$\forall u \in K(t,x),\ g \in \big(A_g^*(1) - L^*(u)\big)h + \partial_x(M_f)$. [by $P = L$]

Testing derivatives (for L of minimal order)

$g \in \partial_x(M_f) \Rightarrow \exists q \in K(t,x),\ A_g^*(1) = L^*(q)$. [by $A_g^*(1) \in \partial_x\mathbb{S} + \mathbb{S}L$]

Consequences of Lagrange's Identity

Lagrange's identity:

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$\forall u \in K(t,x),\ g \in \big(A_g^*(1) - L^*(u)\big)h + \partial_x(M_f)$. [Reduction?]

Testing derivatives (for L of minimal order)

$g \in \partial_x(M_f) \Leftrightarrow \exists q \in K(t,x),\ A_g^*(1) = L^*(q)$. [Algorithm?]

Running Example (continued)

Problem

Integrate $f(n,p,x) = \dfrac{\exp(-px)\,T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

$$F(n,p) := \int_{-1}^{+1} f(n,p,x)\,dx = (-1)^n \pi I_n(p).$$

In operator notation, f is cancelled by all left-linear combinations of:

$$\partial_p + x1, \qquad n\partial_n + (1-x^2)\partial_x + \big(p(1-x^2) - (n+1)x\big)1,$$
$$(1-x^2)\partial_x^2 - (2px^2 + 3x - 2p)\partial_x - (p^2x^2 + 3px - n^2 - p^2 + 1)1.$$

Goal: Find a telescoper such that there is a certificate satisfying

$$\sum_{u,v} c_{u,v}(n,p)\partial_n^u\partial_p^v = \partial_x\left(b(n,p,x)\partial_x + a(n,p,x)1\right)$$

modulo the operators above.

Running Example (continued)

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Integrate $f(n,p,x) = \dfrac{\exp(-px)\,T_n(x)}{\sqrt{1-x^2}}$ w.r.t. x and prove the identity

$$F(n,p) := \int_{-1}^{+1} f(n,p,x)\,dx = (-1)^n \pi I_n(p).$$

In operator notation, f is cyclic, so $h := f$, and it is cancelled by:

$$\partial_p + x1, \qquad n\partial_n + (1-x^2)\partial_x + \big(p(1-x^2) - (n+1)x\big)1,$$
$$L := (1-x^2)\partial_x^2 - (2px^2 + 3x - 2p)\partial_x - (p^2x^2 + 3px - n^2 - p^2 + 1)1.$$

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Reduction-Based CT Algorithm (2018): an Example

$$\int_{-1}^{+1} f(n,p,x)\,dx = F(n,p) = ?$$

$$L_1 := \partial_p - (\dots)\partial_x - (\dots)1,$$

$$L_2 := \partial_n - (\dots)\partial_x - (\dots)1,$$

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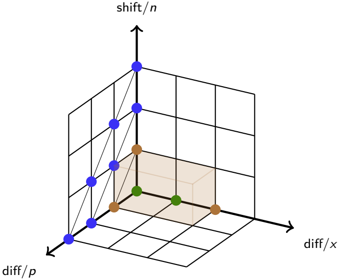

Reduction-Based CT Algorithm (2018): an Example

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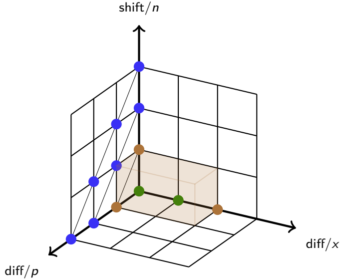

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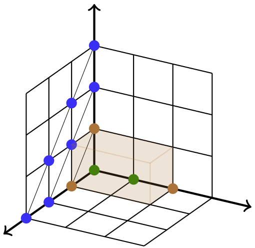

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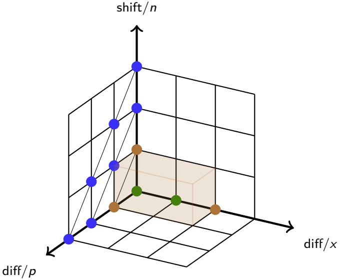

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For those P, $u - v_x \in K(p,n)[x]$ with degree ≤ 3, while

$$L^*(p^2x^0) = p^2x^2 - px - (n^2 + p^2),$$

$$L^*(p^2x^1) = p^2x^3 - 3px^2 - (n^2 + p^2 - 1)x + 2p.$$

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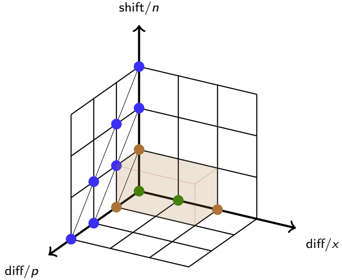

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$$P(f) = (u - v_x)f + \partial_x(\dots) = \left(\mu_P(p,n)x^1 + \lambda_P(p,n)x^0\right) f + \partial_x(\dots).$$

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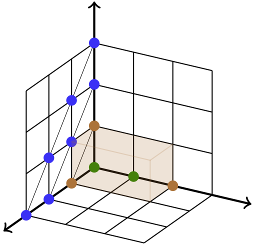

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Linear algebra over $K(p,n)$ finds a basis of telescopers

$$\left(\sum_P c_P P\right)(f) = \partial_x(\dots).$$

Reduction Modulo $L^*(K(x))$

Local decomposition of a rational function $R \in K(x)$

$$R = R_{(\infty)} + \sum_{\alpha} R_{(\alpha)} \text{ for some } R_{(\alpha)} \in \frac{1}{x-\alpha} K(\alpha)[\frac{1}{x-\alpha}] \text{ and } R_{(\infty)} \in K[x].$$

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Local study of the action of L^*

$\exists$ polynomials I_α and I_∞, $\exists$ integers σ_α and σ_∞, such that $\forall s \in \mathbb{Z}$,

$$L^*\big((x-\alpha)^{-s}\big) \underset{x\to\alpha}{=} I_\alpha(-s)(x-\alpha)^{\sigma_\alpha - s} + \mathcal{O}\big((x-\alpha)^{\sigma_\alpha-(s-1)}\big),$$

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Weak reduction strategy

- reduce at finite α (in any order) before at ∞,
- skip monomials for which $I_\alpha(-s-\sigma_\alpha) = 0$ or $I_\infty(-s-\sigma_\infty) = 0$.

Canonical Form Modulo $L^*(K(x))$

Problem: $L^*(K(x))$ does not weakly reduce to $\{0\}$

For $c_0 = I_\alpha(-s-\sigma_\alpha)$ and some c_1, write $R := L^*\big((x-\alpha)^{-s-\sigma_\alpha}\big)$ as

$$R = c_0(x-\alpha)^{-s} + c_1(x-\alpha)^{-(s-1)} + \mathcal{O}((x-\alpha)^{-(s-2)}).$$

- If $c_0 \neq 0$, this reduces to

$$L^*\left((x-\alpha)^{-s-\sigma_\alpha} - (x-\alpha)^{-s-\sigma_\alpha}\right) = 0.$$

- If $c_0 = 0$ and $c_1 \neq 0$, this reduces to some

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Solution

- finitely-many potential obstructions, described by the integer zeros of the I_α and I_∞,
- this can be computed, leading to a canonical-form computation.

```
[chyzak@slowfox (04:21:54) ~]$ maple -b Mgfun.mla -B
    |\^/|     Maple 2018 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2018
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> read "redct.mpl";
> f := ChebyshevT(n,x)/sqrt(1-x^2)*exp(-p*x);
                           ChebyshevT(n, x) exp(-p x)
                      f := --------------------------
                                      2     1/2
                                  (-x   + 1)

> redct(Int(f,x=-1..1),[n::shift,p::diff]);
memory used=3.5MB, alloc=8.3MB, time=0.09
                                           2
            [p D[n] + p D[p] - n, p D[n]  - 2 n D[n] - p - 2 D[n]]

> f := 2*BesselJ(m+n,2*t*x)*ChebyshevT(m-n,x)/sqrt(1-x^2);
                    2 BesselJ(m + n, 2 t x) ChebyshevT(m - n, x)
             f := --------------------------------------------
                                      2     1/2
                                  (-x   + 1)

> redct(Int(f,x),[t::diff, n::shift, m::shift]);
memory used=1189.8MB, alloc=144.8MB, time=9.98
                                            2
[t D[m] + t D[n] + t D[t] - m - n, t D[m]  - 2 m D[m] + t - 2 D[m],

          2
    t D[n]  - 2 n D[n] + t - 2 D[n]]
```

Timings: More than 140 integrals tested

Algorithm	(1)	(2)	(3)	(4)	(5)	(6)	(7)
new (mpl)	13s	$>$ 1h	$>$ 1h	1.5s	1.5s	165s	53s
Chyzak's (mma)	19s	253s	45s	232s	516s	$>$1h	$>$1h
Koutschan's (mma)	1.9s†	2.3s	5.3s	$>$1h	2.3s†	5.4s	2.2s†

$$\int \frac{2J_{m+n}(2tx)T_{m-n}(x)}{\sqrt{1-x^2}}\,dx \quad \text{[diff. } t\text{, shift } n \text{ and } m\text{]}, \tag{1}$$

$$\int_0^1 C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)C_\ell^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx \quad \text{[shift } n,\ m,\ \ell\text{]}, \tag{2}$$

$$\int_0^\infty xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx \quad \text{[diff. } a\text{]}, \tag{3}$$

$$\int \frac{n^2+x+1}{n^2+1}\left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3}\right)^n \sqrt{x^2-5}\, e^{\frac{x^3+1}{x(x-3)(x-4)^2}}\,dx \quad \text{[shift } n\text{]}, \tag{4}$$

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$$\int (x+a)^{\gamma+\lambda-1}(a-x)^{\beta-1}C_m^{(\gamma)}(x/a)C_n^{(\lambda)}(x/a)\,dx \quad \text{[diff. } a\text{, shift } n, m, \beta, \gamma, \lambda\text{]}. \tag{7}$$

†: Heuristic got these faster answers by looking for telescopers of non-minimal orders, yet smaller sizes.

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Need to investigate failures:

- non-mathematical bugs? “not ours”?
- impact of apparent singularities of P^*?

Summary

Approach by solving functional equations (1991+)

↓ see failures to solve as obstructions, recombine obstructions

Primal reduction-based approach (2010+)

↓ work on rational coordinates to simplify singularities, Lagrange's formula

Dual reduction-based approach (2018)