On $(3, 1)$-Regular Graphs with One More Vertex than Edges: A Case Study in Difference-Differential Algebra

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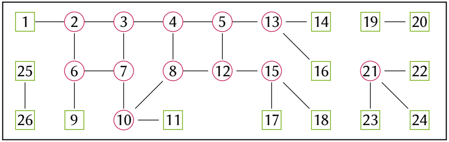
Joint work with Hui Huang and Manuel Kauers

A Conjecture in Enumerative Graph Theory

Conjecture 16 in (Kauers, Koutschan, 2023)

Sequence A339987 of the OEIS, which counts vertex-labelled (3,1)-regular graphs having one more vertex than edges by half the number of vertices, satisfies an explicit, guessed recurrence relation of order 5, valid for all $k \geq 0$.



26 vertices, 25 edges, all degrees in $\{1, 3\}$

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 90, \quad a_4 = 8400, \quad a_5 = 1426950,$$
$$a_6 = 366153480, \quad a_7 = 134292027870$$

A Conjecture in Enumerative Graph Theory

$$
\begin{aligned}
&32(328k^3 + 3300k^2 + 10844k + 11589) \\
&\qquad \times (k+1)(k+2)(2k+1)(2k+3)(2k+5)(2k+7)(2k+9)\, a_k \\
&- 8(2624k^4 + 30664k^3 + 129460k^2 + 232328k + 148119) \\
&\qquad \times (k+2)(2k+3)(2k+5)(2k+7)(2k+9)\, a_{k+1} \\
&- 16(2952k^5 + 40852k^4 + 219308k^3 + 569267k^2 + 712135k + 341634) \\
&\qquad \times (k+3)(2k+5)(2k+7)(2k+9)\, a_{k+2} \\
&+ 8(3936k^5 + 55672k^4 + 306380k^3 + 818282k^2 + 1057879k + 527520) \\
&\qquad \times (k+4)(2k+7)(2k+9)\, a_{k+3} \\
&- 2(2624k^5 + 42472k^4 + 264028k^3 + 786236k^2 + 1117119k + 601452) \\
&\qquad \times (k+5)(2k+9)\, a_{k+4} \\
&+ 3(328k^3 + 2316k^2 + 5228k + 3717)(k+4)(k+6)\, a_{k+5} = 0
\end{aligned}
$$

Guessed by computing a Hermite–Padé approximant

generation of enough terms $\longrightarrow$ guessed ODE $\longrightarrow$ guessed recurrence

Three Proofs by Difference/Differential Algebra

Combinatorial theory: (Gessel, 1987, 1990)

By a representation as a triple-sum

Classical creative telescoping for sums has to be adapted.
Difference (shift) operators with rational coefficients.
(Zeilberger, 1990), (Chyzak, 2000), (Koutschan, 2010)

By a representation as a residue

Uses reduction-based creative telescoping for D-modules.
Differential operators with polynomial coefficients.
(Brochet, Chyzak, Lairez, 2026)

Combinatorial theory: (Read, 1970), (Wormald, 1979)

By combinatorial recursion

Uses differential elimination and creative telescoping for diagonals.
Differential operators with rational coefficients.
(Bostan, Chyzak, Lairez, Salvy, 2018), (van der Hoeven, 2021)

Creative Telescoping for Sums and Integrals

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

Given a relation $a_r(n) u_{n+r,k} + \cdots + a_0(n) u_{n,k} = v_{n,k+1} - v_{n,k}$, summation leads by "telescoping" to

$$a_r(n) U_{n+r} + \cdots + a_0(n) U_n = v_{n,b+1} - v_{n,a} \overset{\text{often}}{=} 0.$$

$$U(t) = \int_a^b u(t,x)\,dx = ?$$

Given a relation $a_r(t) \frac{\partial^r u}{\partial t^r} + \cdots + a_0(t) u = \frac{\partial}{\partial x} v(t,x)$, integrating leads by "telescoping" to

$$a_r(t) \frac{\partial^r U}{\partial t^r} + \cdots + a_0(t) U = v(t,b) - v(t,a) \overset{\text{often}}{=} 0.$$

Adapts easily to $U(t) = \sum_{k=a}^{b} u_k(t)$, $U_n = \int_a^b u_n(x)\,dx$, etc.

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$$a_r(t)\frac{\partial^r U}{\partial t^r} + \cdots + a_0(t)U = v(t,b) - v(t,a) \overset{\text{often}}{=} 0.$$

Telescoper

“Certificate”

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$$U(t) = \int_a^b u(t,x)\, dx = ?$$

Given a relation $a_r(t) \frac{\partial^r u}{\partial t^r} + \cdots + a_0(t) u = \frac{\partial}{\partial x} v(t,x)$, integrating leads by "telescoping" to

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Older algorithms compute telescopers while computing "certificates".
Reduction-based algorithms compute telescopers without computing "certificates".

Ore Algebras

(Chyzak, Salvy, 1998), building on (Ore, 1933)

Difference operators

$\mathbb{Q}(k, u, v, w)\langle S_k, S_u, S_v, S_w\rangle$: skew polynomials in S_k, S_u, S_v, S_w obeying

$$S_k k = (k+1)S_k, \quad \text{and similar relations,}$$
$$S_k \cdot f_{k,u,v,w} = f_{k+1,u,v,w}, \quad \text{and similar actions.}$$

Also $\mathbb{Q}(k)\langle S_k\rangle$, etc.

Differential operators

$\mathbb{Q}(t)[p_1, p_2, p_3, q]\langle \partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \partial_q\rangle$: skew polynomials in $\partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \partial_q$ obeying

$$\partial_{p_1} p_1 = p_1 \partial_{p_1} + 1, \quad \text{and similar relations,}$$
$$\partial_{p_1} \cdot f(p_1, p_2, p_3, q) = \frac{\partial f}{\partial p_1}(p_1, p_2, p_3, q), \quad \text{and similar actions.}$$

Also $\mathbb{Q}(q, t)\langle \partial_q, \partial_t\rangle$, etc.

First Approach:
By a Triple-Sum Representation

A Formula from the Theory of Symmetric Functions

An application of Gessel's theory (1987, 1990)

$r_{m,n} := \#\{(3,1)\text{-regular simple graphs on } n \text{ labelled vertices with } m \text{ edges}\}$

The EGF $$\sum_{m,n=0}^{\infty} r_{m,n} q^m \frac{t^n}{n!} = \langle F(p,q), G(p,t) \rangle \qquad \in \mathbb{Q}[[q,t]],$$

where

$$F(p,q) = \exp\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2 p_2^2 + \tfrac{1}{6}q^3 p_3^2\right) \qquad \in \mathbb{Q}[p_1,p_2,p_3][[q]],$$
$$G(p,t) = \exp\left((p_1 + \tfrac{1}{6}p_1^3 + \tfrac{1}{2}p_1 p_2 + \tfrac{1}{3}p_3)t\right) \qquad \in \mathbb{Q}[p_1,p_2,p_3][[t]],$$

is D-finite w.r.t. q and t.

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is D-finite w.r.t. q and t.

Scalar product (= standard pairing on symmetric functions)

A scalar product on $\mathbb{Q}[p_1,p_2,p_3]$ is defined from the formula

$$\langle p_1^{r_1} p_2^{r_2} p_3^{r_3}, p_1^{s_1} p_2^{s_2} p_3^{s_3} \rangle = \begin{cases} 1^{r_1} r_1!\, 2^{r_2} r_2!\, 3^{r_3} r_3! & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

This extends by bilinearity to $\mathbb{Q}[p_1,p_2,p_3][[q,t]]$.

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The EGF $\displaystyle\sum_{m,n=0}^{\infty} r_{m,n} q^m \frac{t^n}{n!} = \langle F(p,q), G(p,t)\rangle \quad \in \mathbb{Q}[[q,t]],$

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$$G(p,t) = \exp\left((p_1 + \tfrac{1}{6}p_1^3 + \tfrac{1}{2}p_1p_2 + \tfrac{1}{3}p_3)t\right) \quad \in \mathbb{Q}[p_1,p_2,p_3][[t]],$$

is D-finite w.r.t. q and t.

Earlier algorithms for computing scalar products without q

(Chyzak, Mishna, Salvy, 2005)
(Chyzak, Mishna, 2026)

Deriving a Triple Sum (1/2)

We are interested in

$$\sum_{m,n=0}^{\infty} r_{m,n} q^m \frac{t^n}{n!} = \Big\langle \exp\big(f(q,p)\big), \exp\big(g(p)t\big) \Big\rangle$$

where

$$\langle p_1^{r_1} p_2^{r_2} p_3^{r_3}, p_1^{s_1} p_2^{s_2} p_3^{s_3} \rangle = \begin{cases} 1^{r_1} r_1!\, 2^{r_2} r_2!\, 3^{r_3} r_3! & \text{if } r = s, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(q,p) = \tfrac{1}{2} q (p_1^2 - p_2) - \tfrac{1}{4} q^2 p_2^2 + \tfrac{1}{6} q^3 p_3^2, \qquad g(p) = p_1 + \tfrac{1}{6} p_1^3 + \tfrac{1}{2} p_1 p_2 + \tfrac{1}{3} p_3.$$

Deriving a Triple Sum (1/2)

We are interested in

$$r_{m,n} = \Big\langle [q^m] \exp\big(f(q,p)\big), g(p)^n \Big\rangle$$

where

$$\langle p_1^{r_1} p_2^{r_2} p_3^{r_3}, p_1^{s_1} p_2^{s_2} p_3^{s_3} \rangle = \begin{cases} 1^{r_1} r_1!\, 2^{r_2} r_2!\, 3^{r_3} r_3! & \text{if } r = s, \\ 0 & \text{otherwise,} \end{cases}$$

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Deriving a Triple Sum (1/2)

We are interested in

$$r_{m,n} = \Big([q^m]\exp\big(f(q,p)\big) \odot g(1p_1, 2p_2, 3p_3)^n \odot K(p_1)K(p_2)K(p_3)\Big)_{\substack{p_1=1\\p_2=1\\p_3=1}}$$

where

$$p_1^{r_1}p_2^{r_2}p_3^{r_3} \odot p_1^{s_1}p_2^{s_2}p_3^{s_3} = \begin{cases} p_1^{r_1}p_2^{r_2}p_3^{r_3} & \text{if } r = s, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(q,p) = \tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2, \qquad g(p) = p_1 + \tfrac{1}{6}p_1^3 + \tfrac{1}{2}p_1p_2 + \tfrac{1}{3}p_3,$$

$$K(x) = \sum_{\ell=0}^{\infty} \ell!\, x^{\ell}.$$

Deriving a Triple Sum (2/2)

We are interested in

$$r_{m,n} = \Big([q^m] \exp\big(f(q,p)\big) \odot \tilde{g}(p)^n \odot K(p_1)K(p_2)K(p_3) \Big)_{p_1=p_2=p_3=1}$$

where

$$f(q,p) = \tfrac{1}{2} q(p_1^2 - p_2) - \tfrac{1}{4} q^2 p_2^2 + \tfrac{1}{6} q^3 p_3^2, \quad \tilde{g}(p) = p_1 + \tfrac{1}{6} p_1^3 + p_1 p_2 + p_3.$$

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By multiple applications of the binomial theorem:

$$\tilde{g}(p)^n = \sum_{a=0}^{n} \sum_{b=0}^{n-a} \sum_{c=0}^{n-a-b} \binom{n}{a,\, b,\, c,\, n-a-b-c} \frac{1}{2^b 3^b} p_1^{a+3b+c} p_2^c p_3^{n-a-b-c},$$

$$[q^m] \exp\big(f(q,p)\big) = \sum_{u=0}^{\lfloor m/3 \rfloor} \sum_{v=0}^{\lfloor (m-3u)/2 \rfloor} \sum_{w=0}^{m-3u-2v} \frac{1}{u!\, v!\, w!\, (m-3u-2v-w)!} \times \frac{(-1)^{m+u+v+w}}{2^{m-2u} 3^u} p_1^{2w} p_2^{m-3u-w} p_3^{2u}.$$

Deriving a Triple Sum (2/2)

We are interested in

$$a_k = r_{2k-1,2k} = \left([q^{2k-1}] \exp\big(f(q,p)\big) \odot \tilde{g}(p)^{2k} \odot K(p_1)K(p_2)K(p_3) \right)_{p_1=p_2=p_3=1}$$

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$$[q^m] \exp\big(f(q,p)\big) = \sum_{u=0}^{\lfloor m/3 \rfloor} \sum_{v=0}^{\lfloor (m-3u)/2 \rfloor} \sum_{w=0}^{m-3u-2v} \frac{1}{u!\, v!\, w!\, (m-3u-2v-w)!} \times \frac{(-1)^{m+u+v+w}}{2^{m-2u} 3^u} p_1^{2w} p_2^{m-3u-w} p_3^{2u}.$$

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Forcing equal exponents in both sums yields

$$a_k = \sum_{u=0}^{\lfloor (2k-1)/3 \rfloor} \sum_{v=0}^{\lfloor (2k-1-3u)/2 \rfloor} \sum_{w=k-u}^{2k-1-3u-2v} \frac{(-1)^{u+v+w+1}}{2^{k-1-u+w} 3^{2u+w-k}} \times \frac{(2k)!\,(2w)!}{(k+1)!\,(u+w-k)!\,u!\,v!\,w!\,(2k-1-3u-2v-w)!}.$$

Classical Creative Telescoping for Triple Sums

$$a_k = \sum_{u=0}^{\lfloor(2k-1)/3\rfloor} \sum_{v=0}^{\lfloor(2k-1-3u)/2\rfloor} \sum_{w=k-u}^{2k-1-3u-2v} f_{k,u,v,w}$$

What algorithms for creative telescoping compute and promise

(Chyzak, 2000) and (Koutschan, 2010) find a telescoper and certificates,

$$P \in \mathbb{Q}(k)\langle S_k\rangle, \quad Q_1, Q_2, Q_3, C_1, \dots, C_4 \in \mathbb{Q}(k,u,v,w)\langle S_k, S_u, S_v, S_w\rangle,$$

that satisfy

$$P = (S_u - 1)Q_1 + (S_v - 1)Q_2 + (S_w - 1)Q_3 + \sum_{\ell=1}^{4} C_\ell Z_\ell$$

for annihilators of the summand,

$$Z_1 = S_k - \frac{f_{k+1,u,v,w}}{f_{k,u,v,w}}, \quad \dots, \quad Z_4 = S_w - \frac{f_{k,u,v,w+1}}{f_{k,u,v,w}}.$$

Classical Creative Telescoping for Triple Sums

$$a_k = \sum_{u=0}^{\lfloor (2k-1)/3 \rfloor} \sum_{v=0}^{\lfloor (2k-1-3u)/2 \rfloor} \sum_{w=k-u}^{2k-1-3u-2v} f_{k,u,v,w}$$

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that satisfy

$$P \cdot f_{k,u,v,w} = (S_u - 1) \cdot (Q_1 \cdot f_{k,u,v,w}) + (S_v - 1) \cdot (Q_2 \cdot f_{k,u,v,w}) + (S_w - 1) \cdot (Q_3 \cdot f_{k,u,v,w}) + \sum_{\ell=1}^{4} C_\ell \cdot \left(Z_\ell \cdot f_{k,u,v,w}\right).$$

Classical Creative Telescoping for Triple Sums

$$a_k = \sum_{u=0}^{\lfloor (2k-1)/3 \rfloor} \sum_{v=0}^{\lfloor (2k-1-3u)/2 \rfloor} \sum_{w=k-u}^{2k-1-3u-2v} f_{k,u,v,w}$$

Wishful motivation

$$P \cdot f_{k,u,v,w} = (S_u - 1) \cdot (Q_1 \cdot f_{k,u,v,w}) + (S_v - 1) \cdot (Q_2 \cdot f_{k,u,v,w})$$
$$+ (S_w - 1) \cdot (Q_3 \cdot f_{k,u,v,w}) + \sum_{u,v,w} \sum_{\ell=1}^{4} C_\ell \cdot (Z_\ell \cdot f_{k,u,v,w})$$

should imply, after summing over $\{0, \ldots, N\}^3$ with large enough N,

$$P \cdot \sum_{u,v,w=0}^{N} f_{k,u,v,w} = \left(\sum_{v,w=0}^{N} Q_1 \cdot f_{k,u,v,w} \right)_{u=0}^{N+1} + \left(\sum_{u,w=0}^{N} Q_2 \cdot f_{k,u,v,w} \right)_{v=0}^{N+1}$$
$$+ \left(\sum_{u,v=0}^{N} Q_3 \cdot f_{k,u,v,w} \right)_{w=0}^{N+1} + \sum_{u,v,w=0}^{N} 0 = 0.$$

Classical Creative Telescoping for Triple Sums

$$a_k = \sum_{u=0}^{\lfloor (2k-1)/3 \rfloor} \sum_{v=0}^{\lfloor (2k-1-3u)/2 \rfloor} \sum_{w=k-u}^{2k-1-3u-2v} f_{k,u,v,w}$$

What really happens: many ill-shaped auxiliary sums → intractable

$$P \cdot f_{k,u,v,w} = (S_u - 1) \cdot (Q \cdot f_{k,u,v,w}) + (S_v - 1) \cdot (Q_2 \cdot f_{k,u,v,w})$$
$$+ (S_w - 1) \cdot (Q_3 \cdot f_{k,u,v,w}) + \sum_{u,v,w} \sum_{\ell=1}^{4} C_\ell \cdot (Z_\ell \cdot f_{k,u,v,w})$$

- $Q, Q_2, Q_3, C_1, \dots, C_4, Z_1, \dots, Z_4$ have poles,
- absorbing poles by using shifts is not a sufficient proof (any syntactic change in equation requires an independent proof),
- Equation need not hold everywhere, even away from their poles (more poles may have been lost during the computer calculation),
- finite-difference operators do not commute with summation,
- summation range has to be split into sub-ranges.

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Similar observations in the past

- (Chyzak, Mahboubi, Sibut-Pinote, Tassi, 2014)
- (Koutschan, Wong, 2021)

Do not provide a sufficient solution.

Classical Creative Telescoping for Triple Sums

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Do not provide a sufficient solution.

(Bostan, Lairez, Salvy, 2015) is not applicable

a_k is not a binomial sum (non-trivial denominator).

Towards Everywhere-Valid Recurrence Relations

Iverson's bracket notation

Definition:

$$[\![\mathcal{P}]\!] = \begin{cases} 1 & \text{if the predicate } \mathcal{P} \text{ holds,} \\ 0 & \text{if it does not hold.} \end{cases}$$

Nice property: if $c(e)$ is defined everywhere,

$$c(e)\,[\![e = 0]\!] = c(0)\,[\![e = 0]\!].$$

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$$c(e)\,[\![e = 0]\!] = c(0)\,[\![e = 0]\!].$$

Everywhere-defined factorial and "inverse factorial" functions

$$n! = \begin{cases} n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \qquad n¡ = \begin{cases} 1/n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Towards Everywhere-Valid Recurrence Relations

Iverson's bracket notation

Definition:

$$[\![\mathcal{P}]\!] = \begin{cases} 1 & \text{if the predicate } \mathcal{P} \text{ holds,} \\ 0 & \text{if it does not hold.} \end{cases}$$

Nice property: if $c(e)$ is defined everywhere,

$$c(e)\,[\![e = 0]\!] = c(0)\,[\![e = 0]\!].$$

Everywhere-defined factorial and "inverse factorial" functions

$$n! = \begin{cases} n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \qquad n¡ = \begin{cases} 1/n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Everywhere-valid recurrence relations

$$\forall n \in \mathbb{Z},\ n! = n(n-1)! + [\![n = 0]\!]$$

$$\forall n \in \mathbb{Z},\ n¡ = (n+1)(n+1)¡$$

Reformulation as a Sum over Natural Bounds

A triple sum with everywhere-defined summand

$$a_k = \sum_{u\in\mathbb{Z}} \sum_{v\in\mathbb{Z}} \sum_{w\in\mathbb{Z}} f_{k,u,v,w}$$

where

$$f_{k,u,v,w} = (-1)^{u+v+w+1} 2^{1+u-w-k} 3^{k-2u-w}$$
$$\times (2k)!\,(2w)! \times (k+1)_{\text{¡}}\,(u+w-k)_{\text{¡}}\,u_{\text{¡}}\,v_{\text{¡}}\,w_{\text{¡}}\,(2k-1-3u-2v-w)_{\text{¡}}.$$

The summand $f_{k,u,v,w}$ satisfies first-order recurrence relations valid over $\mathbb{Z}^4$. But we won't try to use them directly.

Variant Creative Telescoping

Step 1: Heuristic creative telescoping

Obtain 4-variate rational functions Q_i such that

$$P(k, S_k) - \Delta_u Q_1(k, u, v, w) - \Delta_v Q_2(k, u, v, w) - \Delta_w Q_3(k, u, v, w) \equiv 0 \pmod{Z_1, \dots, Z_4}.$$

"Integer" denominator of, e.g., Q_1: $\quad (k + 3) \prod_{i=2}^{8} (2k + i - 3u - 2v - w)$.

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Step 2: Removal of denominators, by trial and error

Remove denominators by finding suitable shifts such that

$$P(k, S_k) - \Delta_u R_1(k, u, v, w) S_k^9 S_w^9 - \Delta_v R_2(k, u, v, w) S_k^{10} S_w^{10} - \Delta_w R_3(k, u, v, w) S_k^{10} S_w^{10} \equiv 0 \pmod{Z_1, \ldots, Z_4}.$$

Common "integer" denominator of R_1, R_2, R_3: $\quad k + 3$.

Variant Creative Telescoping

There remains to prove $\forall(k, u, v, w) \in \mathbb{Z}^4,\ k \neq -3 \implies z_{k,u,v,w} = 0$ for

$$\begin{aligned} z_{k,u,v,w} := {} & p_5(k) f_{k+5,u,v,w} + p_4(k) f_{k+4,u,v,w} + p_3(k) f_{k+3,u,v,w} \\ & + p_2(k) f_{k+2,u,v,w} + p_1(k) f_{k+1,u,v,w} + p_0(k) f_{k,u,v,w} \\ & - R_1(k, u+1, v, w) f_{k+9,u+1,v,w+9} + R_1(k, u, v, w) f_{k+9,u,v,w+9} \\ & - R_2(k, u, v+1, w) f_{k+10,u,v+1,w+10} + R_2(k, u, v, w) f_{k+10,u,v,w+10} \\ & - R_3(k, u, v, w+1) f_{k+10,u,v,w+11} + R_3(k, u, v, w) f_{k+10,u,v,w+10}. \end{aligned}$$

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Step 3: Rewrite each $f_{k+p,u+q,v+r,w+s}$ in terms of $f_{k+11,u,v,w+12}$ (not $f_{k,u,v,w}$!)

Determine and prove 12 everywhere-valid identities, like

$$\begin{aligned} & \forall(k, u, v, w) \in \mathbb{Z}^4, \\ & \qquad (k+11)(2k+21)(2w+23) f_{k+10,u,v,w+11} = \\ & \qquad\qquad -(k+12)(2k+9-3u-2v-w) f_{k+11,u,v,w+12}. \end{aligned}$$

- Shift $f_{k+11,u,v,w+12}$ determined by trial and error.
- Introduces no new denominator in $z_{k,u,v,w}$.

Variant Creative Telescoping

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&- R_1(k,u+1,v,w)f_{k+9,u+1,v,w+9} + R_1(k,u,v,w)f_{k+9,u,v,w+9} \\
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(k+11)(2k+21)(2w+23)f_{k+10,u,v,w+11} = {}& \\
-(k+12)(2k+9-3u-2v-w)f_{k+11,u,v,w+12}&.
\end{aligned}$$

Proof postponed: new algorithmic idea.

Variant Creative Telescoping

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Step 4: Normalizing the candidate relation

$$z_{k,u,v,w} = \frac{1}{k+3} \frac{\text{polynomial expression}}{\text{polynomial expr. non-zero at integers}} \times f_{k+11,u,v,w+12} = 0$$

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Step 5: Telescoping the telescoping relation

For $k\in\mathbb{Z}\setminus\{-3\}$, summing $z_{k,u,v,w}$ for (u,v,w) over $\mathbb{Z}^3$ yields

$$p_5(k)a_{k+5} + p_4(k)a_{k+4} + p_3(k)a_{k+3} + p_2(k)a_{k+2} + p_1(k)a_{k+1} + p_0(k)a_k = 0.$$

Proving two-term relations between shifts of f

Goal

$$a(k,u,v,w)f_{k+p,u+q,v+r,w+s} - b(k,u,v,w)f_{k+11,u,v,w+12} = 0$$

Proving two-term relations between shifts of f

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$$a(k, u, v, w) f_{k+p,u+q,v+r,w+s} - b(k, u, v, w) f_{k+11,u,v,w+12} = 0$$

Simplifying by

$$f_{k+p,u+q,v+r,w+s} = \text{expression in factorials and inverse factorials},$$
$$n! = n(n-1)! + [\![n = 0]\!], \qquad n¡ = (n+1)(n+1)¡,$$
$$c(e)\,[\![e = 0]\!] = c(0)\,[\![e = 0]\!]$$

leads to a linear combination of terms of the form

$$t := \operatorname{poly}(k, u, v, w) \times \prod_i \ell_i(k, u, v, w)! \times \prod_i \ell'_i(k, u, v, w)¡ \times [\![e = 0]\!].$$

Proving two-term relations between shifts of f

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Fact

$$t \neq 0 \Rightarrow \bigwedge_i \big(\ell_i(k,u,v,w) \geq 0\big) \wedge \bigwedge_i \big(\ell'_i(k,u,v,w) \geq 0\big)$$

Proving two-term relations between shifts of f

Goal

$$a(k,u,v,w)f_{k+p,u+q,v+r,w+s} - b(k,u,v,w)f_{k+11,u,v,w+12} = 0$$

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Fact

$$t = 0 \Leftarrow \bigwedge_i \big(\ell_i(k,u,v,w) \geq 0\big) \wedge \bigwedge_i \big(\ell'_i(k,u,v,w) \geq 0\big) \text{ unsatisfiable}$$

Proving two-term relations between shifts of f

Goal

$$a(k,u,v,w) f_{k+p,u+q,v+r,w+s} - b(k,u,v,w) f_{k+11,u,v,w+12} = 0$$

Simplifying by

$$f_{k+p,u+q,v+r,w+s} = \text{expression in factorials and inverse factorials,}$$
$$n! = n(n-1)! + [\![n=0]\!], \qquad n¡ = (n+1)(n+1)¡,$$
$$c(e)\,[\![e=0]\!] = c(0)\,[\![e=0]\!],$$

then replacing with 0 those unsatisfiable expressions of the form

$$\mathrm{poly}(k,u,v,w) \times \prod_i \ell_i(k,u,v,w)! \times \prod_i \ell'_i(k,u,v,w)¡ \times [\![e=0]\!],$$

proves the goal in all 12 cases of interest.

Proving unsatisfiability: algorithms exist

Satisfiability Modulo Theory + Linear Integer Arithmetic (SMT LIA)

Second Approach:
By a Residue Representation

Deriving a Residue Representation

We are interested in

$$\sum_{m,n=0}^{\infty} r_{m,n} q^m \frac{t^n}{n!} = \Big\langle \exp\big(f(q,p)\big), \exp\big(g(p)t\big) \Big\rangle$$

where

$$f(q,p) = \tfrac{1}{2} q(p_1^2 - p_2) - \tfrac{1}{4} q^2 p_2^2 + \tfrac{1}{6} q^3 p_3^2, \quad g(p) = p_1 + \tfrac{1}{6} p_1^3 + \tfrac{1}{2} p_1 p_2 + \tfrac{1}{3} p_3,$$

$$\langle p_1^{r_1} p_2^{r_2} p_3^{r_3}, p_1^{s_1} p_2^{s_2} p_3^{s_3} \rangle = \begin{cases} 1^{r_1} r_1!\, 2^{r_2} r_2!\, 3^{r_3} r_3! & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

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Formal Laplace transform, formal residue

$$\mathcal{L}(p_1^{r_1} p_2^{r_2} p_3^{r_3}) = \frac{r_1!}{p_1^{r_1+1}} \frac{r_2!}{p_2^{r_2+1}} \frac{r_3!}{p_3^{r_3+1}}, \qquad \operatorname{res}\Bigg(\sum_{r \in \mathbb{Z}^3} c_r p^r\Bigg) = c_{-1,\dots,-1}.$$

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Residue representation

$$\Big\langle \exp(f), \exp(tg) \Big\rangle = \operatorname{res}_{p,q} \left(e^{f(p_1,p_2,p_3,q)} \mathcal{L}\left(e^{tg(1p_1, 2p_2, 3p_3)} \right) \right)$$

Deriving a Residue Representation

We are interested in

$$A(t) := \sum_{k=0}^{\infty} r_{2k-1,2k} \frac{t^{2k}}{(2k)!} = \mathrm{even}_t\, \mathrm{diag}_{q,t} \Big\langle q \exp\big(f(q,p)\big), \exp\big(g(p)t\big) \Big\rangle$$

where

$$f(q,p) = \tfrac{1}{2} q(p_1^2 - p_2) - \tfrac{1}{4} q^2 p_2^2 + \tfrac{1}{6} q^3 p_3^2, \quad g(p) = p_1 + \tfrac{1}{6} p_1^3 + \tfrac{1}{2} p_1 p_2 + \tfrac{1}{3} p_3,$$

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Residue representation

$$A(t) = \mathrm{even}_t\, \mathrm{res}_{p,q} \left(e^{f(p_1,p_2,p_3,q)} \mathcal{L}\Big(e^{q^{-1} t g(1p_1, 2p_2, 3p_3)} \Big) \right)$$

Integration in D-Modules (Brochet, Chyzak, Lairez, 2026)

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Composition of (holonomy-preserving) operations

- $\exp(t\tilde{g}(p))$: a system of annihilators is

$$q\partial_{p_i} - t\frac{d\tilde{g}}{dp_i}\ (1 \le i \le 3), \quad q^2\partial_q + t\tilde{g}, \quad q\partial_t - \tilde{g}$$

- $\mathcal{L}(\ldots)$: change $p_i \to -\partial_{p_i}$ and $\partial_{p_i} \to p_i$, and think modulo $\ker(\mathrm{res}_p)$
- $\exp(f(p)) \times \ldots$: change $\partial_{p_i} \to \partial_{p_i} - \frac{df}{dp_i}$
- $\mathrm{res}_{p,q}(\ldots)$: compute the integral of a module

Integration in D-Modules (Brochet, Chyzak, Lairez, 2026)

$$A(t) = \mathrm{even}_t\, \mathrm{res}_{p,q}\left(e^{f(p_1,p_2,p_3,q)}\,\mathcal{L}\left(e^{q^{-1}tg(1p_1,2p_2,3p_3)}\right)\right)$$

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holonomic system in $\mathbb{Q}(t)[p_1, p_2, p_3, q]\langle\partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \partial_q\rangle$ + action of ∂_t	$\longrightarrow$	$\dfrac{M}{\partial M}$ + action of ∂_t	$\to$	ODE in $\mathbb{Q}(t)\langle\partial_t\rangle$

Integration in D-Modules (Brochet, Chyzak, Lairez, 2026)

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holonomic system in $\mathbb{Q}(t)[p_1, p_2, p_3, q]\langle\partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \partial_q\rangle$ + action of ∂_t $\longrightarrow$ Julia $\rightarrow$ ODE in $\mathbb{Q}(t)\langle\partial_t\rangle$

Hadrien Brochet's `MultivariateCreativeTelescoping.jl`

Validity of the Algorithms

Role of holonomy

- Holonomic D-module $M \to$ its integral $M/\partial M$ is finite-dimensional.
- Holonomic system $\to$ integration algorithm has non-zero output.

(Brochet, Chyzak, Lairez, 2026)

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always:	holonomy $\Leftarrow$ D-finiteness
smooth functions:	holonomy $\Leftrightarrow$ D-finiteness

Validity of the Algorithms

Role of holonomy

- Holonomic D-module $M \longrightarrow$ its integral $M/\partial M$ is finite-dimensional.
- Holonomic system $\longrightarrow$ integration algorithm has non-zero output.

(Brochet, Chyzak, Lairez, 2026)

always:	holonomy $\Leftarrow$ D-finiteness
smooth functions:	holonomy $\Leftrightarrow$ D-finiteness

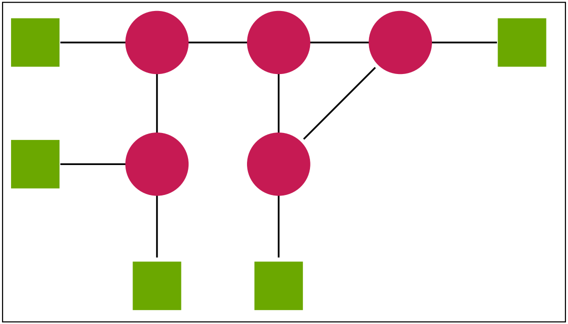
Creative telescoping for D-finite functions is inappropriate here

- D-finite functions $\longrightarrow$ work over $\mathbb{Q}(t, p_1, p_2, p_3, q)$.
- The system contains $p_3 q - t$
  $\longrightarrow$ classical algorithms wongly believe they integrate 0.

~~(Chyzak, 2000), (Koutschan, 2010)~~

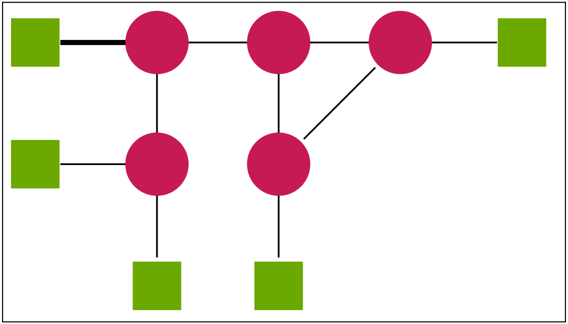
Third Approach:
By a Combinatorial Recurrence

Interrelated Combinatorial Classes



$\mathcal{G}_0$	labelled graphs with vertices of degree 3 or 1 only

Interrelated Combinatorial Classes



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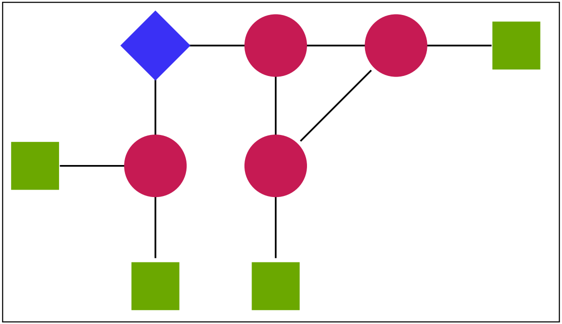

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$\mathcal{G}_1$	labelled graphs with vertices of degree at most 3, including exactly one of degree 2

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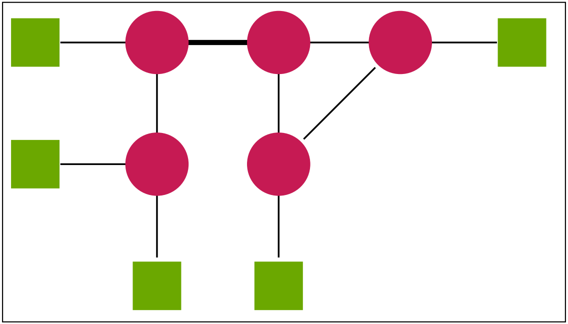

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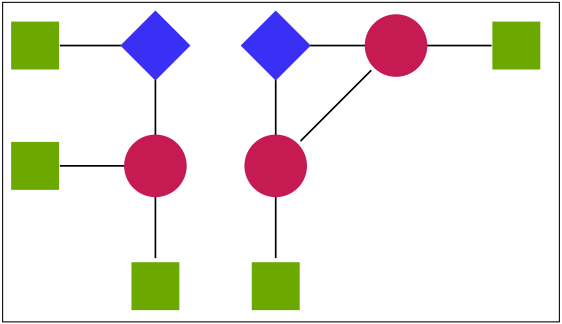

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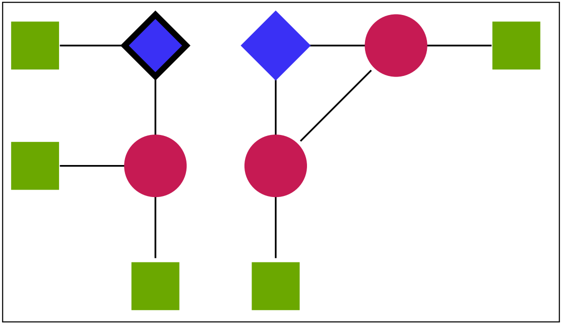

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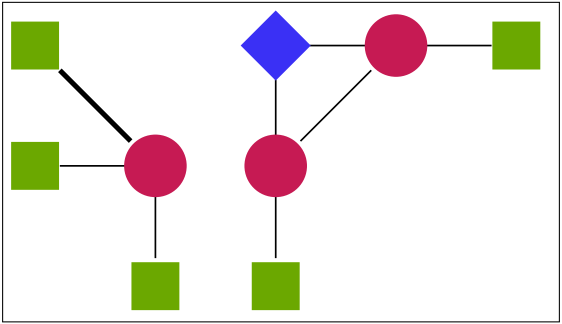

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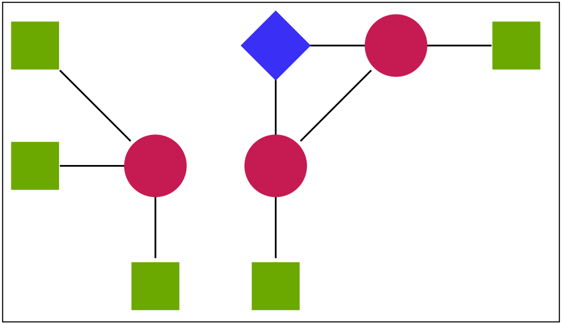

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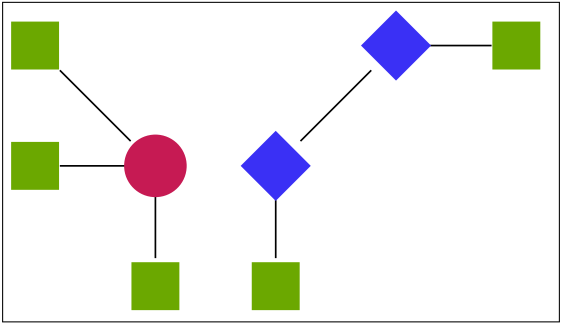

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$\tilde{\mathcal{G}}_2$	labelled graphs with vertices of degree at most 3, including exactly two of degree 2, which are not adjacent

A Differential System in Generating Functions

Four generating functions

$g_{a,b,c}$ = # of labelled graphs with a, b, c vertices of degree 1, 2, 3, $\tilde{g}_{a,2,c} = \dots$

$$G_b(q,t) = \sum_{a,c \geq 0} g_{a,b,c} q^{\frac{a+2b+3c}{2}} \frac{t^{a+b+c}}{(a+b+c)!}, \quad 0 \leq b \leq 2, \qquad \tilde{G}_2(q,t) = \dots$$

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One of five decompositions

element of $\mathcal{G}_0$ with a marked edge $\longrightarrow$ element of $\mathcal{G}_1$ or element of $\tilde{\mathcal{G}}_2$

$$\tfrac{1}{2}(a+3c)g_{a,0,c} = (a+c)g_{a-1,1,c-1} + \tilde{g}_{a,2,c-2}$$

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Similar differential equations for other classes or marking a vertex.

Algebraic Manipulations Get the Recurrence Relation

$$
\begin{aligned}
\partial_q \cdot G_0(q,t) &= tG_1(q,t) + \tilde{G}_2(q,t), \\
G_1(q,t) &= \tfrac{1}{2}q^2t^3G_0(q,t) + q^2t^2G_1(q,t) + q^2tG_2(q,t), \\
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$\downarrow$ elimination: module Gröbner basis over $\mathbb{Q}(q,t)\langle\partial_q,\partial_t\rangle$

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system for G_0 in $\mathbb{Q}(q,t)\langle\partial_q,\partial_t\rangle$

↓ diagonal: creative telescoping for D-finite functions

differential equation for $\displaystyle\sum_{k\geq 0} a_k \frac{t^{2k}}{(2k)!}$

Conclusion

Final Comments

Creative telescoping in computational proofs

- **triple sum**: CT for hypergeometric multi-summation
  traditionally: implicit assumption make proofs incomplete
  here: fix whose generality has to be explored
- **residues**: CT in D-modules for residues
  generally: ensuring a holonomic input system is difficult
  here: polynomial exponential make it simple
- **combinatorial recursion**: CT for the diagonal of a D-finite function
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- **combinatorial recursion**:
  computer enumeration was used to correct proof by cases