Gröbner Bases, Symbolic Summation and Symbolic Integration

[Tutorial]

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Linear Operators

- The Chebyshev polynomials

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$$

satisfy the equations:

$$(1-x^2){T_n}''(x) - x{T_n}'(x) + n^2 T_n(x) = 0,$$
$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0,$$
$$(1-x^2){T_{n+1}}'(x) - (n+1)xT_{n+1}(x) - (n+1)T_n(x) = 0.$$

- Equivalently, we use the operator notation:

$$(1-x^2)D_x^2 - xD_x + n^2,$$
$$S_n^2 - 2xS_n + 1,$$
$$(1-x^2)D_x S_n - (n+1)xS_n - (n+1).$$

$\Longrightarrow$ **Skew polynomials** *[Ore]*.

Operator Algebra

- Differential case: $\mathbb{K}(z)\langle D_z \rangle$

$$(zf(z))' = zf'(z) + f(z) \longleftrightarrow D_z z = z D_z + 1.$$

- Recurrence case: $\mathbb{K}(z)\langle S_z \rangle$

$$(z+1)f(z+1) = (z+1)f(z+1) \longleftrightarrow S_z z = (z+1) S_z.$$

- General framework of **Ore algebras**: $\mathbb{K}(x)\langle \partial \rangle$

$$\partial x = (qx + p)\partial + (sx + r).$$

Extension to the multivariate case: $\mathbb{K}(\mathbf{x})\langle \boldsymbol{\partial} \rangle = \mathbb{K}(x_1, \ldots, x_s)\langle \partial_1, \ldots, \partial_r \rangle$.

- **Annihilating ideals:**

$$\operatorname{Ann} f = \{P(\mathbf{x}, \boldsymbol{\partial}) \in \mathbb{K}(\mathbf{x})\langle \boldsymbol{\partial} \rangle \mid P \cdot f = 0\}.$$

Skew Gröbner Bases

• Easy extension of Buchberger's algorithm to the skew case: focus on left ideals; Gröbner bases still finite (noetherianity).

[Galligo, Takayama, Kandry-Rody and Weispfenning, Kredel]

Direct extension of the FGLM algorithm.

Direct extension to Gröbner bases for (left) modules.

Possible restriction of term orders allowed.

• For an Ore algebra $\mathbb{K}(x_1, \ldots, x_s)\langle\partial_1, \ldots, \partial_r\rangle$:

Zero-dimensional ideal $\longleftrightarrow$ ∂-finite function.

• For an Ore algebra $\mathbb{K}[x_1, \ldots, x_s]\langle\partial_1, \ldots, \partial_r\rangle$:

Dimension theory $\longleftrightarrow$ holonomic functions.

[Bernstein, Kashiwara, Zeilberger]

Closures of ∂-Finite Functions under $+$, $\times$, ∂: The FGLM Algorithm

$\longrightarrow$ Compute an annihilating system for $f+g$, $f \times g$, $\partial \cdot f$ for ∂-finite functions f and g.

• Example: the product

$$h = T_n(x) \times \frac{e^{-ux}}{\sqrt{1-x^2}}.$$

• Ore algebra: $\mathbb{O} = \mathbb{Q}(x,n,u)\langle D_x, S_n, D_u\rangle$.

Term order: total degree in $D_x \succ S_n \succ D_u$.

Gröbner basis for $f = T_n(x)$:

$$D_u, \qquad (x^2-1)D_x - nS_n + nx, \qquad S_n^2 - 2xS_n + 1.$$

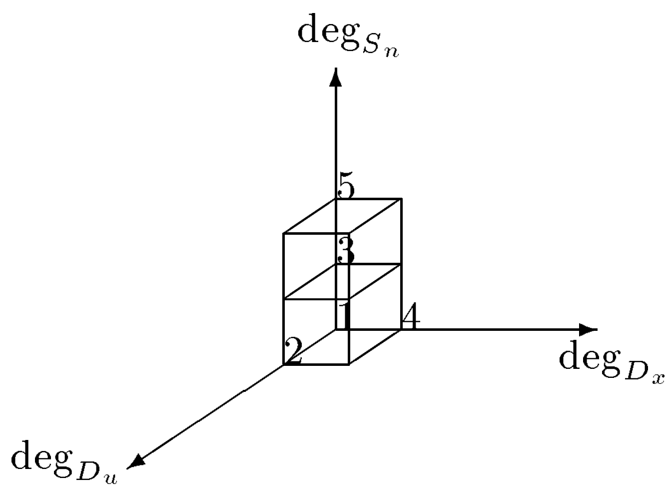
Gröbner basis for $g = \frac{e^{-ux}}{\sqrt{1-x^2}}$:

$$D_u + x, \qquad S_n - 1, \qquad (x^2-1)D_x + ux^2 + x - u.$$

$\Longrightarrow$ **Normal forms** in $\mathbb{O} \cdot f$ and $\mathbb{O} \cdot g$.

- In $\mathbb{O}$:

$$1 \prec \boxed{D_u} \prec S_n \prec \boxed{D_x}$$
$$\prec D_u^2 \prec D_u S_n \prec D_u D_x \prec \boxed{S_n^2} \prec S_n D_x \prec D_x^2 \ldots$$



- **Normal forms** in $\mathbb{O} \cdot f \otimes \mathbb{O} \cdot g$:

$$\begin{aligned}
1 \cdot (f \otimes g) &= f \otimes g, \\
D_u \cdot (f \otimes g) &= (D_u \cdot f) \otimes g + f \otimes (D_u \cdot g) = -x(f \otimes g), \\
S_n \cdot (f \otimes g) &= (S_n \cdot f) \otimes (S_n \cdot g) = (S_n \cdot f) \otimes g, \\
D_x \cdot (f \otimes g) &= (D_x \cdot f) \otimes g + f \otimes (D_x \cdot g) \\
&= (x^2 - 1)^{-1}[n(S_n \cdot f) \otimes g - (ux^2 + (n+1)x - u)(f \otimes g)], \\
S_n^2 \cdot (f \otimes g) &= (S_n^2 \cdot f) \otimes (S_n^2 \cdot g) = 2x(S_n \cdot f) \otimes g - f \otimes g.
\end{aligned}$$

- Gröbner basis **computed** for $f \otimes g$:

$$\boxed{D_u} + x, \quad (x^2 - 1)\boxed{D_x} - nS_n + ux^2 + (n+1)x - u, \quad \boxed{S_n^2} - 2xS_n + 1.$$

$\partial^{-1}|_\Omega$ of ∂-Finite Functions by Creative Telescoping: Gröbner Bases for Elimination

$\longrightarrow$ Compute an annihilating system for the **definite** sum or integral of a ∂-finite function.

• Example: the **definite** integral

$$H = \int_{-1}^{+1} h\,dx = \int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}}\,dx.$$

• Ore algebra: $\mathbb{O} = \mathbb{Q}[n, u, x]\langle D_x, S_n, D_u\rangle$.

Term order: $x \ggg D_x \succ S_n \succ D_u \succ n \succ u$.

Eliminating x yields another Gröbner basis for $\operatorname{Ann} h$ in $\mathbb{O}$:

$$S_n^2 + 2S_nD_u + 1,$$
$$D_x(D_u^2 - 1) - nS_n + uD_u^2 - (n-1)D_u - u, \qquad x + D_u.$$

By integration, we get

$$(S_n^2 + 2S_nD_u + 1) \cdot H = 0,$$
$$[nS_n - uD_u^2 + (n-1)D_u + u] \cdot H = [(D_u^2 - 1) \cdot h]_{x=-1}^{x=+1} = 0.$$

• Success related to the Hilbert dimension of $\operatorname{Ann} h$: here 3; f is holonomic.

$\partial^{-1}|_\Omega$ over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

$\longrightarrow$ Compute an annihilating system for the **definite** sum or integral of a ∂-finite function in the case of **natural boundaries**:

Compute the P's such that

$$[D_x P(n, u, D_x, S_n, D_u) - Q(x, n, u, D_x, S_n, D_u)] \cdot h = 0,$$

or equivalently,

$$P(n, u, D_x, S_n, D_u) \cdot H = [Q(x, n, u, D_x, S_n, D_u) \cdot h]_{x=-1}^{x=+1},$$

without computing the Q's.

- Idea:

if $$\operatorname{Ann} h = \sum_{i=1}^{\ell} \mathbb{Q}[n, u, x]\langle D_x, S_n, D_u\rangle g_i,$$

then $$\operatorname{Ann} h = \lim_{N\to\infty} \sum_{i=1}^{\ell} \sum_{j=0}^{N} \mathbb{Q}[n, u]\langle D_x, S_n, D_u\rangle x^j g_i.$$

- For $N = 3$:

$$1D_u + x, \qquad xD_u + x^2, \qquad x^2D_u + x^3,$$
$$1(-D_x - nS_n - u) + x(n+1) + x^2(u + D_x),$$
$$x(-D_x - nS_n - u) + x^2(n+1) + x^3(u + D_x),$$
$$1(S_n^2 + 1) - x2S_n, \qquad x(S_n^2 + 1) - x^2 2S_n, \qquad x^2(S_n^2 + 1) - x^3 2S_n.$$

- Taking remainder modulo D_x on the left by $p(x)D_x = D_x p(x) - p'(x)$ yields

$$D_u 1 + x, \qquad D_u x + x^2, \qquad D_u x^2 + x^3,$$
$$(-nS_n - u)1 + (n-1)x + ux^2,$$
$$1 + (-nS_n - u)x + (n-2)x^2 + ux^3,$$
$$(S_n^2 + 1)1 - 2S_n x, \qquad (S_n^2 + 1)x - 2S_n x^2, \qquad (S_n^2 + 1)x^2 - 2S_n x^3.$$

- Gröbner basis for $x \ggcurly S_n \succ D_u$:

$$\boxed{(-u - nS_n - (n-1)D_u + uD_u^2)1}, \qquad \boxed{(S_n^2 + 1 + 2D_uS_n)1},$$
$$D_u 1 + x, \qquad (-u - nS_n - (n-1)D_u)1 + ux^2,$$
$$(unS_nD_u + n(n-2)S_n - u + (u^2 + n^2 - 3n + 2)D_u + nu)1 + u^2x^3.$$

$\Longrightarrow$ Same two eliminated polynomials.

Closure of ∂-Finite Functions under Indefinite ∂^{-1}

$\longrightarrow$ Compute **indefinite** sums or integrals in $\mathbb{O} \cdot f$ for a ∂-finite function f.

- Example: compute the **indefinite** integral

$$F = \int f \, dx = \int (P \cdot h) \, dx = \int \left(P \cdot T_n(x) \frac{e^{-ux}}{\sqrt{1 - x^2}} \right) dx,$$

where $P = u^2 D_u^4 + 5u D_u^3 + (4 - 2u^2 - n^2) D_u^2 - 5u D_u + n^2 + u^2 - 2.$

- Gröbner basis for h (total degree in $D_x \succ S_n \succ D_u$):

$$D_u + x, \qquad (x^2 - 1) D_x - n S_n + u x^2 + (n+1) x - u, \qquad S_n^2 - 2x S_n + 1.$$

- Look for $F \in \mathbb{Q}(x, n, u)\langle D_x, S_n, D_u \rangle \cdot h$ under the form

$$F = Q \cdot h = (\phi + \psi S_n) \cdot h.$$

- **Reducing** $D_x Q + P$ and solving first order system yields

$$\phi(x) = (x^2 - 1)(u x^2 - (n+2) x - u), \qquad \psi(x) = n(x^2 - 1).$$

- Decision algorithm, given an algorithm for rational solutions of first order systems.

$\partial^{-1}|_\Omega$ of ∂-Finite Functions : Extension of Zeilberger's Algorithm

$\longrightarrow$ Compute an annihilating system for the **definite** sum or integral of a ∂-finite function.

• Example: the **definite** integral

$$H = \int_{-1}^{+1} h\,dx = \int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}}\,dx.$$

• Set $P = \eta_0 + \eta_1 D_u + \eta_2 D_u^2$, and determine for which η_i's the **indefinite** integral

$$F = \int f\,dx = \int (P \cdot h)\,dx = P \cdot \int h\,dx$$

can be computed by the previous algorithm ($F = Q \cdot h$).

• Solution:

$$P = u^2 + n^2 - uD_u - u^2 D_u^2, \qquad Q = (-ux^2 + nx + u) - nS_n.$$

$$[Q \cdot h]_{x=-1}^{x=+1} = 0 \Longrightarrow P \cdot H = 0.$$

• FGLMization to obtain the Gröbner basis ($S_n \succ D_u$):

$$u^2 D_u^2 + uD_u - u^2 - n^2, \qquad uS_n + uD_u - n.$$