A Solution to a Summation Problem by Feng Qi

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January 15, 2017

On Research Gate, Feng Qi asked[1] for a proof of the identity

$$\sum_{\ell=0}^{m-1} f_{m,\ell} = 0 \tag{1}$$

whenever

$$n \geq 2m, \qquad m \geq 1$$

for

$$f_{m,\ell} = (-1)^\ell \left(m(2m-1) - \frac{(2m-\ell-1)(n+2\ell+1)}{n-\ell+1} \right) \times \frac{(2m-\ell-2)!(2(n-\ell-1))!}{\ell!(2m-2\ell-1)!(n-\ell)!(n-\ell-1)!}.$$

Here, we define this sequence for $0 \leq \ell \leq m-1$, implying that $n-\ell+1$ does not vanish and is thus well defined.

Introduce

$$g_{m,\ell} = -\frac{(-2n+2\ell+1)(-2m+\ell+1)\ell}{\delta_{m,\ell}} f_{m,\ell} \tag{2}$$

where

$$\delta_{m,\ell} = -2\ell m^2 + 2m^2 n + 2\ell^2 - 3\ell m + \ell n + 2m^2 - 3mn + 3\ell - 3m + n + 1,$$

and observe

$$f_{m,\ell} = g_{m,\ell+1} - g_{m,\ell}, \tag{3}$$

provided all three terms are well defined. In view of Equation (2), $g_{m,\ell}$ is defined whenever $\delta_{m,\ell} \neq 0$. To analyze this constraint, setting $n = 2m + p$ and $m = \ell + 1 + q$ for nonnegative p and q yields

$$\delta_{m,\ell} = 2\ell^3 + 2\ell^2 p + 8\ell^2 q + 4\ell pq + 10\ell q^2 + 2pq^2 + 4q^3 + 5\ell^2 + 2\ell p + 11\ell q + pq + 8q^2 + 3\ell + 3q,$$

[1] https://www.researchgate.net/post/Can_one_verify_an_identity_involving_factorials

which is then positive unless $p = q = 0$. Consequently, $g_{m,\ell}$ is well defined whenever $0 \leq \ell \leq m-1$, and Equation (3) holds whenever $0 \leq \ell \leq m-2$.

For $m = 1$, the sum in Equation (1) reduces to $f_{1,0}$, which is seen to be zero. We continue with $m \geq 2$. Isolating the term $f_{m,m-1}$ in the sum in Equation (1) and evaluating the rest as a telescoping sum delivers

$$\sum_{\ell=0}^{m-1} f_{m,\ell} = g_{m,m-1} - g_{m,0} + f_{m,m-1}. \tag{4}$$

The rational function $g_{m,\ell}/f_{m,\ell}$ is zero when $\ell = 0$ and is -1 when $\ell = m-1$, so that the three right-hand terms in Equation 4 add up to zero.

We have thus proved Equation (1).