Introduction to the Mgfun package, a package for Multivariate Generating FUnctions.

The Mgfun package is used to perform calculations on multivariate functions and sequences. In particular, it is well suited for calculations on so-called special functions, including hypergeometric functions and sequences, their q-analogues and holonomic functions. In the philosophy of Mgfun, these functions and sequences are described by systems of linear homogeneous differential or difference equations, or by q-analogues of them. A natural tool that plays a central role in these calculations is a suitable generalization of Groebner bases.

When using the Mgfun package, any function or sequence is represented by a set of linear homogeneous equations it satisfies. When such systems are known for functions f(x,y) and g(x,y), Mgfun can compute a system for the sum f(x,y)+g(x,y), the product f(x,y)g(x,y) or definite integrals of f(x,y) with respect to y. In the case of sequences u[n,m] and v[n,m], it can compute equations for the sum u[n,m]+v[n,m], the product u[n,m]v[n,m] or definite summations of u[n,m] over all m. Mgfun can also deal with mixed cases, such as f[n,m](x,y), or with other types of linear equations the user might want to define.

Using these basic operations, one can: - find closed forms (in terms of hypergeometric functions and sequences and their q-analogues); - compute (ordinary, exponential, Bessel...) generating functions, - extract coefficients of a series (for any orthogonal basis of the algebra of series in a general class); - check identities and prove new ones; - compute diagonals of series.

This worksheet gives an overview of Mgfun through some applications.

Once you have installed Mgfun, the following call should run with no error.

```
> with(Mgfun);
```

$$[algtoholon, annihilators, applyopr, commalg, dependency, fglm, gbasis, hdefint, hdefqsum, hdefsum, hdiag, hilbertdim, hpowerrs, hprod, hprodrs, hsum, hsumrs, hypergeomtoholon, leadcoeff, leading\_term, leadmon, makeopr, oppower, opprod, orealg, qbinomial, qfactorial, qpochhammer, randopr, reduce, reducelist, reducescale, setparam, shiftalg, skewelim, skewgcdex, skewpdiv, skewprem, spoly, termorder, testorder, weylalg]$$

We want Mgfun to be verbose.

```
> infolevel[Mgfun`warnings]:=1:
```

The Legendre polynomials.

As mentioned above, definite summations can be computed using the Mgfun package. As an example, we show how to calculate equations that describe the Legendre polynomials P[n](x). The n-th polynomial of this sequence is defined as the sum for all k of binomial(n,k)^2*(x-1)^k*(x+1)^(n-k)*2^(-n). Let us now compute this sum.

We first declare an algebra of operators to enter equations that describe the summand. We need differential equations with respect to x and recurrence equations with respect to n and k. It will also be possible to enter mixed differential-recurrence equations.

```
> A:=orealg(diff=[Dx,x],shift=[Sn,n],shift=[Sk,k]):
```

Most calculations by Mgfun on multivariate functions make intensive use of Groebner bases computations. Here, we declare an elimination term order to prepare an elimination of k by Groebner basis computation, since k is the index of the summation we want to compute.

```
> T:=termorder(A,lexdeg=[[k],[Dx,Sn,Sk]]):
```

We now enter equations satisfied by the term to be summed.

```
> G:=hypergeomtoholon(binomial(n,k)^2*(x-1)^k*(x+1)^(n-k)*2^(-n),A);
```

$$\begin{aligned} G := [&(-x^2+1)\,Dx + 2\,k + n\,x - n,\\ &(-2\,n^2 - 4\,n + 4\,n\,k - 2 + 4\,k - 2\,k^2)\,Sn + n^2\,x + n^2 + 2\,n\,x + 2\,n + x + 1,\\ &(-k^2\,x - k^2 - 2\,k\,x - 2\,k - x - 1)\,Sk + n^2\,x - n^2 - 2\,n\,k\,x + 2\,n\,k + k^2\,x - k^2] \end{aligned}$$

And we compute equations satisfied by the Legendre polynomials P[n](x).

```
> GL[P]:=hdefsum(G,k=0..infinity,T);
```

```
Mgfun/Holonomy/definitesum:
Assume the summand and its shifts vanish in 0 and infinity
```

$$GL_P := [\,2\,Sn\,n\,x + 3\,Sn\,x - n - 1 - Sn^2\,n - 2\,Sn^2, -n\,x + Sn\,n - Dx\,x^2 + Dx + Sn - x\,]$$

A particular use of summation is to compute generating function. As an example, we give here the generating function of Legendre polynomials. Recall that it is defined as phi(x,y)=sum(P[n](x)y^n,n=0..infinity).

We follow the same method as above, beginning by declaring another algebra and another term order well-suited for elimination of the index of summation n.

```
> A:=orealg(diff=[Dx,x],diff=[Dy,y],shift=[Sn,n]):
> T:=termorder(A,lexdeg=[[n],[Dx,Dy,Sn]]):
> BR:=ratpoly(rational,[x,y]):
```

From a description of P[n](x), we derive a description of P[n](x)y^n.

```
> G:=[op(numer(subs(Sn=Sn/y,GL[P]))),y*Dy-n]:
```

We next derive a description for their generating function phi(x,y).

```
> GL[phi]:=hdefsum(G,n=0..infinity,T);
```

```
Mgfun/Holonomy/definitesum:
Assume the summand and its shifts vanish in 0 and infinity
```

$$GL_\phi := [\,2\,x\,y\,Dx - y^2\,Dx + y - Dx, 2\,x\,y\,Dy + x - y^2\,Dy - y - Dy\,]$$

Solving these differential equations by separation of variables yields a closed form for the generating function.

```
> applyopr(GL[phi][1],phi(x,y),A);
```

$$y\,\phi(\,x,y\,) + (\,-y^2 + 2\,x\,y - 1\,)\left(\frac{\partial}{\partial x}\,\phi(\,x,y\,)\right)$$

```
> pdesolve(",phi(x,y));
```

$$\phi(x,y) = \frac{\_F1(y)}{\sqrt{-y^2+2\,x\,y-1}}$$

Any solution must be of the previous form. The equation in derivatives with respect to y makes it possible to compute _F1(y).

```
> numer(normal(applyopr(GL[FG][2],op(2,"),A)));
```

$$GL_{FG\,2}\,\_F1(y)$$

Since the previous expression is identically zero, _F1(y) is a constant (at least in a neighbourhood of 0) and phi(x,0)=P[0](x)=1 yields the following generating function.

```
> phi:=1/sqrt(1-2*x*y+y^2);
```

$$\phi := \frac{1}{\sqrt{1-2\,x\,y+y^2}}$$

Computing integrals using Mgfun is very similar to computing sums. As an example, we prove that Legendre polynomials form an orthogonal family for the scalar product <f(x),g(x)>=int(f(x)g(x),x=-1..1).

We prove this by integration of the bivariate generating function h of the products J[n](x)J[m](x) between -1 and +1, so as to find the generating function of the scalar products <J[n](x),J[m](x)>. So we first declare another algebra and define h.

```
> A:=weylalg([Dx,x],[Du,u],[Dv,v]):
> h:=subs(y=u,phi)*subs(y=v,phi);
```

$$h := \frac{1}{\sqrt{1-2\,x\,u+u^2}\,\sqrt{1-2\,x\,v+v^2}}$$

```
> G:=hypergeomtoholon(h,A);
```

$$\begin{aligned} G := [&(-1+2\,x\,v-v^2+2\,x\,u-4\,x^2\,u\,v+2\,x\,u\,v^2-u^2+2\,u^2\,x\,v-u^2\,v^2)\,Dx+u \\ &-4\,u\,x\,v+u\,v^2+v+v\,u^2, (-1+2\,x\,u-u^2)\,Du+x-u, \\ &(-1+2\,x\,v-v^2)\,Dv+x-v] \end{aligned}$$

Once again, we will perform an elimination by calculation of a Groebner basis. However, the existing function hdefint of Mgfun is not suited for this computation, since h does not vanish at -1 and +1. Here we need to eliminate the variable of integration, x, by lower-level tools.

```
> T:=termorder(A,lexdeg=[[x],[Dx,Du,Dv]]):
```

The Groebner basis calculation is performed by a call to gbasis. This function is Mgfun's own routine, which performs more efficiently than grobner[gbasis] in the case of commutative calculations. We keep only those polynomials that are free of x.

```
> EL:=collect(remove(has,gbasis(G,T,ratpoly(rational,[u,v])),x),Dx,factor);
```

$$\begin{aligned} EL := [&-2\,Du\,u\,Dv-2\,Du\,u\,Dv\,v^2-2\,Du\,u\,v+2\,Dv\,v\,Du+2\,Dv\,v\,Du\,u^2+2\,Dv\,v\,u \\ &+Du+Du\,u^2+u-Dv-Dv\,v^2-v, (v-1)\,(v+1)\,(u-1)\,(u+1)\,Dx+u\,v^2 \\ &-2\,Dv\,v^2+2\,Dv\,u^2\,v^2+2\,Du\,u^2\,v^2-v+v\,u^2-u-2\,Du\,u^2] \end{aligned}$$

The previous operations are known as Zeilberger's method of creative telescoping. They are usually followed by the substitution Dx=0. This would correspond to cases where h(+1)=h(-1)=0. Here, we need to take h(+1)-h(-1) into account. Let us give the following notation.

> C[0]:=coeff(EL[2],Dx,0);

$$C_0 := u\,v^2 - 2\,Dv\,v^2 + 2\,Dv\,u^2\,v^2 + 2\,Du\,u^2\,v^2 - v + v\,u^2 - u - 2\,Du\,u^2$$

> C[1]:=coeff(EL[2],Dx,1);

$$C_1 := (\,v-1\,)\,(\,v+1\,)\,(\,u-1\,)\,(\,u+1\,)$$

By integration, EL[2] yields the following non-homogeneous equation.

> C[0]*Int(h,x=-1..1)=-C[1]*('h'(x,1)-'h'(x,-1));

$$\begin{aligned}&(\,u\,v^2 - 2\,Dv\,v^2 + 2\,Dv\,u^2\,v^2 + 2\,Du\,u^2\,v^2 - v + v\,u^2 - u - 2\,Du\,u^2\,)\\ &\quad\int_{-1}^{1} \frac{1}{\sqrt{1-2\,x\,u+u^2}\,\sqrt{1-2\,x\,v+v^2}}\,dx =\\ &\quad -(\,v-1\,)\,(\,v+1\,)\,(\,u-1\,)\,(\,u+1\,)\,(\,\mathrm{h}(\,x,1\,)-\mathrm{h}(\,x,-1\,)\,)\end{aligned}$$

Let us compute C[1]*(h(x,1)-h(x,-1)).

> A:=weylalg([Du,u],[Dv,v]):
> simplify(C[1]*normal(subs(x=1,h)-subs(x=-1,h)),symbolic);

$$2\,u + 2\,v$$

Generally, this difference has to be holonomic (i.e., it is always possible to get equations that describe it using the theory of holonomy). In the present case, it is a polynomial. We thus get simple linear operators that cancel on them.

> hypergeomtoholon(",A);

$$[\,(\,-u-v\,)\,Du+1, (\,-u-v\,)\,Dv+1\,]$$

Applying these operators to the right-hand side of the integral equation above yields zero. Therefore, it is the same when we apply them to the left-hand side. Composition of operators translates into product.

> G:=map(collect,[EL[1],op(map(opprod,",C[0],A))],[Du,Dv],distributed,factor);

$$\begin{aligned}G := \Big[&u - v + 2\,(\,u\,v-1\,)\,(\,u-v\,)\,Du\,Dv + (\,-v^2+2\,u\,v-1\,)\,Dv + (\,-2\,u\,v+1+u^2\,)\,Du,\\ &-v\,(\,u+v\,)^2 - 2\,v^2\,(\,u-1\,)\,(\,u+1\,)\,(\,u+v\,)\,Du\,Dv - 2\,v^2\,(\,u^2+2\,u\,v+1\,)\,Dv\\ &+ (\,6\,u\,v - 4\,u^2\,v^2 + 3\,u^2 + v^2 - v\,u^3 - 5\,u\,v^3\,)\,Du\\ &- 2\,u^2\,(\,v-1\,)\,(\,v+1\,)\,(\,u+v\,)\,Du^2, -u\,(\,u+v\,)^2\\ &- 2\,u^2\,(\,v-1\,)\,(\,v+1\,)\,(\,u+v\,)\,Du\,Dv\\ &+ (\,3\,v^2 - 4\,u^2\,v^2 + 6\,u\,v - u\,v^3 - 5\,v\,u^3 + u^2\,)\,Dv - 2\,u^2\,(\,1+v^2+2\,u\,v\,)\,Du\\ &- 2\,v^2\,(\,u-1\,)\,(\,u+1\,)\,(\,u+v\,)\,Dv^2\Big]\end{aligned}$$

Recall that these operators cancel on the integral psi(u,v) we want to determine. Now, we compute a Groebner basis for another purpose, namely to get a normal form for this set of equations.

```
> T:=termorder(A,tdeg=[Du,Dv],max):
> BR:=ratpoly(rational,[u,v]):
> GL[psi]:=collect(gbasis(G,T,BR),[Du,Dv],distributed,factor);
```

$$\begin{aligned}GL_\psi := [&-u\,v\,(u-v)\,(u+v) - v^2\,(-v-3\,v\,u^2+3\,u+u^3)\,Dv \\ &- u\,(-5\,u\,v^3+4\,v^2+3\,u^2\,v^2+4\,v\,u^3-3\,u\,v-3\,u^2)\,Du \\ &- 2\,u^2\,(u-v)\,(u\,v-1)\,(u+v)\,Du^2, \\ &-u+v-2\,(u\,v-1)\,(u-v)\,Du\,Dv+(v^2+1-2\,u\,v)\,Dv+(-u^2-1+2\,u\,v)\,Du, \\ &u\,v\,(u-v)\,(u+v)+v\,(5\,v\,u^3-3\,u^2\,v^2+3\,u\,v-4\,u\,v^3-4\,u^2+3\,v^2)\,Dv \\ &+ u^2\,(-3\,v+3\,u\,v^2-v^3+u)\,Du+2\,v^2\,(u-v)\,(u\,v-1)\,(u+v)\,Dv^2]\end{aligned}$$

To solve this system in psi(u,v), we separate variables. To compute an equation that involves derivatives with respect to u only, we use the function dependency, that searches for a dependency between successive derivatives with respect to u.

```
> eq_Du:=collect(dependency(psi(u,v),u,3,GL,T,BR),Du,factor);
```

$$\begin{aligned}eq\_Du := &-4\,u^2\,(-v-3\,v\,u^2+3\,u+u^3)\,(u-v)\,(u\,v-1)\,Du^3 - 4u(5\,u^5\,v-21\,u^4\,v^2 \\ &- 3\,u^4+12\,u^3\,v^3+34\,v\,u^3-29\,u^2\,v^2-15\,u^2+6\,u\,v^3+15\,u\,v-4\,v^2)Du^2+(\\ &-17\,u^5\,v+78\,u^4\,v^2+3\,u^4-21\,u^3\,v^3-129\,v\,u^3+89\,u^2\,v^2+45\,u^2-27\,u\,v^3 \\ &- 30\,u\,v+9\,v^2)Du - v\,(u^4-6\,v\,u^3-3\,u^2\,v^2+15\,u^2-10\,u\,v+3\,v^2)\end{aligned}$$

For symmetry reasons, a similar equation could be obtained for derivatives with respect to v. At this stage, we have to solve to get a solution with parameter v. Maple's dsolve does not return any solution. So we look for a function of the form sigma(uv). We change variables by z=uv.

```
> collect(numer(subs([u=z/v,Du=v*Dz],eq_Du)),Dz,factor);
```

$$\begin{aligned}&-4\,z^2\,v\,(v^4+3\,z^2\,v^2-3\,z\,v^2-z^3)\,(-z+v^2)\,(z-1)\,Dz^3 - 4z\,v(5\,z^5-21\,z^4\,v^2-3\,z^4 \\ &+ 12\,z^3\,v^4+34\,z^3\,v^2-29\,z^2\,v^4-15\,z^2\,v^2+6\,z\,v^6+15\,z\,v^4-4\,v^6)Dz^2-(17\,z^5 \\ &- 78\,z^4\,v^2-3\,z^4+21\,z^3\,v^4+129\,z^3\,v^2-89\,z^2\,v^4-45\,z^2\,v^2+27\,z\,v^6+30\,z\,v^4 \\ &- 9\,v^6)v\,Dz - v\,(z^4-6\,z^3\,v^2-3\,z^2\,v^4+15\,z^2\,v^2-10\,z\,v^4+3\,v^6)\end{aligned}$$

Now, we are working in the (z,v) plane, so we set v=1 to get an equation satisfied by any solutions of the form sigma(uv).

```
> collect(primpart(subs(v=1,"),Dz),Dz,factor);
```

$$-4\,z^2\,(z-1)^2\,Dz^3-4\,z\,(5\,z-4)\,(z-1)\,Dz^2+(-17\,z^2+30\,z-9)\,Dz-z+3$$

Maple is now able to solve.

```
> A:=weylalg([Dz,z]):
> dsol:=dsolve(-applyopr("",sigma(z),A)",sigma(z));
```

$$dsol := \sigma(z) = \frac{\_C1}{\sqrt{z}} + \frac{\_C2\,\mathrm{arctanh}\,(\sqrt{z})}{\sqrt{z}} + \frac{\_C3\,(-\ln(z)+2\ln(z-1))}{\sqrt{z}}$$

Let us compute an asymptotic series at 0 to be able to use initial conditions in 0, so as to determine the integration constants `C1, `C2 and `C3.

> series(op(2,dsol),z,2);

$$\frac{\_C1 + \_C3\left(\ln\left(\frac{1}{z}\right) - 2\,I\,\mathrm{csgn}(\,I\,(\,z-1\,)\,)\,\pi\right)}{\sqrt{z}} + \_C2 - 2\,\_C3\,\sqrt{z} + \frac{1}{3}\,\_C2\,z + \mathrm{O}(\,z^{3/2}\,)$$

Since <P[0](x),P[0](x)>=2, i.e., sigma(z)=2, we find the generating function of the scalar products.

> sigma:=2*arctanh(sqrt(z))/sqrt(z):
> subs(z=u*v,sigma);

$$2\,\frac{\mathrm{arctanh}\,(\sqrt{u\,v})}{\sqrt{u\,v}}$$

This proves that the family of Legendre polynomials is orthogonal. It is now possible to compute the squares of the modules ——P[n](x)——^2=<P[n](x),P[n](x)>. We first reduce the order of the equation that defines GF. To do so, we use the method of indetermined coefficients to get a second order differential equation.

> solve(–coeffs(numer(normal(a*sigma+b*diff(sigma,z)+c*diff(sigma,z,z))),ar
> ctanh(sqrt(z)))",–a,b,c");

$$\{\,b = 5\,a\,z - 3\,a, c = 2\,a\,z^2 - 2\,a\,z, a = a\,\}$$

> DE`sigma:=normal(primpart(subs(",a+b*Dz+c*Dz^2),Dz));

$$DE\_sigma := 1 + 5\,Dz\,z - 3\,Dz + 2\,Dz^2\,z^2 - 2\,Dz^2\,z$$

We easily check this equation.

> normal(applyopr(DE`sigma,sigma,A));

$$0$$

The following series expansion makes us conjecture that there is a closed form for the coefficients, namely 2/(2*n+1). We now proceed to prove this conjecture.

> series(sigma,z,10);

$$2 + \frac{2}{3}\,z + \frac{2}{5}\,z^2 + \frac{2}{7}\,z^3 + \frac{2}{9}\,z^4 + \frac{2}{11}\,z^5 + \frac{2}{13}\,z^6 + \frac{2}{15}\,z^7 + \frac{2}{17}\,z^8 + \frac{2}{19}\,z^9 + \mathrm{O}(\,z^{19/2}\,)$$

The differential equation that defines sigma can easily be translated in terms of coefficients. To do so, we use Salvy and Zimmermann's gfun package to derive a recurrence equation satisfied by the coefficients of sigma.

> with(share):

See ?share and ?share,contents for information about the share l
ibrary

> readshare(gfun,analysis):

> with(gfun):

> diffeqtorec(applyopr(DE`sigma,f(z),A),f(z),u(n));

$$(\,2\,n+1\,)\,\mathrm{u}(\,n\,) + (\,-2\,n-3\,)\,\mathrm{u}(\,n+1\,)$$

And Maple is able to solve this recurrence equation.

```
> rsolve(–",u(0)=2",u(n));
```

$$2\,\frac{1}{2\,n+1}$$

A particular case of calculation of integrals is taking the coefficients of a series by Cauchy's formula. We exemplify it by computing again the squares of the modules ——P[n](x)——ˆ2=<P[n](x),P[n](x)> in this more direct way.

We first divide the generating function GF of the squares of the modules by xˆ(n+1).

```
> A:=orealg(diff=[Dz,z],shift=[Sn,n]):
> numer(normal(applyopr(collect(DE˙sigma,Dz,factor),zˆ(n+1)*f(z),A)/zˆ(n+1)
> ));
```

$$6\,\mathrm{f}(\,z\,)\,z+2\,z\,\mathrm{f}(\,z\,)\,n^2+7\,z\,\mathrm{f}(\,z\,)\,n+4\,\left(\frac{\partial}{\partial z}\,\mathrm{f}(\,z\,)\right)\,z^2\,n+9\,\left(\frac{\partial}{\partial z}\,\mathrm{f}(\,z\,)\right)\,z^2+2\,\left(\frac{\partial^2}{\partial z^2}\,\mathrm{f}(\,z\,)\right)\,z^3$$
$$-\,2\,\mathrm{f}(\,z\,)\,n^2-5\,\mathrm{f}(\,z\,)\,n-4\,\left(\frac{\partial}{\partial z}\,\mathrm{f}(\,z\,)\right)\,z\,n-7\,\left(\frac{\partial}{\partial z}\,\mathrm{f}(\,z\,)\right)\,z-2\,\left(\frac{\partial^2}{\partial z^2}\,\mathrm{f}(\,z\,)\right)\,z^2-3\,\mathrm{f}(\,z\,)$$

```
> collect(subs([diff(f(z),z,z)=Dzˆ2,diff(f(z),z)=Dz,f(z)=1],"),Dz,factor);
```

$$2\,z^2\,(\,z-1\,)\,Dz^2+z\,(\,9\,z+4\,z\,n-7-4\,n\,)\,Dz+(\,2\,n+3\,)\,(\,z\,n-n+2\,z-1\,)$$

We eliminate z since the integration is with respect to this variable. Note the trick of integrating from 1 to 1! This means integration on any contour from and to 1 on which the integrand is well defined.

```
> T:=termorder(A,lexdeg=[[z],[Dz,Sn]],max):
> hdefint(["",z*Sn-1],z=1..1,T);
```

Mgfun/Holonomy/definiteintegral:
Assume the integrand and its derivatives vanish in 1 and 1

$$[\,2\,n-2\,Sn\,n-3\,Sn+1\,]$$

And Maple is able to solve the corresponding recurrence.

```
> rsolve(–applyopr(op("),u(n),A),u(0)=2",u(n));
```

$$2\,\frac{1}{2\,n+1}$$