SOME IDENTITIES THAT CAN BE PROVED USING MGFUN

1. Cassini's identity on the Fibonacci numbers

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^n,$$

with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

2. A double sum identity

$$\sum_{i,j=0}^{\infty} \binom{n}{i}\binom{n}{j}\binom{i+j}{i,j} = \sum_{n=0}^{\infty} \binom{n}{i}^2 4^i.$$

This identity was left to me by Philippe Flajolet as an exercise.

3. An identity between Franel and Apéry numbers

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3.$$

Several proofs of this identity were given in [6]. Both sides of this equation satisfy the equation

$$(n+2)^3 u_{n+2} - \left((n+2)^3 + (n+1)^3 + 4(2n+3)^3\right) u_{n+1} + (n+1)^3 u_n.$$

This operator was used by Apéry in his proof of the irrationality of $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$.

4. Combinatorial sums with non-natural boundaries

The following examples were given in [7].

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^n, \qquad \sum_{k=0}^{n} \frac{(2n-2k)!(2k)!}{k!(k+1)!(n-k)!^2} = \binom{2n+1}{n}.$$

5. A challenging sum with non-natural boundaries

The following identity was suggested by William Steiger <steiger@cs.princeton.edu> as a challenge for CAS.

$$\sum_{k=1}^{n-r} \frac{1}{k}\binom{n-k}{r} = \binom{n}{r}\sum_{k=r+1}^{n}\frac{1}{k}.$$

Both expressions are the probability that in a random permutation, 1 is the first element smaller than the minimum of the first r elements.

6. Generating function of the Jacobi polynomials

$$\sum_{n=0}^{\infty} P_n^{(a,b)}(x)y^n = \frac{1}{\sqrt{1-2yx+y^2}\left(1-y+\sqrt{1-2yx+y^2}\right)^a\left(1+y+\sqrt{1-2yx+y^2}\right)^b},$$

where $P_n^{(a,b)}(x)$ are the Jacobi polynomials. [Here, Maple's `dsolve` is not powerful enough to solve the differential equation found for the left-hand side, but the right-hand satisfies this equation with the right initial conditions.]

7. The Mehler formula for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}},$$

where $H_n(x)$ are the Hermite polynomials. See [4] for a combinatorial proof or [1] for a WZ-based proof.

8. Doetsch identity for Hermite polynomials

$$\int_{y=-\infty}^{\infty} \frac{(1+xy+2y^2)\exp\left(\frac{x^2y^2}{1+2y^2}\right)}{y^{n+1}(1+2y^2)^{\frac{3}{2}}}dx = H_n(x),$$

where $H_n(x)$ are the Hermite polynomials. See [3] for a combinatorial proof or [1] for a WZ-based proof.

9. A Rogers-Ramanujan identity

The following identity is a finite version due to Andrews of one of the famous Rogers-Ramanujan identities:

$$\sum_k \frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}} = \sum_k \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}.$$

10. Dixon's identity and its q-analogue

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \binom{a+b+c}{a,b,c},$$

and

$$\sum_k (-1)^k q^{k(3k+1)/2} \binom{a+b}{a+k}_q \binom{a+c}{c+k}_q \binom{b+c}{b+k}_q = \frac{(q;q)_{a+b+c}}{(q;q)_a (q;q)_b (q;q)_c}.$$

11. A doubly infinite sum of Ramanujan's

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n x^n}{(b;q)_n} = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}.$$

Here, we found a q recurrence equation for the left-hand side. Resolution would need an implementation of the q-Petkovšek algorithm in Maple, which is not available. However, the right-hand side satisfies this equation with the right initial conditions.

12. An identity between an exp-log integral and a harmonic sum

$$\int_0^1 \frac{x \ln^2 x \exp(-px)}{1-x} dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \psi''(n+1) p^n}{n!} - 2\,{}_3F_3\left(\begin{matrix}1,1,1\\2,2,2\end{matrix}; -p\right),$$

where ψ'' denotes the second derivative of $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ and satisfies

$$-\frac{\psi''(n+1)}{2} = \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3},$$

and the ${}_3F_3$-hypergeometrique function satisfies

$$-2\,{}_3F_3\left(\begin{matrix}1,1,1\\2,2,2\end{matrix}; -p\right) = -\frac{\int_1^p \frac{E_1(u)}{u} dp + 2\gamma \ln p + \ln^2 p - \frac{385019}{216000}}{p},$$

where γ and E_1 denote Euler's constant $\gamma = 0.5772156649\ldots$ and the exponential integral $E_1(z) = \int_z^\infty \frac{\exp(-t)}{t} dt$ respectively.

13. Abel's identity

$$\sum_{k=0}^{n} \binom{n}{k} (k+r)^{k-1} (n-k+s)^{n-k} = \frac{(n+r+s)^n}{r}.$$

A WZ-based proof can be found in [2, 5]; the more general holonomic approach also extend to Abel-type identities.

14. A sum suggested on the news by Eric Sparla <sparla@mathematik.uni-stuttgart.de>

$$\sum_{j=0}^{d} \binom{d}{j} \binom{d-r-j-1}{r-j} = \frac{2^r}{r!} \prod_{i=0}^{r-1} d-1-2i.$$

15. A problem suggested on the news by Eric Sparla <sparla@mathematik.uni-stuttgart.de>

The sum $u_{l,m}$

$$\sum_{i=0}^{m} (-2)^i \binom{2l-i}{l} \binom{m}{i}$$

satisfies

$$u_{j,2k+1} = 0, \text{ for } 0 \leq k < j, \qquad \text{and} \qquad u_{l,2l+1} = (-4)^l.$$

16. A double sum suggested on the news by Len Trigg <trigg@jane.cs.waikato.ac.nz>

$$\sum_{k,l\in\mathbb{N}} \frac{(2k+2l+i+j)!}{(k+i)!(l+j)!k!l!} \left(\frac{x}{4}\right)^{k+l} = \binom{i+j}{i,j} {}_4F_3\left(\begin{matrix} \frac{i+j}{2}+1, \frac{i+j}{2}+1, \frac{i+j+1}{2}, \frac{i+j+1}{2} \\ i+j+1, i+1, j+1 \end{matrix}; 4x\right).$$

This double sum appears when modeling a simple 5-instruction machine which goes to North, East, South or West with probability $x/4$ each and stops with probability $1-x$.

References

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