Introduction to the Mgfun Package

The **Mgfun** package consists of procedures to compute with special functions and combinatorial sequences that are implicitly defined as solutions of systems of linear difference-differential equations. Its syntax is very close to that of the **gfun** package. In the same spirit as with the latter, a typical use of **Mgfun** to compute a sum or integral is the following. First one obtains systems that define the summand or integrand by calls to dfinite_expr_to_sys or by combining known systems by using ‘sys+sys‘ and ‘sys*sys‘. Then one derives the sum or integral by a call to sum_of_sys or int_of_sys. Finally one has to solve the system to obtain a closed form (if one exists).

```
> with(Mgfun);
```

[*dfinite_expr_to_diffeq*, *dfinite_expr_to_rec*, *dfinite_expr_to_sys*, *diag_of_sys*, *int_of_sys*, *pol_to_sys*, *sum_of_sys*, *sys*sys*, *sys+sys*]

In the sequel, we exemplify the use of the package by working out the evaluation of a sum and of two integrals.

Closed Form Evaluation of $\sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)\, u^n}{n!}$

This is the Mehler formula for Hermite polynomials. We first input the summand.

```
> expr := HermiteH(n, x) HermiteH(n, y) u^n / n!
```

$$expr := \frac{\mathrm{HermiteH}(n, x)\,\mathrm{HermiteH}(n, y)\,u^n}{n!}$$

Since the sum is over n, we need the dependency of the summand in terms of n. In this session, we focus on the dependency of the sum in u, and leave x and y as parameters, but other choices could have been made. The following call yields a system that traces the dependency of the summand in n and u.

```
> sys[1]:=dfinite_expr_to_sys(expr,f(n::shift,u::diff));
```

$$sys_1 := \{(16\,u^4 + 16\,u^4\,n)\,\mathrm{f}(n, u) - 16\,u^3\,x\,y\,\mathrm{f}(n+1, u) + (n+4)\,\mathrm{f}(n+4, u) + (-8\,u^2\,n + 8\,u^2\,x^2 - 20\,u^2 + 8\,u^2\,y^2)\,\mathrm{f}(n+2, u) - 4\,u\,y\,x\,\mathrm{f}(n+3, u), -n\,\mathrm{f}(n, u) + u\left(\frac{\partial}{\partial u}\mathrm{f}(n, u)\right)\}$$

The summation itself is performed by the following call. Viewing the sum as a formal power series in u, we have that the summand goes to 0 as n tends to infinity. At the other boundary, the recurrence in n proves that the summand is 0 for negative n. The use of the option *natural_boundaries* is therefore fully justified.

```
> sys[2]:=sum_of_sys(sys[1],n=0..infinity,natural_boundaries);
```

$$sys_2 := \{ (16\,u^3 + 8\,u\,y^2 + 8\,u\,x^2 - 4\,u - 4\,y\,x - 16\,u^2\,y\,x)\,\mathrm{f}(u) + (1 - 8\,u^2 + 16\,u^4)\left(\frac{\partial}{\partial u}\mathrm{f}(u)\right)\}$$

There only remains to solve this differential equation.

```
> dsolve(sys[2] union {f(0)=1},f(u));
```

$$\mathrm{f}(u) = -\frac{I\,\mathbf{e}^{\left(1/2\,\frac{-4\ln(-1+4\,u^2)\,u^2 - 8\,u\,y\,x + 2\,x^2 + 2\,y^2 + \ln(-1+4\,u^2)}{(2\,u+1)\,(2\,u-1)}\right)}}{\mathbf{e}^{(-x^2-y^2)}}$$

```
> simplify(expand(%),symbolic);
```

$$\mathrm{f}(u) = -\frac{I\,\mathbf{e}^{\left(4\,\frac{u\,(u\,x^2 + u\,y^2 - y\,x)}{(2\,u+1)\,(2\,u-1)}\right)}}{\sqrt{-1+4\,u^2}}$$

In other words, we have obtained:

$$\sum_{n=0}^{\infty} \frac{H_n(x)\,H_n(y)\,u^n}{n!} = \frac{\mathbf{e}^{\left(\frac{4\,u\,(x\,y - u\,x^2 - u\,y^2)}{1-4\,u^2}\right)}}{\sqrt{1-4\,u^2}}.$$

Closed Form Evaluation of $\int_0^\infty u^n \mathbf{e}^{\left(-\frac{u z \left(1-\frac{1}{u^2}\right)}{2}\right)} (\mathrm{BesselJ}(n, z) + u\, \mathrm{BesselJ}(n-1, z))\, dz$

We study this integral when parameter values make it well-defined, that is to say for $1 \le n$ and $|u| < 1$.
Here again, we first input the integrand and make our choice of variables with respect to which we want to study the dependency. This has to contain the integration variable z.
Here, we made the choice to leave no parameter.

> $expr :=$

$$u^n \mathbf{e}^{\left(-\frac{u z \left(1-\frac{1}{u^2}\right)}{2}\right)} (\mathrm{BesselJ}(n, z) + u\, \mathrm{BesselJ}(n-1, z))$$

```
> sys[1]:=dfinite_expr_to_sys(expr,f(n::shift,z::diff,u::diff));
```

$$sys_1 := \{ (2 z^2 u^2 + z^2 + u^4 z^2 - 4 u^3 n z + 4 u^2 n + 4 u^2 n^2 - 4 u z - 4 n u z) \mathrm{f}(n, z, u) + (4 u^4 z + 4 z u^2 - 8 u^3 n) \left(\frac{\partial}{\partial u} \mathrm{f}(n, z, u) \right) + 4 u^4 \left(\frac{\partial^2}{\partial u^2} \mathrm{f}(n, z, u) \right), (z u^2 - 2 u n) \mathrm{f}(n, z, u) + z \mathrm{f}(n+1, z, u) + u z \left(\frac{\partial}{\partial z} \mathrm{f}(n, z, u) \right) + u^2 \left(\frac{\partial}{\partial u} \mathrm{f}(n, z, u) \right), (-12 u^2 n^2 + 4 u^3 n z + 4 u^2 n + 4 n u z + u^4 z^2 + 2 z^2 u^2 - 4 u z + z^2) \mathrm{f}(n, z, u) + 4 z^2 u^2 \left(\frac{\partial^2}{\partial z^2} \mathrm{f}(n, z, u) \right) + (4 z^2 u^3 - 4 u z^2 + 4 z u^2) \left(\frac{\partial}{\partial z} \mathrm{f}(n, z, u) \right) + (8 u^3 n - 4 u^3) \left(\frac{\partial}{\partial u} \mathrm{f}(n, z, u) \right) \}$$

The integration itself takes place in the following call. The justification that we may use the option *takayama_algo*, and thus, that we may perform an integration over natural boundaries is the following. We will consider this integral for u in a neighbourhood of 0, so that the exponential term is exponentially small when z tends to infinity. At 0, the integrand has valuation $2\,n$ in u, so that any linear combination of the integrand and its derivatives with polynomial function coefficients will be 0 at $u = 0$, provided n becomes sufficiently large.

```
> sys[2]:=int_of_sys(sys[1],z=0..infinity
  ,takayama_algo);
```

$$sys_2 := \{ (7 + 20\,n + 12\,n^2 - u^2)\,\mathrm{f}(n, u) + (-4 - 12\,n^2 - 20\,n)\,\mathrm{f}(n+1, u) + (-7\,u - 8\,u\,n + u^3)\left(\frac{\partial}{\partial u}\mathrm{f}(n, u)\right) + (u^4 + u^2)\left(\frac{\partial^2}{\partial u^2}\mathrm{f}(n, u)\right) + (8\,u\,n + 4\,u)\,\mathrm{D}_2(f)(n+1, u), (9 - u^4\,n + 15\,n - 10\,u^2\,n - 3\,u^2 - 4\,n^3 - 8\,n^2 - 24\,u^2\,n^2 - 12\,n^3\,u^2)\,\mathrm{f}(n, u) + (12\,u^2\,n + 24\,u^2\,n^2 - 24\,n + 12\,n^3\,u^2 - 20\,n^2 - 4\,n^3)\,\mathrm{f}(n+1, u) + (u^5\,n - 2\,u^3\,n - 9\,u + 3\,u^3 - 3\,u\,n)\left(\frac{\partial}{\partial u}\mathrm{f}(n, u)\right) + (3\,u^4 + 3\,u^2 + u^2\,n + n\,u^6 + 2\,u^4\,n)\left(\frac{\partial^2}{\partial u^2}\mathrm{f}(n, u)\right) + (8\,n^3 + 28\,n^2 + 12\,n)\,\mathrm{f}(n+2, u), (-9 - 19\,n^2 - 29\,n + u^2 + 8\,n^3 + 4\,n^4 + 3\,u^2\,n + 5\,u^2\,n^2)\,\mathrm{f}(n, u) + (8\,n - 4\,n^4 - 8\,n^3 + 4\,n^2)\,\mathrm{f}(n+1, u) +$$

$$(-u^3 + 21\,u\,n - 3\,u^3\,n + 9\,u - 5\,u^3\,n^2 + 15\,u\,n^2)\left(\frac{\partial}{\partial u}\,\mathrm{f}(n, u)\right) + (3\,u^4\,n - 5\,u^2\,n^2 - 4\,u^2 - 9\,u^2\,n - 5\,u^4\,n^2 + 2\,u^4)\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(n, u)\right) + (u^5 + 2\,u^3\,n + u^3 + 2\,u^5\,n)\left(\frac{\partial^3}{\partial u^3}\,\mathrm{f}(n, u)\right)\}$$

The above system cancels the integral for sufficiently large n. (But unfortunately, we do not know at this point any lower bound for the valid n.)

At this point, we face a non-deterministic behaviour of Maple: the above output of *int_of_sys* varies, according to the system on which this session is run. Avoiding this non-determinism would require that the user inputs some constraints that are not expressable at the level of the Mgfun package, but only at the lower-level of the Holonomy package (this relates to choices of term orders in Groebner bases, see the Groebner package). Due to this, we provide two ways to solve the system, depending on the output of the following two Maple commands:

```
> map(indets,sys[2],function);
```

$$\{\{\mathrm{f}(n, u), \mathrm{D}_2(f)(n+1, u), \frac{\partial}{\partial u}\,\mathrm{f}(n, u), \mathrm{f}(n+1, u), \frac{\partial^2}{\partial u^2}\,\mathrm{f}(n, u)$$

$$\}, \{ f(n,u), \frac{\partial}{\partial u} f(n,u), f(n+1,u), f(n+2,u), \frac{\partial^2}{\partial u^2} f(n,u) \},$$

$$\{ \frac{\partial^3}{\partial u^3} f(n,u), f(n,u), \frac{\partial}{\partial u} f(n,u), f(n+1,u), \frac{\partial^2}{\partial u^2} f(n,u) \} \}$$

```
> if nops(remove(has,%,[D,diff]))=1 then
  "Case A" else "Case B" fi;
```

"Case B"

Please, directly enter the section with title indicated by the last output above.

Case A

In this situation, a recurrence is already available in the system. It is readily solved.

```
> eq[0]:=collect(eval(subs(f=unapply(f
  (n),n,u),op(remove(has,sys[2],diff))
  )),f,factor);
```

$$eq_0 := -u^2 f(n) + (u-1)(u+1) f(n+1) + f(n+2)$$

```
> normal(LREtools[hypergeomsols](eq[0]
  ,f(n),{},output=basis),expanded);
```

$$\_C_1 + \_C_2 (-u^2)^n$$

```
> solve(expand({subs(n=1,%)=2*u,subs(n
  =2,%)=2*u,subs(n=3,%)=2*u}),{_C[1],_
  C[2]});
```

$$\{ \_C_1 = 2u, \_C_2 = 0 \}$$

```
> rsol:=normal(subs(%,%%),expanded);
```

$$rsol := 2\,u$$

Case B

In this situation, we proceed to eliminate all shifts in n so as to obtain a purely differential relation.

```
> eq[1]:=op(select(has,sys[2],diff(f(n,u),u$3)));
```

$$eq_1 := (-9 - 19\,n^2 - 29\,n + u^2 + 8\,n^3 + 4\,n^4 + 3\,u^2\,n + 5\,u^2\,n^2)\,\mathrm{f}(n, u) + (8\,n - 4\,n^4 - 8\,n^3 + 4\,n^2)\,\mathrm{f}(n+1, u) + (-u^3 + 21\,u\,n - 3\,u^3\,n + 9\,u - 5\,u^3\,n^2 + 15\,u\,n^2)\left(\frac{\partial}{\partial u}\mathrm{f}(n, u)\right) + (3\,u^4\,n - 5\,u^2\,n^2 - 4\,u^2 - 9\,u^2\,n - 5\,u^4\,n^2 + 2\,u^4)\left(\frac{\partial^2}{\partial u^2}\mathrm{f}(n, u)\right) + (u^5 + 2\,u^3\,n + u^3 + 2\,u^5\,n)\left(\frac{\partial^3}{\partial u^3}\mathrm{f}(n, u)\right)$$

```
> eq[2]:=op(select(has,sys[2],D[2](f)(n+1,u)));
```

$$eq_2 := (7 + 20\,n + 12\,n^2 - u^2)\,\mathrm{f}(n, u) + (-4 - 12\,n^2 - 20\,n)\,\mathrm{f}(n+1, u)$$

$$+(-7\,u-8\,u\,n+u^3)\left(\frac{\partial}{\partial u}\,\mathrm{f}(n,u)\right)$$

$$+(u^4+u^2)\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(n,u)\right)$$

$$+(8\,u\,n+4\,u)\,\mathrm{D}_2(f)(n+1,u)$$

```
> eq[3]:=op(select(has,sys[2],f(n+2,u)
  ));
```

$$eq_3 := (9-u^4\,n+15\,n-10\,u^2\,n-3\,u^2-4\,n^3-8\,n^2-24\,u^2\,n^2-12\,n^3\,u^2)\,\mathrm{f}(n,u)+(12\,u^2\,n+24\,u^2\,n^2-24\,n+12\,n^3\,u^2-20\,n^2-4\,n^3)\,\mathrm{f}(n+1,u)$$

$$+(u^5\,n-2\,u^3\,n-9\,u+3\,u^3-3\,u\,n)\left(\frac{\partial}{\partial u}\,\mathrm{f}(n,u)\right)$$

$$+(3\,u^4+3\,u^2+u^2\,n+n\,u^6+2\,u^4\,n)\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(n,u)\right)$$

$$+(8\,n^3+28\,n^2+12\,n)\,\mathrm{f}(n+2,u)$$

[Procedure to replace var in expr with its value solved from relation.]

```
> plug:=proc(relation,expr,var)

  eval(subs(var=solve(relation,var),ex
  pr))
  end:
```

[Procedure that collects in each function call.]

```
> normalize:=proc(expr)

  collect(expr,indets(expr,function),distributed,factor)
  end:
> eq[4]:=normalize(plug(diff(eq[1],u),
  eq[2],D[2](f)(n+1,u)));
```

$$eq_4 := \frac{u^4 (2n+1)^2 (u^2+1) \left(\frac{\partial^4}{\partial u^4} \mathrm{f}(n,u)\right)}{n (n-1)(n+2)(n+1)} + ((-14 n + 35 n^4 - 30 n^3 - 47 n^2 + 44 n^5 + 12 n^6 + 12 u^2 n - u^2 n^4 + 18 n^3 u^2 + 23 u^2 n^2 + 2 u^2) \mathrm{f}(n,u)) / (n (n-1)(n+2)(n+1)) + u^2 (3 n + 21 u^2 n - 48 n^3 u^2 - 8 u^2 n^2 + 10 n^2 + 12 n^3 + 1 + 7 u^2 + u^2 n^4 + n^4) \left(\frac{\partial^2}{\partial u^2} \mathrm{f}(n,u)\right) / (n (n-1)(n+2)(n+1)) + u (3 n^2 - 12 u^2 n - 3 n^4 - 18 n^3 u^2 - 23 u^2 n^2 + 6 n - 2 u^2 - 6 n^3 + u^2 n^4) \left(\frac{\partial}{\partial u} \mathrm{f}(n,u)\right) / (n (n-1)(n+2)(n+1)) - u^3 (2 n + 1) (1 - 7 u^2 - 13 u^2 n + 3 n + 5 n^2 + 5 u^2 n^2) \left(\frac{\partial^3}{\partial u^3} \mathrm{f}(n,u)\right)$$

$$/ (n (n-1) (n+2) (n+1)) + (-4 - 12 n^2 - 20 n) \mathrm{f}(n+1, u)$$

```
> eq[5]:=normalize(plug(eq[1],eq[4],f(
n+1,u)));
```

$$eq_5 := \frac{u^4 (2n+1)^2 (u^2+1) \left(\frac{\partial^4}{\partial u^4} \mathrm{f}(n, u)\right)}{n (n-1) (n+2) (n+1)} - \frac{(2n+1)^3 (-6n + 2u^2 n - 9 - u^2) \mathrm{f}(n, u)}{n (n-1) (n+2) (n+1)} + (2n+1)^2 (4n^2 + 4u^2 n^2 + 12n - 12u^2 n + 5 + 5u^2) u^2 \left(\frac{\partial^2}{\partial u^2} \mathrm{f}(n, u)\right) / (n (n-1) (n+2) (n+1)) + \frac{(2n+1)^3 (-6n + 2u^2 n - 9 - u^2) u \left(\frac{\partial}{\partial u} \mathrm{f}(n, u)\right)}{n (n-1) (n+2) (n+1)} - 2 \frac{(2n+1)^2 (2n + 2u^2 n + 1 - 3u^2) u^3 \left(\frac{\partial^3}{\partial u^3} \mathrm{f}(n, u)\right)}{n (n-1) (n+2) (n+1)}$$

The following is a purely differential equation satisfied by the integral.

```
> eq[5]:=normalize(primpart(subs(f(n,u
)=f(u),eq[5]),diff(f(u),u$4)));
```

$$eq_5 := u\,(2\,n+1)\,(-6\,n+2\,u^2\,n-9-u^2)\left(\frac{\partial}{\partial u}\,\mathrm{f}(u)\right)+ u^2\,(4\,n^2+4\,u^2\,n^2+12\,n-12\,u^2\,n+5+5\,u^2)\left(\frac{\partial^2}{\partial u^2}\,\mathrm{f}(u)\right)-(2\,n+1)\,(-6\,n+2\,u^2\,n-9-u^2)\,\mathrm{f}(u) + u^4\,(u^2+1)\left(\frac{\partial^4}{\partial u^4}\,\mathrm{f}(u)\right) - 2\,u^3\,(2\,n+2\,u^2\,n+1-3\,u^2)\left(\frac{\partial^3}{\partial u^3}\,\mathrm{f}(u)\right)$$

There only remains to solve it. First, a general solution is obtained by a call to Maple's general dsolve.

```
> dsol:=op(2,dsolve(eq[5],f(u)));
```

$$dsol := \frac{\_C1\;u}{u^2+1}+\frac{\_C2\;u^3}{u^2+1}+\frac{\_C3\;u^{(2\,n+3)}}{u^2+1}+\frac{\_C4\;u^{(2\,n+1)}}{u^2+1}$$

Next, we use initial conditions.

```
> solve({seq(subs(n=i,dsol)=simplify(int(subs(n=i,expr),z=0..infinity),symbolic),i=1..4)},{_C1,_C2,_C3,_C4});
```

$$\{\_C2=\_C2,\ \_C3=\_C3,\ \_C4=-\_C3\;u^2,\ \_C1=-\_C2\;u^2+2\,u^2+2\}$$

```
> normal(subs(%,dsol),expanded);
```

$$2\,u$$

We have just obtained:

$$\int_0^\infty u^n e^{\left(-\frac{u z\left(1-\frac{1}{u^2}\right)}{2}\right)} (\mathrm{BesselJ}(n, z) + u\, \mathrm{BesselJ}(n-1, z))\, dz$$

$= 2\, u$ whenever $1 \leq n$.

Chebyshev Coefficients of $x\, e^x$

Again, we input the integrand.

$$> \quad expr := \frac{x\, \mathbf{e}^x\, \mathrm{ChebyshevT}(n, x)}{\sqrt{1-x^2}}$$

The following differential difference system keeps track of all the dependencies.

```
> sys[1]:=dfinite_expr_to_sys(expr,f(n::shift,x::diff));
```

$$sys_1 := \{ (-x^2\, n - 1 - x + x^3)\, \mathrm{f}(n, x) + (x - x^3) \left(\frac{\partial}{\partial x} \mathrm{f}(n, x)\right) + x\, n\, \mathrm{f}(n+1, x), (-x^2\, n^2 + x^4 - x^3 - x^2 - 2\, x - 2)\, \mathrm{f}(n, x) + (-2\, x^4 + x^3 + 2\, x^2 + 2\, x) \left(\frac{\partial}{\partial x} \mathrm{f}(n, x)\right) + (x^4 - x^2) \left(\frac{\partial^2}{\partial x^2} \mathrm{f}(n, x)\right) \}$$

The following call computes a recurrence satisfied by the residue at any singularity of any solution of the above system.

```
> sys[2]:=int_of_sys(sys[1],x=residues);
```

$$sys_2 := \{ f(n) - f(n+4) - 2\,n\, f(n+1) + (-2\,n-8)\, f(n+3) \}$$

This fourth order recurrence has no simple solution.

```
> rec[4]:=op(sys[2]):
> LREtools[hypergeomsols](rec[4],f(n),{},
  output=basis);
```

$$0$$

This is no completely clear at the time why the following second order recurrence, also satisfied by the integrals, cannot be found by holonomic methods.

$$> \; rec_2 := (n^2+n+1)\, f(n+2) + 2\,(n+1)^3\, f(n+1) - (n^2+3\,n+3)\, f(n)$$

In other words, the integral of the following Z is zero.

```
> Z:=eval(subs(f=unapply(expr,n),rec[2]))
  ;
```

$$Z := \frac{(n^2+n+1)\, x\, \mathbf{e}^x\, \mathrm{ChebyshevT}(n+2, x)}{\sqrt{1-x^2}} + 2\,\frac{(n+1)^3\, x\, \mathbf{e}^x\, \mathrm{ChebyshevT}(n+1, x)}{\sqrt{1-x^2}} - \frac{(n^2+3\,n+3)\, x\, \mathbf{e}^x\, \mathrm{ChebyshevT}(n, x)}{\sqrt{1-x^2}}$$

This is easily checked numerically.

```
> for i from 0 to 3 do
```

```
'Z'(n=i)=Int(normal(eval(subs(n=i,subs(
ChebyshevT=orthopoly[T],Z)))),x=-1..1);
     "is approximately equal
to:",evalf(op(2,%),30)
od;
```

$$Z(n=0) = \int_{-1}^{1} 2\,\frac{x\,\mathbf{e}^{x}\,(x^2 - 2 + x)}{\sqrt{1-x^2}}\,dx$$

"is approximately equal to:", $.46\ 10^{-30}$

$$Z(n=1) = \int_{-1}^{1} 4\,\frac{x\,\mathbf{e}^{x}\,(3\,x^3 - 4\,x + 8\,x^2 - 4)}{\sqrt{1-x^2}}\,dx$$

"is approximately equal to:", $.784\ 10^{-29}$

$$Z(n=2) = \int_{-1}^{1} 2\,\frac{x\,\mathbf{e}^{x}\,(28\,x^4 - 41\,x^2 + 10 + 108\,x^3 - 81\,x)}{\sqrt{1-x^2}}\,dx$$

"is approximately equal to:", $.4777\ 10^{-28}$

$Z(n=3) =$

$$\int_{-1}^{1} 8 \frac{x\, \mathbf{e}^{x} (26\, x^{5} - 43\, x^{3} + 16\, x + 128\, x^{4} - 128\, x^{2} + 16)}{\sqrt{1 - x^{2}}}\, dx$$

"is approximately equal to:", $.8792\ 10^{-28}$