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On some combinatorial sequences associated to invariant theory



Alin Bostan^{a,1}, Jordan Tirrell^b, Bruce W. Westbury^b,
Yi Zhang^{c,2}

^a Inria, Université Paris-Saclay, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France

^b Department of Mathematics and Computer Science, Washington College, USA

^c Department of Foundational Mathematics, School of Science, Xi'an Jiaotong-Liverpool University, Suzhou, 215123, China

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ABSTRACT

We study the enumerative and analytic properties of some sequences constructed using tensor invariant theory. The octant sequences are constructed from the exceptional Lie group G_2 and the quadrant sequences from the special linear group $SL(3)$. In each case we show that the corresponding sequences are related by binomial transforms. The first three octant sequences and the first four quadrant sequences are listed in the On-Line Encyclopedia of Integer Sequences (OEIS). These sequences all have interpretations as enumerating two-dimensional lattice walks but for the octant sequences the boundary conditions are unconventional. These sequences are all P-recursive and we give the corresponding recurrence relations. In all cases the associated differential operators are of third order and have the remarkable property that they can be solved to give closed formulae for the ordinary generating functions in terms of classical Gaussian hypergeometric functions. Moreover, we show that the octant sequences and the quadrant sequences are related by the branching rules for the inclusion of $SL(3)$ in G_2 .

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E-mail addresses: Alin.Bostan@inria.fr (A. Bostan), jtirrell2@washcoll.edu (J. Tirrell), Bruce.Westbury@gmail.com (B.W. Westbury), Yi.Zhang03@xjtlu.edu.cn (Y. Zhang).

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1. Introduction

We study two families of sequences, each of them constructed using tensor invariant theory. The theory is summarized in Section 2. The first family contains the *octant sequences*. The first octant sequence is constructed from the seven-dimensional fundamental representation of the exceptional simple algebraic group G_2 . This sequence was studied initially by Kuperberg in [31] and subsequently by Westbury in [42]. We will refer to this sequence as T_3 . The second family contains the *quadrant sequences*. The first quadrant sequence is constructed from the direct sum of the three-dimensional vector representation of $SL(3)$ and its dual. The sequences are defined as the dimension of the subspace of invariant tensors in the tensor powers of the representation. In both cases we extend to a family of sequences by adding copies of the trivial representation to the initial representation. Combinatorially, it means that each family consists of the iterated binomial transforms of the first sequence.

The first three sequences of the octant family have entries in the On-Line Encyclopedia of Integer Sequences (OEIS). They are displayed in Fig. 1. The sequence A059710 is the first octant sequence T_3 .

To our knowledge, the sequence T_3 has not appeared previously in the combinatorics literature. In Section 3 we prove the following combinatorial interpretations of this sequence.

Theorem 1.1. *The sequence T_3 enumerates:*

- *hesitating tableaux of height 2, empty shape, and no singleton;*
- *set partitions with no singleton and no enhanced 3-crossing;*
- *sequences $(x_1, x_2, \dots, x_n)$ such that $1 \leq x_i < i$ with no weakly decreasing subsequence of length 3.*

These all follow from known combinatorial interpretations of the binomial transform of T_3 ; they are corollaries to Theorem 3.1 in Section 3.

The starting point for Theorem 1.1 is the interpretation of the sequences in the first family as lattice walks in the plane. This interpretation is used in [42]. These lattice walks have six nonzero steps and one zero step corresponding to the seven weights of the representation. These walks are restricted to the region $0 \leq y \leq x$. There is an additional constraint which is unusual from the standard theory of lattice walks, namely, that the zero step is forbidden on the line $x = y$.

Let E_3 be the second octant sequence A108307 in Fig. 1. Using this interpretation, we have the following theorem.

Theorem 1.2. *The sequence E_3 is the binomial transform of the sequence T_3 , i.e., for $n \geq 0$,*

$$E_3(n) = \sum_{i=0}^n \binom{n}{i} T_3(i).$$

This result provides an unexpected connection between the invariant theory of G_2 and the combinatorics of set partitions.

In Section 3.3 we study the ordinary generating functions of these sequences. First we derive the following recurrence relation for T_3 .

Theorem 1.3. *The sequence T_3 is determined by the initial conditions $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and the recurrence relation*

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0 \quad (1)$$

satisfied for all $n \geq 0$.

The recurrence relation in Theorem 1.3 was conjectured by Mihailovs [42, §3] and we give three independent proofs in Section 3.3. The corresponding linear differential equation, satisfied by the

a	0	1	2	3	4	5	6	7	8	9
A059710	1	0	1	1	4	10	35	120	455	1792
A108307	1	1	2	5	15	51	191	772	3320	15032
A108304	1	2	5	15	52	202	859	3930	19095	97566

Fig. 1. The first sequences of the family of octant sequences, their OEIS tags and their first ten terms.

a	0	1	2	3	4	5	6	7	8
A151366	1	0	2	2	12	30	130	462	1946
A236408	1	1	3	9	33	131	561	2535	11971
A001181	1	2	6	22	92	422	2074	10754	58202
A216947	1	3	11	47	225	1173	6529	38265	233795

Fig. 2. The first four quadrant sequences, their OEIS tags and their first nine terms.

generating function $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$, can be solved to give a closed formula for $\mathcal{T}(t)$ in terms of the Gaussian hypergeometric function ${}_2F_1$. The differential equations and recurrence relations for the other two sequences in Fig. 1 are derived in [9] and closed formulae are given in the OEIS.

The first four sequences of the quadrant sequences also have entries in the OEIS; they are shown in Fig. 2. The most well-known of these sequences is the third one, which enumerates *Baxter permutations* [2].

These sequences also have interpretations as lattice walks. These walks are restricted to the quadrant $0 \leq x, y$. These are walks in six weights of the representation. This interpretation follows from the Pieri rule. For $GL(3)$, representations are indexed by partitions with three rows. Then applying the Pieri rule gives:

$$V_{[1]} \otimes V_\lambda \cong \oplus_\mu V_\mu \quad V_{[1,1]} \otimes V_\lambda \cong \oplus_\nu V_\nu$$

where the first sum is over all partitions μ obtained by adding a single box to λ and the second sum is over all partitions ν obtained by adding two boxes in different rows to λ . Then the restriction to $SL(3)$ is given by mapping the partition $[\lambda_1, \lambda_2, \lambda_3]$ to the lattice point in the plane $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$.

The first sequence is studied in [8], where it is defined in terms of lattice walks. The identification of the third sequence with the sequence enumerating Baxter permutations can be proved by several methods. One method is to show that they both satisfy the same recurrence relation and initial conditions. Another method is to prove that they both enumerate axis walks in the octant, using [11,12,16] or our derivation utilizing branching rules. The fourth sequence was introduced by Marberg in [37]. It is defined as the number of noncrossing 2-coloured set partitions; a lattice path interpretation is given in [37], where a differential equation and a recurrence relation can also be found. The second sequence does not have a published reference, but it is known to enumerate a family of unlabelled graphs called *planar st-graphs* [17]. We identify this sequence with the OEIS entry by giving a combinatorial interpretation and by showing that the binomial transform is the third sequence in Section 4. Moreover, the second and fourth sequences are both given by axis walks in the corresponding octant.

The ordinary generating functions for these sequences are studied in Section 4.4. We give a recurrence relation for the k th quadrant sequence, which includes k as a parameter. A closed formula for the generating function of the Baxter sequence is given in the OEIS without a reference. We give a different, but equivalent, closed formula.

These two families of (octant and quadrant) sequences are related since $SL(3)$ is a maximal subgroup of G_2 . Furthermore, the restriction of the seven-dimensional fundamental representation is the direct sum of the two three-dimensional representations and one copy of the trivial representation. More generally, the restriction of the representation for the k th sequence in octant family is

the $(k+1)$ st sequence in the quadrant family. It means that these two sequences are related by the branching rules for the inclusion of $SL(3)$ in G_2 . It has been observed several times that the Baxter numbers enumerate walks which end on an axis and bijections which establish this are given in [11, Proposition 23], [16, Theorem 3], [45, Theorem 1.2]. In the last section we give an interpretation of this result using representation theory. It gives a new proof that these sets are equinumerous but it is not bijective. However, this proof does extend the result. First, the analogous result relates the k th quadrant sequence and the k th octant sequence. It also follows combinatorially from the known result using the binomial transforms. Second, our proof also extends this result by giving an analogous result for walks ending at a point other than the origin.

This is an expanded version of [6]. The abbreviated version was used in [40] to find the radius of convergence of sequence A060049 and proving [31, Conjecture 8.2].

2. Invariant theory

In this section we describe the way in which the representation theory of algebraic groups gives rise to combinatorially interesting sequences. For the basic notions in the theory of linear algebraic groups, the reader is invited to consult the book [20].

Let G be a reductive complex algebraic group, i.e., G has a trivial unipotent radical. Let V_λ be the irreducible representation of G with highest weight λ .

Definition 2.1. Let V be a (finite-dimensional) representation of G and λ a dominant weight. The sequence associated to (G, V, λ) , denoted $\mathbf{a}_{V,\lambda}$, is the sequence whose n th term is the multiplicity of V_λ in the tensor power $\otimes^n V$.

Our main interest is in the case $\lambda = 0$. In this case V_λ is the trivial representation and the n th term of the sequence $\mathbf{a}_{V,\lambda}$ is the dimension of the subspace of invariant tensors in $\otimes^n V$. The sequence $\mathbf{a}_{V,0}$ is denoted by $\mathbf{a}_V$.

Example 2.2. The simplest example is to take $G = SL(2)$ and V to be the two-dimensional defining representation. The odd terms of the sequence $\mathbf{a}_V$ are zero and the even terms are the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$.

It is very classical that the generating function for the sequence of Catalan numbers is *algebraic*. In our context, this can be seen as a special case of the following general result.

Proposition 2.3. Take $G = SL(2)$ and V to be any representation. Then the generating function of the sequence $\mathbf{a}_V$ is algebraic.

Proof. The character $\chi(V)$ of V is a Laurent polynomial in x invariant under $x \leftrightarrow x^{-1}$. Then the n th term of the sequence $\mathbf{a}_V$ is the constant term of the Laurent polynomial $\chi(V)^n(1 - x^2)$. Equivalently, the generating function of the sequence $\mathbf{a}_V$ is the diagonal of the bivariate rational function $\frac{x - x^{-1}}{1 - tx\chi(V)}$. The result now follows by applying the Pólya–Furstenberg theorem [21,38], see also [41, Theorem 6.3.3]. $\square$

Next we discuss the case where V is a fundamental representation of a classical simple Lie algebra of rank two. These sequences have all been studied and are known to be *P-recursive*, or *holonomic*: they satisfy linear recurrences with polynomial coefficients (in the index n).

Example 2.4. Take $G = SL(3)$ and V to be the three-dimensional defining representation. In the sequence $\mathbf{a}_V$ every third term is non-zero and the other terms are zero. Taking every third term gives the sequence of three-dimensional Catalan numbers (A005789), whose n th term is given by $2(3n)!/(n!(n+1)!(n+2)!)$ and matches the number of standard tableaux of rectangular shape (n, n, n) .

This can be seen as follows. Let V be the vector representation of $GL(n)$. Then Schur–Weyl duality gives the decomposition of $\otimes^r V$ as $\oplus_\lambda S_\lambda \otimes V_\lambda$ where λ is a partition of size r with at most n non-zero parts and S_λ is an irreducible representation of the symmetric group on r points. There are

several constructions (due to Young) of S_λ , which take the vector space with basis the set of standard tableaux of shape λ and give an action of the symmetric group. Now restrict from $GL(n)$ to $SL(n)$. The restriction of the representation V_λ is trivial if λ has rectangular shape and is non-trivial otherwise.

Example 2.5. Take $G = Sp(4)$ of type C_2 and V to be the four-dimensional defining representation. Under the isomorphism $Sp(4) \cong Spin(5)$, we see that V is the spin representation of the spin group $Spin(5)$. The odd terms in the sequence $\mathbf{a}_V$ are zero and the even terms are those of the sequence $C_n C_{n+2} - C_{n+1}^2$ (A005700). This sequence counts 3-noncrossing perfect matchings on $2n$ points and also nested pairs of Dyck paths of length $2n$.

The theory of Kashiwara crystals (see [36, Ch. 2], or [26, Ch. 4]) gives combinatorial interpretations of these sequences. There is a crystal associated to each representation. The crystal associated to the irreducible representation V_λ is denoted by C_λ . These are the connected crystals. Crystals have a tensor product with the property that the crystal associated to the tensor product of representations is the tensor product of the crystals associated to the representations. The trivial crystal is C_0 and has one element. Moreover, the crystal associated to the direct sum of representations is the disjoint union of the crystals associated to the representations.

Let C be a crystal. The elements of $\otimes^n C$ are words of length n in the elements of C . The analogue of the subspace of invariant tensors is the subset of invariant words in $\otimes^n C$.

The crystal version of Definition 2.1 is:

Definition 2.6. Let C be a (finite) crystal of G and λ a dominant weight. The sequence associated to (G, C, λ) , denoted $\mathbf{a}_{C,\lambda}$, is the sequence whose n th term is the multiplicity of the crystal C_λ in the tensor power $\otimes^n C$.

Proposition 2.7. If C is the crystal of the representation V then $\mathbf{a}_{C,\lambda} = \mathbf{a}_{V,\lambda}$ for all dominant weights λ .

In this paper we only apply this to the sum of the two fundamental representations of $SL(3)$ and to the seven-dimensional fundamental representation of G_2 . For the fundamental representations of $SL(3)$ this follows from classical results by interpreting a standard or semistandard tableau as a sequence of partitions. For the G_2 case this result has been used in [31,42]. We give an outline of the general case which is an application of the tensor product rule for crystals. This is [25, Theorem 2].

Proof. A tensor product of highest weight representations has a canonical vector space decomposition as

$$V_\lambda \otimes V_\mu \cong \bigoplus_v A_{\lambda,\mu}^v \otimes V_v.$$

Similarly the tensor product of crystals has a set decomposition

$$C_\lambda \otimes C_\mu \cong \bigsqcup_v B_{\lambda,\mu}^v \otimes C_v.$$

The decomposition rule of [35] asserts that $\dim A_{\lambda,\mu}^v = |B_{\lambda,\mu}^v|$.

Then the n th terms of the two sequences are equal. This is proved by induction on n using the above for the inductive step. $\square$

2.1. Binomial transform

In this subsection we recall some basic properties of the binomial transform. Identities associated to the binomial transform are given for instance in [10].

The *binomial transform operator* is denoted by $\mathcal{B}$. This is a linear operator acting on sequences. Given the sequence $\mathbf{a}$ with n th term $a(n)$, the *binomial transform of $\mathbf{a}$* is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n th term is

$$\sum_{i=0}^n \binom{n}{i} a(i).$$

Binomial transforms arise naturally for sequences associated to the representations of reductive complex algebraic groups.

Lemma 2.8. Assume that $\mathbf{a}_{V,\lambda}$ is the sequence associated to (G, V, λ) as specified in Definition 2.1. Then $\mathbf{a}_{V \oplus \mathbb{C}, \lambda} = \mathcal{B}\mathbf{a}_{V, \lambda}$.

Proof. Expanding the tensor product gives

$$\otimes^n(V \oplus \mathbb{C}) \cong \oplus_{i=0}^n \mathbb{C}^{(i)} \otimes (\otimes^i V).$$

This can be proved by induction on n as in the binomial theorem.

Now comparing invariant tensors on both sides gives the result. $\square$

The binomial transform also arises naturally for lattice walks restricted to a domain.

Lemma 2.9. Assume that a sequence $\mathbf{a}$ enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ is given by adding a new step corresponding to the zero element of the lattice; that is, S is replaced by the disjoint union $S \sqcup \{0\}$ without changing the domain.

Proof. A path of length n with steps $S \sqcup \{0\}$ has i steps S and j steps 0 with $i + j = n$. Taking the steps in S and ignoring the 0 steps gives a path of length i in S . Each path of length i in S appears $\binom{n}{j}$ times corresponding to the partitions of a set of size n into two disjoint subsets of size i and j . $\square$

The following is a corollary to both Lemmas 2.8 and 2.9.

Corollary 2.10. Assume that $\mathbf{a}_{C,\lambda}$ is the sequence associated to (G, C, λ) as specified in Definition 2.6. Let $*$ be the trivial crystal. Then

$$\mathbf{a}_{C \sqcup *} = \mathcal{B}\mathbf{a}_{C, \lambda}.$$

We can also consider iterations of the binomial transform.

Lemma 2.11. Given the sequence $\mathbf{a}$ with n th term $a(n)$. For $k \in \mathbb{Z}$, the k th binomial transform of $\mathbf{a}$ is given by

$$(\mathcal{B}^k \mathbf{a})(n) = \sum_{i=0}^n k^{n-i} \binom{n}{i} a(i) \text{ for each } n \in \mathbb{N}.$$

The binomial transform can also be regarded as an operator on the generating function of a sequence. Let $G(t) = \sum_{n=0}^{\infty} a(n)t^n$ be the generating function of the sequence $\mathbf{a}$. We denote the generating function of $\mathcal{B}^k \mathbf{a}$ by $\mathcal{B}^k G$. Then we have the following immediate lemma.

Lemma 2.12. For $k \in \mathbb{Z}$, the k th binomial transform of $G(t)$ is

$$(\mathcal{B}^k G)(t) = \frac{1}{1 - k t} G\left(\frac{t}{1 - k t}\right).$$

3. Octant sequences

In this section we consider a sequence associated to the representation theory of the exceptional simple algebraic group G_2 by the construction in Definition 2.1. Then we relate this sequence and its binomial transforms to known sequences of hesitating tableaux of height 2 and vacillating tableaux of height 2. These were introduced in [13].

The algebraic group G_2 has dimension 14 and rank 2. It can be constructed as the automorphism group of the (complex) octonions. By construction, the octonions are then an eight-dimensional representation of G_2 . The unit spans a one-dimensional invariant subspace and the imaginary octonions are an invariant complementary subspace. This seven-dimensional representation of G_2 is a fundamental representation.

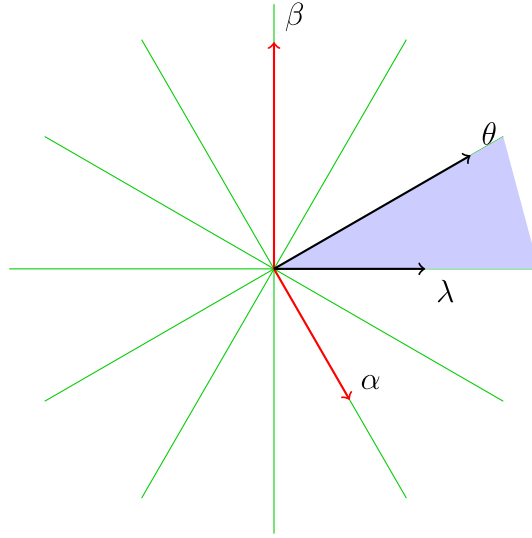


Fig. 3. G_2 root system.

3.1. Root systems

The root system of G_2 (see [20, Lecture 22]) is shown in Fig. 3 with the fundamental chamber shaded. The two simple roots are α and β with α short and β long. The two fundamental weights are λ and θ . The highest weight representation with highest weight λ is the seven-dimensional representation we are interested in. The highest weight representation with highest weight θ is the adjoint representation, so θ is the highest root.

These are related by

$$\lambda = 2\alpha + \beta, \quad \theta = 3\alpha + 2\beta.$$

The element ρ is defined in terms of the roots as half the sum of the positive roots. The six positive roots are

$$\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta.$$

This gives $\rho = 5\alpha + 3\beta$.

The element ρ is also defined in terms of the weights as the sum of the fundamental weights. This gives $\rho = \lambda + \theta$.

3.2. Lattice walks

In this subsection, we give a combinatorial proof of Theorem 1.2. Let G be G_2 , let V be the seven-dimensional fundamental representation, and C be the associated crystal. We now apply the general theory of Section 2 to this case.

The highest weight words in tensor powers of C correspond to lattice walks in the weight lattice of G_2 restricted to the dominant chamber. These lattice walks are studied by Kuperberg [31] and by Westbury [42]. Since V has dimension 7, C has cardinality 7, and the seven steps are shown in Fig. 4, where the coordinates are with respect to the basis of fundamental weights. One can make it more explicit by taking the intersection of the Euclidean lattice in three-dimensional space with the plane $x + y + z = 0$. Then, we can take the short simple root $\alpha = (0, 1, -1)$, the long simple root $\beta = (1, -2, 1)$, the fundamental weight $\lambda = (1, 0, -1)$, and the fundamental weight $\theta = (2, -1, -1)$.

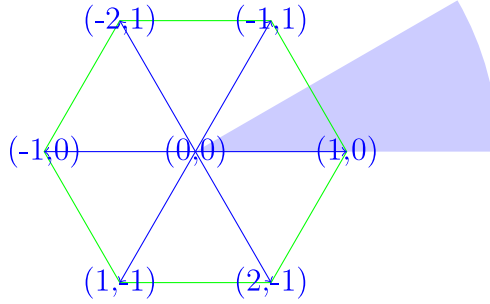


Fig. 4. Steps in weight lattice.

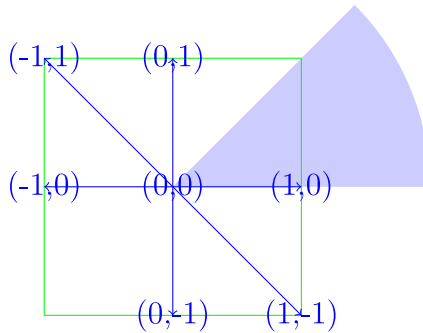


Fig. 5. Steps in octant.

The highest weight words correspond with these steps, restricted to the dominant chamber and with the extra condition that the step $(0, 0)$ is not permitted at a weight $(0, k)$ for $k \geq 0$.

A *hesitating tableau* of semilength n is a walk in the Young lattice with n steps. Each step is one of the following pairs of moves on the Young lattice:

- do nothing, add a cell
- remove a cell, do nothing
- add a cell, remove a cell

The *shape* is the final partition.

A hesitating tableau of *height* h is a hesitating tableau such that every partition in the sequence has height at most h .

There is a lattice walk interpretation of hesitating tableaux of height 2 given by Bousquet-Mélou and Xin in [9].

A hesitating tableau of height 2 can be interpreted as a walk in $\mathbb{Z}^2$ by identifying partitions with at most two nonzero rows with the set

$$\{(x, y) \in \mathbb{Z}^2 | x \geq y \geq 0\}.$$

There are 8 steps which are shown in Fig. 5. There are eight steps since there are:

- two ways to do nothing then add a cell,
- two ways to remove a cell then do nothing
- four ways to remove a cell then add a cell

Two of the four ways to remove a cell then add a cell give the step $(0, 0)$, namely add a cell on the first row then remove this cell and add a cell on the second row then remove this cell. It is always

allowed to add and then remove a cell on the first row. The step which adds and removes a cell on the second row is not permitted on the line $x = y$.

Proof of Theorem 1.2. We compare the two descriptions of the walks. The steps of the walks for the sequence T_3 are shown in Fig. 4 and the steps of the walks for the sequence E_3 are shown in Fig. 5.

In order to compare these, we first make the change of coordinates $(x, y) \rightarrow (x + y, x)$, which maps the steps from Fig. 4 to those of Fig. 5. This identifies the six non-zero steps, as well as the two domains.

By Lemma 2.9, it remains to compare the zero steps. There is one zero step in the T_3 case and two in the E_3 case. In the E_3 case we have the zero step which adds and then removes a cell on the second row. The boundary condition is that this step is not allowed on the line $x = y$. After the change of coordinates, this is the same boundary condition as the zero step in the T_3 case. The second zero step in the E_3 case adds and then removes a cell in the first row. This is always allowed. $\square$

Theorem 3.1. For each $n \geq 0$ and $r, s \geq 0$ there is a correspondence between highest weight words of weight (r, s) and length n in the crystal $C \coprod *$ and hesitating tableaux of height 2, semilength n and shape $(r + s, s)$.

Proof. The proof is the observation that the two descriptions of the lattice walks agree under a simple change of coordinates. The point (r, s) in the weight lattice is dominant if $r, s \geq 0$. This point corresponds to the two part partition $(r + s, s)$.

It is straightforward to check that under this change of coordinates, the eight steps correspond, the two domains correspond, and the extra condition on a step of weight $(0, 0)$ also corresponds. $\square$

There is a variation of Theorem 3.1 for vacillating tableaux. A *vacillating tableau* of semilength n is an excursion in the Young lattice with n steps. Each step is one of the following pairs of moves on the Young lattice:

- do nothing twice
- do nothing, add a cell
- remove a cell, do nothing
- remove a cell, add a cell

A vacillating tableau of height h is a vacillating tableau such that every partition in the sequence has height at most h .

There is a lattice walk interpretation of vacillating tableaux of height 2 also given in [9]. There are nine steps since there are six steps which change the shape and three steps which leave the shape unaltered.

Vacillating tableaux were introduced in [13] in the context of set partitions. A *set partition* will mean a partition of the ordered set

$$[n] = \{1, 2, \dots, n\}$$

into non-empty pairwise disjoint subsets. The *standard representation* of a set partition is a graph with vertex set $[n]$. The edges are arcs connecting pairs of elements in a block which are adjacent in the numerical order on the block.

A *singleton* in a set partition is a block with one element.

A *k-crossing* in a set partition is a k -subset of arcs, $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$i_1 < i_2 < \dots < i_k < j_1 < j_2 < \dots < j_k$$

An *enhanced k-crossing* in a set partition is a k -subset of arcs, $(i_1, j_1), \dots, (i_k, j_k)$ such that

$$i_1 < i_2 < \dots < i_k \leq j_1 < j_2 < \dots < j_k$$

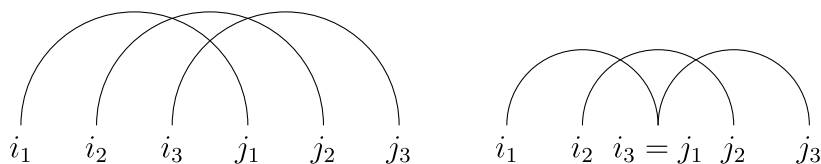


Fig. 6. A 3-crossing and an enhanced 3-crossing.

Theorem 3.2. For each $n \geq 0$ and $r, s \geq 0$ there is a correspondence between highest weight words of weight (r, s) and length n in the crystal $C[\downarrow * \downarrow *]$ and vacillating tableaux of height 2, semilength n and shape $(r + s, s)$.

Proof. The proof is essentially the same as the proof of Theorem 3.1. $\square$

The main innovation in [13] is the construction of a bijection between set partitions of $[n]$ and vacillating tableaux of semilength n and empty shape. The set partition has a k -crossing if and only if some partition in the vacillating tableau has length at least k . In particular, for $k = 3$, this is a bijection between 3-noncrossing set partitions and vacillating tableaux of height 2 and empty shape. There is a correspondence between hesitating tableaux of semilength n , empty shape and height h and set partitions of n with no enhanced $h + 1$ -crossing.

This is proved by Lin in [33] and by Marberg in [37, Proposition 5.8]. In [24] the sequence A108307 is called $NC_{0,3}$, the sequence A108304 is called $NC_{1,3}$; the binomial transform relation between them is given in [24,33].

A corollary is that there is a correspondence between invariant words of length n in C and set partitions of $[n]$ with no singleton and no enhanced 3-crossing.

There is another combinatorial interpretation based on [33,43].

Definition 3.3. An inversion sequence of length n is a sequence $(x_1, x_2, \dots, x_n)$ such that $1 \leq x_i \leq i$.

A singleton in the inversion sequence $(x_1, x_2, \dots, x_n)$ is an $i \in [n]$ such that $x_i = i$.

Proposition 3.4. For $n \geq 0$ there is a bijection between set partitions of $[n]$ with no enhanced 3-crossing and no singletons and inversion sequences of length n with no weakly decreasing subsequence of length 3 and no singletons.

Proof. There is a bijection in [43] between set partitions of $[n]$ with no enhanced 3-crossing and inversion sequences of length n with no weakly decreasing subsequence of length 3. This bijection preserves singletons. $\square$

3.3. Algebra

In this subsection, we give three proofs of Theorem 1.3, with different flavours. The first two proofs utilize Theorem 1.2 and a result of Bousquet-Mélou and Xin [9] on partitions that avoid 3-crossings. The last one is a direct proof which relies on the connection with G_2 walks.

The first proof is based on Theorem 1.2, on Proposition 2 in [9] and on the method of creative telescoping [28,46] for the summation of (bivariate) holonomic sequences.

Proposition 3.5 ([9, Proposition 2]). The number $E_3(n)$ of partitions of $[n]$ having no enhanced 3-crossing is given by $E_3(0) = E_3(1) = 1$, and for $n \geq 0$,

$$\begin{aligned} &8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) \\ &- (n+8)(n+7)E_3(n+2) = 0. \end{aligned} \quad (2)$$

Equivalently, the associated generating function $\mathcal{E}(t) = \sum_{n \geq 0} E_3(n)t^n$ satisfies

$$t^2(1+t)(1-8t) \frac{d^2}{dt^2} \mathcal{E}(t) + 2t(6-23t-20t^2) \frac{d}{dt} \mathcal{E}(t) + 6(5-7t-4t^2) \mathcal{E}(t) - 30 = 0.$$

First Proof of Theorem 1.3. By Theorem 1.2, we have

$$T_3(n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n, k) = (-1)^{n-k} \binom{n}{k} E_3(k)$. By Proposition 3.5, and by the closure properties for holonomic functions, it follows that $f(n, k)$ is holonomic. Thus, we can apply Chyzak's algorithm [28] for creative telescoping to derive a recurrence relation for T_3 . In particular, using Koutschan's Mathematica package `HolonomicFunctions.m` [29] that implements Chyzak's algorithm, we find exactly the recurrence equation in Theorem 1.3. $\square$

The detailed calculation involved in the above proof can be found in [7].

The second proof is also based on Proposition 3.5 and on Theorem 1.2, namely on the relation between the generating functions of $T_3(n)$ and of $E_3(n)$ implied by Theorem 1.2.

Second Proof of Theorem 1.3. Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. By Theorem 1.2 and Lemma 2.12,

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

We know from Proposition 3.5 a differential equation for $\mathcal{E}(t)$. By (univariate) closure properties of D-finite functions, we deduce a differential equation for $\mathcal{T}(t)$, and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence in Theorem 1.3. $\square$

Let G be a reductive complex algebraic group and V be a representation of G . If G is connected then the sequence $\mathbf{a}_V$ associated to (G, V) can sometimes be written as an algebraic residue. The general principle is discussed by Gessel and Zeilberger in [23] and in detail by Grabiner and Magyar in [25].

Let K be the character of V and let Δ be given by

$$\Delta = \sum_{w \in W} \varepsilon(w) [w(\rho) - \rho].$$

Here W is the Weyl group, $\varepsilon: W \rightarrow \{\pm 1\}$ is the sign character and ρ is half the sum of the positive roots. These are both elements of the group ring of the root lattice of G . If we choose a basis of the root lattice then these become Laurent polynomials where the number of indeterminates is the rank of G .

For the representations that appear in this paper, the n th term of the sequence $\mathbf{a}_V$ is the constant term in the Laurent polynomial ΔK^n and the generating function is the algebraic residue of the rational function $\Delta/(xy - txyK)$.

Example 3.6. In Example 2.2, the Laurent polynomials are

$$\Delta = (1 - x^{-2}), \quad K = (x - x^{-1}).$$

Then the constant term of the Laurent polynomial

$$(1 - x^{-2})(x - x^{-1})^{2n}$$

is

$$\binom{2n}{n} - \binom{2n}{n+2}.$$

which is a well known expression for the n th Catalan number C_n . This can be proved directly using the reflection principle.

It is a general result that an algebraic residue is D-finite, see [34]. This shows that the generating functions of these sequences are D-finite and hence the sequences are P-recursive.

The third proof of [Theorem 1.3](#) relies on G_2 walks and the method in [4].

The following definition is given in [30] in the paragraph following Conjecture 3.3.

Definition 3.7. The n th term $T_3(n)$ is the constant term of the Laurent polynomial ΔK^n , where

$$K = 1 + x + y + xy + x^{-1} + y^{-1} + (xy)^{-1},$$

and Δ is the Laurent polynomial

$$\begin{aligned} \Delta = & x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ & + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2). \end{aligned}$$

Third Proof of Theorem 1.3. Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. By [Definition 3.7](#), $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $\Delta/(1-tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $\Delta/(xy - txyK)$, and thus it is D-finite [34] and is annihilated by a linear differential operator that cancels the contour integral of $\Delta/(xy - txyK)$ over a cycle. Using the integration algorithm for multivariate rational functions in [5] we compute the following operator of order 6, that cancels $\mathcal{T}(t)$:

$$\begin{aligned} L_6 = & t^5(t+1)(7t-1)(2t+1)^2\partial^6 + \\ & 3t^4(2t+1)(168t^3+211t^2+40t-11)\partial^5 + \\ & 6t^3(2100t^4+3475t^3+1616t^2+79t-61)\partial^4 + \\ & 6t^2(11200t^4+17400t^3+7556t^2+268t-273)\partial^3 + \\ & 36t(4200t^4+6100t^3+2442t^2+54t-77)\partial^2 + \\ & 36(3360t^4+4540t^3+1646t^2+16t-35)\partial + \\ & 20160t^3+25200t^2+8064t, \end{aligned}$$

where $\partial = \frac{\partial}{\partial t}$ denotes the derivation operator with respect to t .

The operator L_6 factors as $L_6 = QL_3$, where

$$Q = (2t+1)t^3\partial^3 + (24t+13)t^2\partial^2 + 6(12t+7)t\partial + 48t+30,$$

and

$$\begin{aligned} L_3 = & t^2(2t+1)(7t-1)(t+1)\partial^3 + \\ & 2t(t+1)(63t^2+22t-7)\partial^2 + \\ & (252t^3+338t^2+36t-42)\partial + 28t(3t+4). \end{aligned} \quad (3)$$

This shows that $f(t) := L_3(\mathcal{T}(t))$ is a solution of the differential operator Q . Hence, by denoting $f(t) = \sum_{n \geq 0} f_n t^n$, one deduces that for all $n \geq 0$ we have

$$2(n+2)f_n + (n+6)f_{n+1} = 0.$$

On the other hand, from $\mathcal{T}(t) = 1 + t^2 + O(t^3)$, it follows that $f_0 = 0$, therefore $f(t) = 0$, in other words $\mathcal{T}(t)$ is also a solution of L_3 . From there, deducing the recurrence relation of [Theorem 1.3](#) is immediate. $\square$

3.4. Closed formulae

In this subsection, we give two closed formulae for the generating function $\mathcal{T}(t)$ of sequence T_3 ([A059710](#)) in terms of classical Gaussian hypergeometric functions ${}_2F_1$.

The operator L_3 in (3) factors as $L_3 = L_2 L_1$, where

$$L_2 = t^2 (2t + 1) (7t - 1) (t + 1) \partial^2 + \frac{t(t + 1)P_1}{P} \partial + \frac{P_2}{P^2},$$

and

$$L_1 = \partial - \frac{\frac{d}{dt}(P/t^5)}{P/t^5}$$

with

$$P_1 = 3136t^6 + 7560t^5 + 5744t^4 + 1592t^3 - 90t^2 - 131t - 9,$$

$$P_2 = 131712t^{11} + 719712t^{10} + 1626800t^9 + 2014088t^8 + 1498264t^7 + 665620t^6 + 146508t^5 - 4560t^4 - 11138t^3 - 2663t^2 - 244t - 7$$

and

$$P = 28t^4 + 66t^3 + 46t^2 + 15t + 1.$$

Clearly, the operator L_1 has a basis of solutions formed of $\{P/t^5\}$. Let $\{f_1, f_2\}$ be a basis of solutions of L_2 . Then one can show that the solution $\mathcal{T}(t)$ of L_3 is

$$\frac{P}{t^5} \cdot \int (C_1 f_1 + C_2 f_2) \frac{t^5}{P} dt,$$

for some constants C_1 and C_2 to be determined.

Using algorithms for solving second order differential equations, as described in [4], one deduces that f_1 and f_2 have hypergeometric expressions. After identifying the constants C_1 and C_2 , we derive the following explicit formula for $\mathcal{T}(t)$. Define the polynomials:

$$S = (7t - 1)(t + 1)(2t + 1)^2,$$

$$U = (11t - 1)(46t^3 - 78t^2 + 15t - 1),$$

and

$$V = 11870t^7 - 6934t^6 - 13371t^5 + 1115t^4 + 1112t^3 - 300t^2 + 29t - 1,$$

as well as the rational function

$$\phi = \frac{27(t + 1)t^2}{(1 - t)^3}.$$

Then the generating function $\mathcal{T}(t)$ is equal to

$$\frac{P}{30t^5} \int \frac{S}{P^2(t - 1)^2} \left[U {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}; \phi \right) + \frac{V}{(1 - t)^3} {}_2F_1 \left(\frac{4}{3}, \frac{5}{3}; \phi \right) \right] dt.$$

Note that the above indefinite integrals have no constant term. This expression can be further simplified by introducing the additional rational functions

$$R_1 = \frac{(t + 1)^2 (214t^3 + 45t^2 + 60t + 5)}{t - 1}$$

and

$$R_2 = 6 \frac{t^2 (t + 1)^2 (101t^2 + 74t + 5)}{(t - 1)^2}.$$

Then the simpler expression for $\mathcal{T}(t)$ is

$$\mathcal{T}(t) = \frac{1}{30t^5} \left[R_1 \cdot {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}; \phi \right) + R_2 \cdot {}_2F_1 \left(\frac{2}{3}, \frac{4}{3}; \phi \right) + 5P \right].$$

Following the approach in [4, §3.3], one can obtain an alternative expression for $\mathcal{T}(t)$ by using an approach with a more geometric flavour. The key point is that the denominator of $W/(xy - txyK)$ is a family of *elliptic curves*, thus integrating $W/(xy - txyK)$ over a small torus amounts to computing the periods of the two forms (of the first and the second kind). Working out the details, this approach yields an expression for $\mathcal{T}(t)$ in terms of the Weierstrass invariant

$$g_2 = (t - 1)(25t^3 + 21t^2 + 3t - 1)$$

and of the j -invariant

$$J = \frac{(t - 1)^3 (25t^3 + 21t^2 + 3t - 1)^3}{t^6 (1 - 7t)(2t + 1)^2 (t + 1)^3}$$

of our family of curves. As in [4], we introduce the expression

$$H(t) = \frac{1}{g_2^{1/4}} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1728}{J}\right).$$

Then the generating function $\mathcal{T}(t)$ is equal to

$$\frac{P}{6t^5} + \frac{(7t - 1)(2t + 1)(t + 1)}{360t^5} \left((155t^2 + 182t + 59)(11t + 1)H(t) + (341t^3 + 507t^2 + 231t + 1)(5t + 1)H'(t) \right).$$

Moreover, one can use the methods in [4] to deduce that the generating function $\mathcal{T}(t)$ is a transcendental power series and that the asymptotic behaviour of its n th coefficient $T_3(n)$ is

$$T_3(n) \sim C \cdot \frac{7^n}{n^7}, \quad \text{where } C = \frac{4117715}{864} \frac{\sqrt{3}}{\pi} \approx 2627.6.$$

4. Quadrant sequences

In this section we consider a family of sequences associated to the representation theory of the simple algebraic group $SL(3)$ by the construction [Definition 2.1](#). The algebraic group $SL(3)$ has dimension 8 and rank 2. Let V be the three-dimensional vector representation and V^* be the dual representation. These are the two fundamental representations of $SL(3)$.

Definition 4.1. For $k \geq 0$, the quadrant sequences S_k are the sequence associated to $(SL(3), V \oplus V^* \oplus k\mathbb{C})$.

By [Lemma 2.8](#), those sequences are also related by binomial transforms.

By the general theory in [Proposition 2.7](#) these sequences enumerate walks in the weight lattice of $SL(3)$. Equivalently, these are walks in the positive quadrant which is our reason for calling them *quadrant sequences*.

The n th term of the sequence S_0 enumerates A_2 -webs with n boundary points, see [31]. If we use the representation theory of $GL(3)$ (instead of $SL(3)$) then we can translate this to a combinatorial interpretation in terms of rectangular semistandard tableaux with three rows.

Let $\text{SST}(\lambda; \alpha)$ be the set of semistandard tableaux of shape λ and weight α . Here λ is a partition, α is a composition and $|\lambda| = |\alpha|$.

Proposition 4.2. The first three quadrant sequences have the following interpretations in terms of rectangular semistandard tableaux.

- (1) The n th term of the sequence S_0 is the number of rectangular semistandard tableaux with three rows whose weight, α , is a composition of length n such that $\alpha_i \in \{1, 2\}$ for $1 \leq i \leq n$.
- (2) The n th term of the sequence S_1 is the number of rectangular semistandard tableaux with three rows whose weight, α , is a composition of length n such that $\alpha_i \in \{0, 1, 2\}$ for $1 \leq i \leq n$.

- (3) The n th term of the sequence S_1 is the number of rectangular semistandard tableaux with three rows whose weight, α , is a composition of length n such that $\alpha_i \in \{1, 2, 3\}$ for $1 \leq i \leq n$.
- (4) The n th term of the sequence S_2 is the number of rectangular semistandard tableaux with three rows whose weight, α , is a composition of length n such that $\alpha_i \in \{0, 1, 2, 3\}$ for $1 \leq i \leq n$.

Proof. There are easy bijections with the respective walks. A semistandard tableaux can be regarded as a sequence of partitions where the i th partition is the shape given by taking the cells whose entry is at most i . For a semistandard tableau with (at most) three rows this gives a sequence of vectors $(\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Each such vector maps to the point $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$. This constructs a map from semistandard tableaux with three rows to walks in the two-dimensional lattice which stay in the non-negative quadrant. This map gives all bijections. $\square$

Proposition 4.3. The first three quadrant sequences have the following expressions.

- (1) The n th term of the sequence S_0 is

$$\sum_{m \geq 0} \sum_{\substack{a, b, c \geq 0 \\ a+b=n \\ a+2b=3m}} \binom{n}{a, b} |\text{SST}(3^m; 1^a 2^b)|$$

- (2) The n th term of the sequence S_1 is

$$\sum_{m \geq 0} \sum_{\substack{a, b, c \geq 0 \\ a+b+c=n \\ b+2c=3m}} \binom{n}{a, b, c} |\text{SST}(3^m; 0^a 1^b 2^c)|$$

- (3) The n th term of the sequence S_1 is

$$\sum_{m \geq 0} \sum_{\substack{a, b, c \geq 0 \\ a+b+c=n \\ a+2b+3c=3m}} \binom{n}{a, b, c} |\text{SST}(3^m; 1^a 2^b 3^c)|$$

- (4) The n th term of the sequence S_2 is

$$\sum_{m \geq 0} \sum_{\substack{a, b, c, d \geq 0 \\ a+b+c=nb+2c+3d=3m}} \binom{n}{a, b, c, d} |\text{SST}(3^m; 1^b 2^c 3^d)|$$

Proof. These follow directly from [Proposition 4.2](#) together with the result that $|\text{SST}(\lambda; \alpha)| = |\text{SST}(\lambda; \alpha')|$ if α' is a reordering of α . $\square$

Note that it is clear from these expressions that S_1 is the binomial transform of S_0 and that S_2 is the binomial transform of S_1 . This uses the observation that $|\text{SST}(3^m; 1^a 2^b)| = |\text{SST}(3^{m+c}; 1^a 2^b 3^c)|$ for all $a, b, c, m \geq 0$ such that $a + 2b = m$.

Theorem 4.4. The quadrant sequences S_0, S_1, S_2, S_3 are identical to the sequences in the second family specified in [Fig. 2](#).

Proof. The sequence S_0 appears as sequence [A151366](#). This sequence is defined in [\[8\]](#) using lattice walks.

The sequence S_1 appears as sequence [A236408](#). There is no published reference for this sequence and we discuss this sequence in [Section 4.2](#).

The sequence S_2 appears as sequence [A001181](#). This is the sequence of Baxter permutations. These two sequences can be seen to be the same by noting that [Proposition 4.3](#) together with the known formula for the number of rectangular partitions agrees with the formula in [\[15\]](#) for the number of Baxter permutations. Alternatively one can check that both sequences satisfy the same

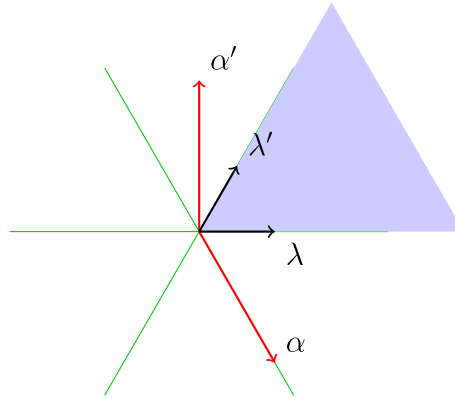


Fig. 7. A_2 root system.

recurrence relation and initial conditions. A bijective proof is given by noting that there is an easy bijection using the combinatorial interpretation in Proposition 4.2 and [18, Theorem 2]. Alternative proofs are given in [11, Proposition 23], and [16, Theorem 3].

The sequence S_3 appears as sequence A216947. This sequence is defined in [37] using 2-coloured noncrossing set partitions. This paper also gives an easy bijection with the lattice paths. $\square$

Definition 2.1 generalizes to give sequences $\mathbf{a}_{V \oplus V^* \oplus k \mathbb{C}, \lambda}$. Here $\lambda = (r, s)$ for $r, s \geq 0$ and the sequence enumerates lattice walks which start at $(0, 0)$ and end at (r, s) .

4.1. Root systems

The root system of $SL(3)$ is shown in Fig. 7 with the fundamental chamber shaded. The two simple roots are α and α' . The two fundamental weights are λ and λ' . The fundamental representations are the three-dimensional vector representation and its dual.

The element ρ is

$$\rho = \alpha + \alpha' = \lambda + \lambda'.$$

All four entries in the OEIS give a recurrence relation for each quadrant sequence. These recurrence relations all agree with, or are consequences of, the recurrence relations in Section 4.4. If the recurrence relation in the OEIS has a proof then we can check that the initial terms agree and so we have an alternative proof that the two sequences agree. If the recurrence relation in the OEIS is conjectural then our results prove that these recurrence relations are satisfied by the sequence.

As a consequence of the identification of S_2 with the Baxter sequence we can give a formula for the terms of S_k . This formula is obtained by substituting the formula (1) in [15] for the terms in the Baxter sequence into the formula in Lemma 2.11 for iterated binomial transforms.

Other known results that are consequences are: A bijective proof of the following is given in [11, Proposition 23] and [16, Theorem 3], and in a recent paper by Yan [44].

Proposition 4.5. *The set of hesitating tableaux of length n which end with a single row partition is equinumerous with the set of Baxter permutations on $[n]$.*

A bijective proof of the following can be found in [16, Corollary 14].

Proposition 4.6. *The set of vacillating tableaux of length n which end with a single row partition is equinumerous with the set of 2-coloured noncrossing set partitions of $[n]$.*

4.2. Increasing maps

In this subsection we give a combinatorial interpretation of the sequence S_1 . This depends on the closely related combinatorial interpretations of the Baxter sequence in [1] (mosaic floor plans), [3] (plane bipolar orientations), and [32] (diagonal rectangulations). All three of these interpretations are equivalent to oriented st-maps.

A *map* is a connected graph embedded in the plane with no edge-crossings, considered up to isotopy. The vertices and edges of the map are those of the graph. The faces of the map are the connected components of the complement of the graph in the plane. The *outer face* is unbounded, the inner faces are bounded.

An *oriented map* is a map together with an orientation of each edge. An oriented map is an *st-map* if it has a unique source, s , and a unique sink, t , both on the outer face.

An *increasing map* is an oriented map together with an increasing labelling of the vertices. If the map has r vertices then the labelling, ℓ , is a bijection between the vertices and the set $\{1, 2, \dots, r\}$. A labelling is increasing if whenever there is a directed path from vertex v to vertex w then $\ell(v) < \ell(w)$. Let O be an increasing st-map. Then $\ell(s) = 1$ and $\ell(t) = r$.

A *simple increasing st-map* is an increasing st-map whose underlying graph has no multiple edges.

Proposition 4.7. *Let $a(n)$ be the number of increasing st-maps with n edges and let $a_S(n)$ be the number of simple increasing st-maps with n edges. Then*

$$a(n+1) = \sum_{k=0}^n \binom{n}{k} a_S(k+1)$$

Proof. There is a bijection between simple increasing st-maps with each edge labelled by a positive integer and increasing st-maps since we can replace each edge labelled by m with m copies of the same edge. This is a bijection between increasing st-maps with n edges and pairs consisting of a simple increasing st-map with k edges and a labelling of the edges by positive integers whose sum is n . The result follows from the observation that the number of sequences of positive integers of length k with sum n is $\binom{n+1}{k+1}$. $\square$

One corollary of Proposition 4.7 is that the conjectured recurrence relation for A236408 is correct.

Plane bipolar orientations arise in the unpublished notes by James Cranch, *Pasting Diagrams*, as the morphisms in a strict monoidal category.

First we construct the category. The set of objects is $\mathbb{N}$ so we have an object $[n]$ for each $n \in \mathbb{N}$. Let O be an oriented st-map. One of the oriented paths going from s to t has the outer face on its right: this path is the *right border* of O , and its length is the *right outer degree* of O . The *left border* and *left outer degree* is defined similarly. An oriented st-map, O , is a morphism $[l] \rightarrow [r]$ where l is the left outer degree of O and r is the right outer degree. Composition of O and O' is only defined if the right outer degree of O is the left outer degree of O' . In this case, the composite $O \circ O'$ is the oriented st-map given by identifying the right border of O with the left border of O' . The identity morphism of $[n]$ is a path with n edges.

This category has a tensor product. On objects this is just $[n] \otimes [m] = [n+m]$ so the monoid of objects is $\mathbb{N}$. The tensor product $O \otimes O'$ is always defined and is the plane bipolar orientation given by identifying the sink of O with the source of O' . It is straightforward to check that this tensor product is strictly associative and that these are compatible in the sense that they satisfy the interchange condition $(O \circ O') \otimes (P \circ P') = (O \otimes P) \circ (O' \otimes P')$. Hence this is a strict monoidal category, $\mathcal{P}$. Simple plane bipolar orientations are the morphisms of a strict monoidal subcategory, $\mathcal{P}_S$.

The monoidal category $\mathcal{P}$ is the free strict monoidal category whose monoid of objects is $\mathbb{N}$ with a morphism $\alpha(n, m): [n] \rightarrow [m]$ for $n, m > 0$. The monoidal subcategory $\mathcal{P}_S$ is the free strict monoidal category whose monoid of objects is $\mathbb{N}$ with a morphism $\alpha(n, m): [n] \rightarrow [m]$ for $n, m > 0$ with $\alpha(1, 1)$ omitted. The morphism $\alpha(n, m)$ consists of a single inner face with n edges on the left border and m edges on the right border.

4.3. Branching rules

In this subsection we use the branching rules for the inclusion of $SL(3)$ in G_2 to relate the two families of sequences.

The interpretation using invariant theory gives a connection between the octant sequences and the quadrant sequences. This uses the fact that $SL(3)$ is a subgroup of G_2 , in fact, a maximal subgroup. Then the restriction of the representation $V \oplus k\mathbb{C}$ from G_2 to $SL(3)$ is the representation $V \oplus V^* \oplus (k+1)\mathbb{C}$. This implies that each quadrant sequence $\mathbf{a}_{V \oplus k\mathbb{C}, \lambda}$ can be written as a linear combination of the octant sequences $\mathbf{a}_{V \oplus k\mathbb{C}, \mu}$ with coefficients which are independent of k . These coefficients are the branching rules for the inclusion of $SL(3)$ in G_2 . A combinatorial description of the branching rules for the inclusion

$$SL(3) \rightarrow G_2$$

is given in [27,39].

A special case of this is the result that walks in the quadrant which end at $(0, 0)$ correspond to walks in the octant which end on the x -axis, see [11,11,12,16,45].

The generating function for the branching rules for the inclusion of the maximal subgroup

$$SL(3) \rightarrow G_2$$

are given by Gaskell and Sharp in [22, (2.3)]. Let $V(r, s)$ be a highest weight representation of G_2 with highest weight (r, s) and let $U(p, q)$ be a highest weight representation of $SL(3)$ with highest weight (p, q) . Denote the multiplicity of $U(p, q)$ in the restriction of $V(r, s)$ to $SL(3)$ by $m_{(p,q)}^{(r,s)}$.

Proposition 4.8. *The generating function*

$$\sum_{r,s,p,q \geq 0} m_{(p,q)}^{(r,s)} x^p y^q X^r Y^s$$

is the rational function

$$\frac{(1-X)^{-1} - xyY(1-xyY)^{-1}}{(1-xX)(1-yX)(1-xY)(1-yY)} \quad (4)$$

Theorem 4.9. *Let V be a representation of G_2 and $V \downarrow$ the restriction to $SL(3)$. Then the number of axis-walks of length n for V is the number of excursions for $V \downarrow$.*

Proof. Putting $x = 0, y = 0$ in the generating function (4) gives

$$\sum_{r,s,p,q \geq 0} m_{(0,0)}^{(r,s)} X^r Y^s.$$

Putting $x = 0, y = 0$ in the rational function gives $(1-X)^{-1}$. This shows that

$$m_{(0,0)}^{(r,s)} = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Example 4.10. First we extend each octant sequence to a sequence of functions on the dominant weights of G_2 , or, equivalently, on the octant $y \geq 0, x \geq y$. It follows from Lemma 2.11 that these can be written as polynomials in k .

Then the octant sequences are given by

$$\begin{array}{c|c|c} 1 & k & 1 \\ \hline 1+k^2 & 1+2k & 1 \end{array} \quad \begin{array}{c|c|c} 1 & 2+2k & 2 \\ \hline 1+3k+k^3 & 4+3k+3k^2 & 3+3k \end{array} \quad \begin{array}{c|c|c} 1 & 2 & 1 \\ \hline 1 & 1 & 1 \end{array}$$

Then summing the entries on the bottom row gives the sequence

$$1, 1 + k, 3 + 2k + k^2, 9 + 9k + 3k^2 + k^3.$$

This proves [Propositions 4.5](#) and [4.6](#).

4.4. Recurrence equations

In this subsection we give recurrence relations for the sequences S_k . A recurrence relation for S_3 is given by Marberg in [[37](#), § 4].

Lemma 4.11. *Let $\mathcal{G}_k$ be the generating function of S_k , where $k \geq 0$. Then $\mathcal{G}_k$ is the constant coefficient $[x^0 y^0]$ of $W/(1 - tK)$, where*

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x} \quad (5)$$

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2 y^2 + y^3 - \frac{y^2}{x}. \quad (6)$$

Let $C_2(n)$ be the n th term of the sequence [A216947](#) in the second family. Marberg [[37](#), Theorem 1.7] showed that $C_2(n)$ is the constant term $[x^0 y^0]$ of $W\tilde{K}^n$, where $\tilde{K} = K|_{k=3}$, the Laurent polynomials K and W are specified in (5) and (6). By [Lemma 4.11](#), the sequence S_3 is identical to the sequence [A216947](#). Therefore, we have:

Proposition 4.12 ([\[37, Theorem 1.7\]](#)). *The n th term $C_2(n)$ of the sequence S_3 is given by $C_2(0) = 1$, $C_2(1) = 3$ and for $n \geq 0$:*

$$\begin{aligned} & (n+5)(n+6) \cdot C_2(n+2) - 2(5n^2 + 36n + 61) \cdot C_2(n+1) \\ & + 9(n+1)(n+4) \cdot C_2(n) = 0, \end{aligned}$$

Equivalently, the associated generating function $\mathcal{G}_3(t) = \sum_{n \geq 0} C_2(n)t^n$ satisfies

$$\begin{aligned} & 72\mathcal{G}_3(t) + 4(-61 + 117t)\frac{d}{dt}\mathcal{G}_3(t) + 2(15 - 184t + 234t^2)\frac{d^2}{dt^2}\mathcal{G}_3(t) \\ & + 2t(-6 + 7t)(-1 + 9t)\frac{d^3}{dt^3}\mathcal{G}_3(t) + (-1 + t)t^2(-1 + 9t)\frac{d^4}{dt^4}\mathcal{G}_3(t) = 0. \end{aligned}$$

Next, for $k = 0, 1, 2, 3$, we prove a uniform recurrence equation for the sequence S_k . It is given by a single formula with k as a parameter. Moreover, we show that S_0, S_1, S_2, S_3 are identical to the sequences in the second family.

Theorem 4.13. *For $k = 0, 1, 2, 3$, the n th term $a(n)$ of the sequence S_k satisfies the following recurrence equation:*

$$\begin{aligned} & (-3 + k)^2(-2 + k)(6 + k)(1 + n)(2 + n)a(n) + \\ & - 2(-3 + k)(2 + n)(-60 + 8k + 8k^2 - 18n + 3kn + 2k^2n)a(n+1) + \\ & (-342 - 174k + 114k^2 - 195n - 70kn + 54k^2n - 27n^2 - 6kn^2 + 6k^2n^2)a(n+2) \\ & + 2(57 - 70k + 16n - 24kn + n^2 - 2kn^2)a(n+3) \\ & + (n+7)(n+8)a(n+4) = 0. \end{aligned} \quad (7)$$

Proof. By [Lemma 2.8](#), sequences S_2, S_1, S_0 are the first, second, and third inverse binomial transforms of S_3 , respectively. Thus, it follows from [Lemma 2.12](#) that the generating function of S_k is

$$\mathcal{G}_k(t) = \frac{1}{1 + (3 - k)t} \cdot \mathcal{G}_3\left(\frac{t}{1 + (3 - k)t}\right) \quad \text{for } k = 0, 1, 2, 3,$$

where $\mathcal{G}_3(t)$ is the generating function of S_3 . Regarding k as a parameter in the above expression, we deduce the differential equation for $\mathcal{G}_k(t)$ by using [Proposition 4.12](#) and closure properties of D-finite functions. By converting the differential equation for $\mathcal{G}_k(t)$, we get the corresponding recurrence equation for the sequence $a(n)$, which is exactly the recurrence equation in the claim. $\square$

The argument of [8, Proposition 10] gives rise to a proof for the recurrence relation of [A151366](#) stated in the OEIS. For the Baxter sequence [A001181](#), the recurrence relation can be verified by utilizing its explicit formula in terms of binomial coefficients and the creative telescoping method. The details for the verification can be found in [7]. The recurrence equation for [A216947](#) is proven in [37]. As far as we know, the stated recurrence relation in the OEIS for [A236408](#) is only conjectured. Next, we give a unified proof for all those recurrence relations.

Corollary 4.14. *The recurrence relations stated in the OEIS for the sequences in the second family specified in [Fig. 2](#) are true. Moreover, the sequences in the second family are related by binomial transforms.*

Proof. By [Theorem 4.4](#), we see that the quadrant sequences S_0, S_1, S_2 and S_3 are identical to the sequences in the second family specified in [Fig. 2](#). In (7), by setting k to 0, 1, 2, 3, we find that the corresponding recurrence operators are left multiples [14, page 618] of those of [A216947](#), [A001181](#), [A236408](#), and [A151366](#) specified in the OEIS, respectively. To verify that they satisfy those recurrence relations, we just need to check finitely many initial terms. The details of the verification can be found in [7]. Since S_k 's are related by binomial transforms, so are sequences in the second family. $\square$

Closed formulae for these sequences can be obtained by the same methods as in [Section 3.4](#). Define the function

$$H(x) = {}_2F_1\left(\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix}; \frac{27x^2}{(1-2x)^3}\right).$$

Then the ordinary generating function of S_2 is equal to

$$\frac{(x+1)^2(1-8x)}{3x^2(1-2x)^2} \left(H + \frac{12(1+20x-8x^2)(1-2x)}{(x+1)} H' \right) - \frac{(3x^2-x+1)}{(3x^2)}$$

The above formula is equivalent to the following formula for [A001181](#) in the OEIS.

$$\begin{aligned} & -1 + \frac{1}{3x^2} \left[(x-1) + (1-2x) {}_2F_1\left(\begin{matrix} -\frac{2}{3} & \frac{2}{3} \\ 1 \end{matrix}; \frac{27x^2}{(1-2x)^3}\right) \right. \\ & \left. - \frac{(8x^3-11x^2-x)}{(1-2x)^2} {}_2F_1\left(\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 2 \end{matrix}; \frac{27x^2}{(1-2x)^3}\right) \right]. \end{aligned}$$

A closed formula for the generating function of S_k follows by substitution in [Lemma 2.12](#). The asymptotics is stated in [15] and a detailed account is given in [19, Theorem 1].

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