

Efficient Algorithms for Mixed Creative Telescoping

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ABSTRACT

Creative telescoping is a powerful computer algebra paradigm initiated by Doron Zeilberger in the 90's— for dealing with definite integrals and sums with parameters. We address the mixed continuous–discrete case, and focus on the integration of bivariate hypergeometric–hyperexponential terms. We design a new creative telescoping algorithm operating on this class of inputs, based on a Hermite-like reduction procedure. The new algorithm has two nice features: it is efficient and it delivers, for a suitable representation of the input, a minimal-order telescoper. Its analysis reveals tight bounds on the sizes of the telescoper it produces.

CCS Concepts

•Computing methodologies → Algebraic algorithms;

Keywords

symbolic integration; creative telescoping; hypergeometric–hyperexponential term; Hermite reduction

1. INTRODUCTION

Context. *Creative telescoping* is an algorithmic approach introduced in computer algebra by Zeilberger [23, 24, 21] to address definite summation and integration for a large class of functions and sequences involving parameters.

In this article, we focus on the mixed continuous–discrete case. Given a term $F_n(x)$ that is both hypergeometric (i.e., $F_{n+1}(x)/F_n(x)$ is a rational function) and hyperexponential (i.e., $F'_n(x)/F_n(x)$ is a rational function), the question is to find a linear recurrence relation satisfied by the sequence of integrals $I_n = \int_{\gamma} F_n(x)dx$ over a domain γ where $F_n(x)$ is integrable. To do this, the method of creative telescoping looks for polynomials $c_0(n), \dots, c_r(n)$, not all zero, and for a rational function $Q(n, x)$ such that $G_n(x) = Q(n, x)F_n(x)$

satisfies the telescoping relation

$$L(F_n(x)) \stackrel{\text{def}}{=} \sum_{i=0}^r c_i(n)F_{n+i}(x) = G'_n(x). \quad (1)$$

The recurrence operator $L = \sum_{i=0}^r c_i(n)S_n^i$ in the shift operator S_n is called a *telescoper* for $F_n(x)$, and the rational function $Q(n, x)$ is called a *certificate* for the telescoper L . The integer r is the *order* of L and $\max_j \deg c_j$ is its *degree*.

Throughout the article, the ground field, denoted by $\mathbf{k}$, is assumed to be of characteristic zero. Under suitable additional assumptions, $L(I_n) = 0$ is a recurrence relation satisfied by the sequence of integrals $I_n = \int_{\gamma} F_n(x)dx$.

A subtle point is that not all hypergeometric–hyperexponential terms admit telescopers. A criterion for deciding when this is the case has been given only recently [7, Section 6]: a hypergeometric–hyperexponential term is telescopic if and only if it can be written as the sum of a derivative and of a *proper term*. In our context, proper terms are of the form

$$P(n, x) \cdot H(x)^n \cdot \exp\left(\int \frac{S(x)}{T(x)}\right) \cdot \Upsilon(n), \quad (2)$$

for $P \in \mathbf{k}[n, x]$, $H \in \mathbf{k}(x)$, $S, T \in \mathbf{k}[x]$ and Υ hypergeometric.

Several methods are known for computing a telescoper L and the corresponding certificate Q , even in much greater generality [1, 21, 11, 20, 2, 15]. Despite a very rich research activity during the last decade, not much is known about the complexity of creative telescoping methods when applied to an input of the type (2). In particular, few estimates are available in the literature on the order and degree of the minimal-order telescoper.

If $\Upsilon(n) \neq 1$, one can compute a telescoper of the form (1) for the rest and then multiply the coefficient $c_i(n)$ by the rational function $\Upsilon(n)/\Upsilon(n+i)$; this process does not affect the minimality of the telescoper. Thus we focus on the case

$$F_n(x) = P(n, x) \cdot H(x)^n \cdot \exp\left(\int \frac{S(x)}{T(x)}\right). \quad (3)$$

Previous work. Among the various classes of creative telescoping methods, three different approaches allow to obtain bounds on the sizes of the telescoper and the certificate, and to control the algorithmic complexity. The first approach is based on elimination techniques [23, 21, 22]; it generally yields pessimistic bounds, not very efficient algorithms and telescopers of non-minimal order. The second approach, initiated by Apagodu and Zeilberger [2], provides sharp bounds on the order and degree of telescopers, more efficient algorithms, but does not provide information on telescopers of minimal

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ISSAC'16, July 19–22, 2016, Waterloo, ON, Canada

ACM ISBN 978-1-4503-4380-0/16/07...\$15.00.

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DOI: <http://dx.doi.org/10.1145/2930889.2930907>

order and works under a restrictive genericity assumption. It was successfully applied by Kauers and co-authors [10, 9, 14] to (bivariate) hyperexponential terms and hypergeometric terms. The third approach, based on Hermite-like reduction, is the only one that computes telescopers of minimal order, while guaranteeing a good control on sizes and complexity. It has been introduced by Bostan *et alii* for the integration of bivariate rational functions [3], then extended to bivariate hyperexponential functions [4], multivariate rational functions [5, 17] and recently adapted to summation [8] for bivariate hypergeometric terms. The present work is part of the on-going effort in this direction.

Contributions. We present the first Hermite-style algorithm in the mixed (continuous–discrete) setting. Our approach is inspired by the proof of Manivel’s lemma [12, §2.3], originally designed in connection with the so-called *polynomial rigidity conjecture*.

Our algorithm works on terms of the form (3). Its input is $F_n(x) = P(n, x)\Phi(n, x)$, given by

$$P(n, x), \quad \frac{\Phi'}{\Phi}(n, x) = \frac{A(n, x)}{B(x)} = n \frac{H'(x)}{H(x)} + \frac{S(x)}{T(x)}, \quad (*)$$

i.e., a polynomial in $\mathbf{k}[n, x]$ and a rational function in $\mathbf{k}(n, x)$ of a special form given either in lowest terms (A/B with $\gcd(A, B) = 1$ and $\deg_n B = 0$) or decomposed as the sum of the logarithmic derivative of a rational function $H \in \mathbf{k}(x)$ multiplied by n and another rational function $S/T \in \mathbf{k}(x)$. Note that, for a given term, several choices are available for P and Φ . If we further assume that Φ'/Φ has no positive integer residue, they become unique and we call them the *minimal decomposition* of $F_n(x)$.

Our main results consist in bounds on the order and degree of telescopers for $F_n(x)$, that are summarized in Theorem 1 below (see Theorems 6 and 15 for more precise statements). A complexity analysis of the algorithms leading to these bounds is conducted in section 4 (see in particular Theorem 21).

In the statement of the following theorem, $\deg_x H$ and $\deg_n P$ denote the maximum of the degrees of their numerator and denominator, and d_H is the degree of H at infinity.

Theorem 1 *Given a decomposition as in (*), $F_n(x)$ admits a telescoper of order r bounded by δ , where*

$$\delta := \max(\deg_x A, \deg_x B - 1), \quad (4)$$

and degree bounded by

$$r(\deg_x H + \deg_n P) + \deg_x P.$$

Moreover, if the decomposition is minimal then these bounds apply to a minimal-order telescoper for $F_n(x)$.

Algorithm MixedCT (§2.3) produces a telescoper with these properties. If d denotes an upper bound on the degrees of the numerator and denominator of H , and of all the polynomials in (), and if all these polynomials are square-free, then the telescoper has arithmetic size $O(d^3)$ and MixedCT computes it using at most*

$$\begin{cases} \tilde{O}(d^{\omega+1}) & \text{if } d_H < 0, \\ \tilde{O}(d^5) & \text{if } d_H \geq 0 \end{cases}$$

arithmetic operations in $\mathbf{k}$, where ω denotes a feasible matrix multiplication exponent for $\mathbf{k}$.

Note that one can always choose the decomposition of $F_n(x)$ in (*) to be minimal. Indeed, we may write $\Phi = Q\tilde{\Phi}$ where Q is a polynomial and $\tilde{\Phi}'/\tilde{\Phi}$ has no positive integer residue. We then have a minimal decomposition $F_n(x) = (PQ)\tilde{\Phi}$.

The proof of Theorem 1 is achieved through two main ingredients: *confinement* and *Hermite reduction*. Confinement is the property that given Φ , any polynomial P can be reduced modulo derivatives to a polynomial R of degree at most $\delta - 1$: $P\Phi = R\Phi + \Gamma'$ for some Γ . It gives a finite dimensional vector space over $\mathbf{k}(n)$, where the computation will be confined. Hermite reduction is more classical. In this context, it consists in an algorithm that performs a reduction $P\Phi(n+1, x) = R\Phi(n, x) + \Gamma'$ for some Γ . By iterated application of both these operations, all $P(n+i, x)\Phi(n+i, x)$ for $i = 0, \dots, \delta$ can be rewritten in a vector space of dimension δ over $\mathbf{k}(n)$. Thus by (polynomial) linear algebra there exists a non-trivial linear relation between them, i.e., a telescoper. A more careful study of the increase of the degrees in n during these rewritings gives the degree bound.

Notation. In all that follows, $\mathbf{k}$ and $\mathbb{K}$ will denote fields of characteristic 0. In our applications we will often set $\mathbb{K} = \mathbf{k}(n)$. We denote by $\mathbb{K}[x]_d$ the set of polynomials in $\mathbb{K}[x]$ of degree less than d .

Rational functions are always written in reduced form, with monic denominator. Thus the numerator and denominator are defined without ambiguity. If k is the degree of the numerator and ℓ the degree of the denominator of a rational function F , we say that F has *rational degree* (k, ℓ) , that we denote $\text{Rdeg}(F) = [k]/[\ell]$. We also define the *regular degree* $\text{deg}(F)$ of F as $\text{deg}(F) = \max(k, \ell)$. Finally, the *degree at infinity* of F is defined as $\text{deg}^\infty(F) = k - \ell$. The variable with respect to which the degree is taken will be indicated as a subscript when there is an ambiguity.

A polynomial is called *square-free* when its gcd with its derivative is trivial. The *square-free decomposition* of a monic polynomial $Q \in \mathbb{K}[x]$ is a factorization $Q = Q_1^1 \cdots Q_m^m$, with $Q_i \in \mathbb{K}[x]$ monic and square-free, the Q_i ’s pairwise coprime and $\deg_x(Q_m) > 0$. The *square-free part* of Q is the polynomial $Q^* = Q_1 \cdots Q_m$.

Structure of the article. Section 2 gives the main properties of the confinement and of the mixed Hermite-like reduction, leading to the bound on the order of the telescoper. Section 3 gives a bound on the degree of the telescoper and analyzes the evolution of the degrees during the reductions, preparing the complexity analyses in Section 4. We conclude the article in Section 5 with a few applications of our implementation and experiments on the actual growth of the minimal-order telescopers.

2. ALGORITHMS AND ORDER BOUND

In this section, we introduce the algorithms with just enough information to prove their correctness and to obtain a bound on the order of the telescoper they compute. A more thorough analysis of the degrees is in the next section.

2.1 Confinement

In terms of integrals, the operation of confinement writes

$$\int_\gamma P(n, x)\Phi(n, x) dx = \int_\gamma R(n, x)\Phi(n, x) dx,$$

with R a polynomial of degree smaller than δ from Eq (4). This transformation is based on the following lemma.

Lemma 2 Let Φ , A and B be as in Eq. (★), and δ as in Eq. (4). Then for any polynomial $P \in \mathbb{K}[x]$, there exist unique polynomials R and Q in $\mathbb{K}[x]$ with $\deg R \leq \delta - 1$ and $\deg Q \leq \deg P - \delta$ such that

$$P\Phi = R\Phi + (QB\Phi)'. \quad (5)$$

Thus all polynomial multiples of Φ can be written modulo derivatives on a vector space of dimension δ .

PROOF. Equation (5) rewrites

$$QA + (QB)' = P - R. \quad (6)$$

Denote $d = \deg(P) - \delta$, and consider the linear map

$$f : \mathbb{K}[x]_{d+1} \rightarrow \mathbb{K}[x]_{d+1} \quad Q \mapsto (QA + (QB)') \operatorname{div} x^\delta,$$

where $u \operatorname{div} v$ denotes the quotient in the Euclidean division of u by v . For $m \leq d$, $f(x^m)$ has degree at most m . Its coefficient of degree m equals either $\operatorname{lc} A \neq 0$ if $\delta = \deg A > \deg B - 1$, or $m + \delta + 1 \neq 0$ if $\delta = \deg B - 1 > \deg A$, or $\operatorname{lc} A + m + \delta + 1$ if $\deg A = \deg B - 1$. For this case to occur, it is necessary that $\deg S < \deg T$. In that case, write $H'/H = U/V$ with U, V polynomials such that $\deg U = \deg V - 1$. Then $A/B = (nUT + VS)/(VT)$, and

$$\deg(VS) \leq \deg V + \deg T - 1 = \deg(UT).$$

From this we see that $\operatorname{lc}(A)$ depends on n and we deduce that $f(x^m)$ has degree exactly m in all cases.

Thus f is an isomorphism. It follows that Equation (6) is equivalent to Q being the unique polynomial such that $f(Q) = P \operatorname{div} x^\delta$, and then R is $P - QA - (QB)'$. The degrees of Q and R directly follow from the construction. $\square$

Algorithm Confinement implements the proof of Lemma 2. Its correctness follows from the fact that the relation used in the loop is obtained by extracting the coefficient of $x^{i+\delta}$ in Eq. (6).

The key to the minimality in Theorem 1 is the following property of the confinement.

Proposition 3 With the notation of (★), further assume that Φ'/Φ has no positive integer residue. Then, for any polynomial $R \in \mathbb{K}[x]$ such that $\deg_x R < \delta$

$$\exists K \in \mathbb{K}(x) \quad R\Phi = (K\Phi)' \Leftrightarrow R = 0.$$

PROOF. Only the direct implication is not obvious. If such a K exists, the equation rewrites

$$R = K' + K \frac{A}{B}.$$

If K is a polynomial, this equality can only be satisfied if B divides K , in which case the result is a direct consequence of the uniqueness in Lemma 2. Now assume that K has a pole x_0 of order $v > 0$. Then the equation shows that A/B must have a simple pole at x_0 with residue v , which contradicts the assumption on the residues of Φ'/Φ . $\square$

2.2 Hermite Reduction

In terms of integrals, our Algorithm HermiteReduction lets one change H^{n+1} into H^n in the integral, writing

$$\int_\gamma P(n, x)H(x)\Phi(n, x) dx = \int_\gamma \tilde{P}(n, x)\Phi(n, x) dx,$$

for some polynomial $\tilde{P}$. It relies on a sequence of elementary steps (BasicReduction) based on the following lemma.

Algorithm Confinement(P, F)

Input A polynomial $P \in \mathbb{K}[x]$, a rational function $F = A/B$ with $\gcd(A, B) = 1$.
Output A polynomial $R \in \mathbb{K}[x]$ of degree less than $\max(\deg(A), \deg(B) - 1)$ such that $P = R + (QB)' + QA$ for some $Q \in \mathbb{K}[x]$.

$\delta \leftarrow \max(\deg(A), \deg(B) - 1);$
 $d \leftarrow \deg(P) - \delta;$
Write $A = \sum_i a_i x^i$, $B = \sum_i b_i x^i$, $P = \sum_i p_i x^i$;
for $i \leftarrow d$ to 0 by -1 **do**
 $c \leftarrow a_\delta + (\delta + i + 1)b_{\delta+1};$
 $q_i \leftarrow \frac{1}{c} \left(p_{\delta+i} - \sum_{j=1}^{\delta+1} q_{i+j} a_{\delta-j} - (\delta + i + 1) \sum_{j=1}^{\delta+1} q_{i+j} b_{\delta+1-j} \right);$
 $Q \leftarrow \sum_{i=0}^d q_i x^i;$
return $P - (QB)' - QA.$

Lemma 4 Let Φ , A and B be as in Eq. (★). Then for any G dividing B and satisfying $\gcd(G, A + B') = 1$, there exist polynomials Q and R in $\mathbb{K}[x]$ such that

$$P\Phi = RG\Phi + (QB\Phi)'. \quad (7)$$

PROOF. The hypothesis $\gcd(G, A + B') = 1$ implies the existence of polynomials $U, Q \in \mathbb{K}[x]$ such that $P = UG + Q(A + B')$. Then the derivative $(QB\Phi)'$ expands as

$$\frac{(QB\Phi)'}{\Phi} = Q'B + Q(A + B') = (Q'B/G - U)G + P,$$

which has exactly the form of Equation (7). $\square$

The crucial condition $\gcd(G, A + B') = 1$ required to apply this lemma does not hold for arbitrary A and B and divisor G of B , but when G is square-free, it is a consequence of the presence of n in Eq. (★), as shown in the following.

Lemma 5 Let Φ , A and B be as in Eq. (★). Then for any square-free polynomial G in $\mathbb{k}[x]$ dividing the denominator of H , $\gcd(G, A + B') = 1$. This is also true if A/B is replaced by the reduced form of $A/B + iG'/G$ for some $i \in \mathbb{Z}$.

PROOF. If $H' = 0$ then the denominator of H is 1 and then $G = 1$ and the property holds.

Otherwise, let first G be an irreducible factor of the denominator of H , so that there exist an integer k and polynomials

Algorithm BasicReduction(P, F, G, k)

Input A polynomial $P \in \mathbb{K}[x]$, a rational function $F = A/B \in \mathbb{K}(x)$, a square-free factor G of B s.t. $\gcd(G, A + B' + iBG'/G) = 1$ for all $i \in \mathbb{Z}$, a positive integer k .
Output A polynomial $R \in \mathbb{K}[x]$ such that $P = G^k(R + q' + Fq)$ for some $q \in \mathbb{K}[x][G^{-1}]$.

$R \leftarrow P;$
for $i \leftarrow 1$ to k **do**
 $C \leftarrow A + (i - k - 1)BG'/G + B';$
Write $R = QC + VG$ with $\deg Q < \deg G$;
 $R \leftarrow (R - Q'B - QC)/G; \quad \triangleright \frac{P}{G^k} \Phi = \frac{R}{G^{k-i}} \Phi + (q\Phi)'$
return $R.$

Algorithm HermiteReduction($P, H, S/T$)

Input A polynomial $P \in \mathbf{k}(n)[x]$,
two rational functions H and S/T in $\mathbf{k}(x)$.
Output A polynomial $R \in \mathbf{k}(n)[x]$ such that
 $P = R/H + nQH'/H + QS/T + Q'$ for some
 $Q \in \mathbf{k}(n, x)$.

Compute the square-free decomposition of the denominator of H : $g = g_1 g_2^2 \dots g_m^m$;
Compute the corresponding partial fraction decomposition: $H = U + \sum_{k=1}^m U_k/g_k^k$;
 $R \leftarrow PU$;
 $K \leftarrow S/T + (n-1)H'/H$;
for $k \leftarrow 1$ **to** m **do**
 $r_k \leftarrow \text{BasicReduction}(PU_k, K, g_k, k)$;
return $R + r_1 + \dots + r_m$.

$H_1, H_2 \in \mathbf{k}[x]$ such that

$$\frac{A}{B} = nk \frac{G'}{G} + n \frac{H_1}{H_2} + \frac{S}{T},$$

with $\gcd(G, H_2) = 1$ and $B = \text{lcm}(G, H_2, T)$. Write $B = G^\nu \tilde{B}$ with $\gcd(G, \tilde{B}) = 1$. Then

$$A = nkG'G^{\nu-1}\tilde{B} + nH_1G'(\tilde{B}/H_2) + S(G^\nu\tilde{B}/T).$$

Reducing $A + B'$ modulo G then yields

$$A + B' \equiv (nk + \nu)G'G^{\nu-1}\tilde{B} + S(G^\nu\tilde{B}/T) \pmod{G}. \quad (8)$$

Since G does not depend on n , $G \mid A + B'$ would imply that both $G \mid G'G^{\nu-1}\tilde{B}$ and $G \mid SG^\nu\tilde{B}/T$. The first condition implies $\nu > 1$, which forces that $G^\nu \mid T$, making $G \mid S$ necessary too, a contradiction. This reasoning also proves the result when adding integer multiples of G'/G to the fraction A/B , which adds an integer to $nk + \nu$ in Eq. (8).

If G is only assumed to be a square-free divisor of the denominator of H , then the property holds for each of its irreducible factors, and thus $A + B'$ is invertible modulo their product G by the Chinese remainder theorem. $\square$

Correctness of BasicReduction and HermiteReduction. Algorithm **HermiteReduction** treats each square-free factor of the denominator of H separately, while the second part of the lemma is used in Algorithm **BasicReduction** to deal with multiplicities. If G is a square-free factor of multiplicity k , then Lemma 4 is used successively with A/B the reduced form of the logarithmic derivatives of $\Phi/G^k, \Phi/G^{k-1}, \dots, \Phi/G$, thus rewriting $P\Phi/G^k$ as $R\Phi$ up to a derivative.

2.3 Mixed Creative Telescoping

Combining confinement and Hermite reduction gives the final result.

Theorem 6 *Let P be a polynomial in $\mathbf{k}[n, x]$ and Φ , A and B as in Eq. (★) and δ as in Eq. (4). Then, $F_n = P(n, x)\Phi$ admits a telescoper of order bounded by δ .*

PROOF. By Lemma 4, Algorithm **HermiteReduction** can be used to rewrite all the shifts $F_n, F_{n+1}, F_{n+2}, \dots$ under the form $R\Phi$ modulo derivatives, with $R \in \mathbf{k}(n)[x]$. The necessary condition to apply the algorithm is satisfied at each step thanks to Lemma 5. By Lemma 2, $F_n, F_{n+1}, \dots, F_{n+\delta}$

Algorithm MixedCT($P, H, S/T$)

Input A polynomial $P \in \mathbf{k}[n, x]$,
two rational functions H and S/T in $\mathbf{k}(x)$.
Output A r -tuple $(c_0, \dots, c_{r-1})$ such that
 $P(n+r, x)H^r - \sum_{i=0}^{r-1} c_i P(n+i, x)H^i =$
 $nQH'/H + QS/T + Q'$ for some $Q \in \mathbf{k}(n, x)$.

$F \leftarrow S/T + (n-1)H'/H$;
 $R_0 \leftarrow \text{Confinement}(P \mid_{n \mapsto n-1}, F)$;
for $k \leftarrow 0, \dots$ **do**
 if $\text{rank}_{\mathbf{k}(n)}(R_0, R_1, \dots, R_k) < k+1$ **then**
 Solve $\sum_{i=0}^{k-1} c_i R_i = R_k$ for $c_0, \dots, c_{k-1}$ in $\mathbf{k}(n)$;
 return $(c_0, \dots, c_{k-1})$.
 $P \leftarrow R_k \mid_{n \mapsto n+1}$;
 $P \leftarrow \text{HermiteReduction}(P, H, S/T)$;
 $R_{k+1} \leftarrow \text{Confinement}(P, F)$;

are linearly dependent modulo derivatives. A linear relation between them provides a telescoper of order at most δ . $\square$

Algorithm **MixedCT** implements that proof. In practice, for efficiency purposes, the reduction of F_{n+i} is obtained by applying **HermiteReduction** to the shift of the confined reduction of F_{n+i-1} .

Theorem 7 *With the notation of Theorem 6, further assume that Φ'/Φ does not have any positive integer residue. Then Algorithm **MixedCT**($P, H, S/T$) computes a minimal-order telescoper for $P\Phi$.*

PROOF. Consider the minimal-order monic telescoper

$$L(n, S_n) = S_n^r - \sum_{i=0}^{r-1} c_i(n) S_n^i$$

of $P\Phi$ and its certificate $C \in \mathbf{k}(n, x)$ such that

$$L(n, S_n) \cdot (P\Phi) = (C\Phi)'.$$

By Lemma 2, there exist R and K satisfying

$$L(n, S_n) \cdot (P\Phi) = R\Phi + (K\Phi)',$$

where $K \in \mathbf{k}(n, x)$ and

$$R = R_r - \sum_{i=0}^{r-1} c_i(n) R_i$$

is a linear combination of the R_i 's of Algorithm **MixedCT**. It follows that $R\Phi = ((C-K)\Phi)'$ and Proposition 3 then implies that $R = 0$. Thus, this linear combination is detected by the algorithm, producing the output $(c_0, \dots, c_{r-1})$. $\square$

Certificates. Algorithm **MixedCT** as given here computes the certificate, although not in a normalized form. We chose to only output the telescoper, but it would be possible to return the certificate as well (and normalize it or not).

3. DEGREE BOUNDS

We now review more precisely the algorithms and obtain bounds on the degrees at each step. The first part of this section consists of technical results that are needed for the complexity analysis in the next section. Then, at the end of section 3.3, we give a bound on the degree of the telescoper produced by Algorithm **MixedCT**.

3.1 Confinement

Lemma 8 *Let A and B be as in Eq. (★) and let P be a polynomial in $\mathbf{k}(n)[x]$. Let R be the polynomial returned by Algorithm Confinement. Then $\text{Rdeg}_n R - \text{Rdeg}_n P$ is at most*

$$\begin{cases} (\deg_x P - \deg_x A + 1) \frac{[1]}{[1]}, & \text{if } \deg_x B \leq \deg_x A + 1, \\ \left\lceil \left[\frac{\deg_x P - \deg_x B + 1}{\deg_x B - \deg_x A - 1} \right] + 1 \right\rceil / [0] & \text{otherwise.} \end{cases}$$

PROOF. The proof is a case by case analysis of Algorithm Confinement; we use its notation.

When $\delta = \deg_x A$, the recurrence for q_i has a summand $a_{\delta-1}q_{i+1}$ except when $i = d$, while c has a_δ for summand. Thus by induction, the degree in n of the numerator and denominator of q_{d-i} increase by 1 at each step. Since there are $d+1$ steps, $\text{Rdeg}_n R - \text{Rdeg}_n P$ is bounded by $[d]/[d+1]$. The result for R then follows from Equation (6).

When $\delta = \deg_x B - 1$, the coefficients $a_{\delta-j}$ are zero for $j < \delta - \deg_x A$ and c has degree 0 in n . By induction on i , q_{d-i} has degree that changes (by increases of 1) only when $i \equiv 0 \pmod{\delta - \deg_x A}$. More explicitly, $\text{Rdeg}_n q_{d-i} - \text{Rdeg}_n P$ is bounded by $\lceil [i/(\delta - \deg_x A)] \rceil / [0]$. Again, the conclusion for R follows from Equation (6). $\square$

3.2 Hermite Reduction

In order to track the degrees in n of the polynomials involved in Algorithms BasicReduction and HermiteReduction, we need to look deeper into the modular inversions involved. This is done in the next lemma, using the same notation as in the discussion preceding Lemma 5.

Lemma 9 *Let A , B and H be as in Eq. (★). Let G be an irreducible factor of the denominator of H , and ν the G -adic valuation of B . Let P be in $\mathbf{k}(n)[x]$ and let Q be the polynomial such that $\deg_x Q < \deg_x(G)$ and $P \equiv Q(A + B') \pmod{G}$. Then $\text{Rdeg}_n Q - \text{Rdeg}_n P$ is bounded by*

$$\begin{cases} [0]/[0], & \text{if } \nu > 1; \\ [0]/[1], & \text{if } \nu = 1 \text{ and } G \nmid T; \\ [\deg_x(G) - 1]/[\deg_x(G)], & \text{if } \nu = 1 \text{ and } G \mid T. \end{cases}$$

PROOF. The existence of Q follows from Lemma 5. Notice first that $\text{Rdeg}_n Q - \text{Rdeg}_n P = \text{Rdeg}_n (A + B')^{-1}$ (where $(A + B')^{-1}$ denotes an inverse mod G). Indeed, writing $p(n)P = P_0(x) + \dots + P_d(x)n^d$ with $p(n)$ the denominator of P , we see that $p(n)Q = (A + B')^{-1}(P_0 \bmod G) + \dots + (A + B')^{-1}n^d(P_d \bmod G)$ has rational degree in n at most $[d]/[0] + \text{Rdeg}_n (A + B')^{-1}$.

Thus we just need to bound $\text{Rdeg}_n (A + B')^{-1}$. To do so, we take a closer look at Equation (8). If $\nu > 1$, the equation becomes

$$A + B' \equiv S \frac{\tilde{B}}{T/G^\nu} \pmod{G},$$

so that $(A + B')^{-1}$ does not depend on n in this case.

If $\nu = 1$ and $G \nmid T$, the equation becomes

$$A + B' \equiv (nk + 1)G' \tilde{B}.$$

Then, $(A + B')^{-1} \equiv (G' \tilde{B})^{-1}/(nk + 1)$ has rational degree $[0]/[1]$. Finally, if $\nu = 1$ and $G \mid T$, the result follows from Lemma 10 below. $\square$

Lemma 10 *Let A, B, C be polynomials in $\mathbf{k}[x]$ such that C is relatively prime with at least one of A or B . Let $U, V \in \mathbf{k}(n)[x]$ be such that*

$$1 = U(An + B) + VC, \quad \text{with } \deg_x U < \deg C. \quad (9)$$

Then $\text{Rdeg}_n U$ is bounded by $[\deg C - 1]/[\deg C]$.

PROOF. Let $R(n) \neq 0$ denote the resultant with respect to x of C and $An + B$. Then $R = S(An + B) + TC$ for some S, T in $\mathbf{k}[n, x]$ with $\deg_x S < \deg C$. Moreover, by Cramer's rule applied to the Sylvester matrix of C and $An + B$, we have $\deg R \leq \deg C$ and $\deg_n S < \deg C$.

Now denote by $q(n)$ the monic denominator of U , and by $\tilde{U} \in \mathbf{k}[n, x]$ its numerator. We need to prove that $\deg q \leq \deg C$ and $\deg_n \tilde{U} < \deg C$. Equalities

$$1 = U(An + B) + VC, \quad 1 = S/R(An + B) + (T/R)C$$

imply by subtraction that C divides $(An + B)(U - S/R)$ in $\mathbf{k}(n)[x]$. Since C is coprime with $An + B$, this implies that C divides $U - S/R$. As $\deg C < \deg_x(U - S/R)$, this shows that $U = S/R$. In particular, q divides R . It follows that $\deg q \leq \deg R \leq \deg C$ and $\deg_n \tilde{U} = \deg_y(qS/R) \leq \deg_n S < \deg C$, which concludes the proof. $\square$

Lemma 11 *With the same notation as in Lemma 9, and assuming that G has multiplicity k in the denominator of H , the output R of BasicReduction($P, A/B, G, k$) satisfies*

$$\deg_x R \leq \max(\deg_x P - k \deg_x G, \deg_x A - 1, \deg_x B - 2),$$

and $\text{Rdeg}_n R - \text{Rdeg}_n P$ is bounded by

$$\begin{cases} k[1]/[0], & \text{if } \nu > 1; \\ k[1]/[1], & \text{if } \nu = 1 \text{ and } G \nmid T; \\ k \deg_x(G)[1]/[1], & \text{if } \nu = 1 \text{ and } G \mid T. \end{cases}$$

PROOF. Both bounds follow from Lemma 9. For the degree in n , the polynomial Q of the algorithm is obtained from a modular inverse as above, that is multiplied by R , and then by C that has rational degree $[1]/[0]$ in n . The bound directly follows in the first and third cases since the condition on ν and G is preserved at each step. In the second case, the condition $G \nmid T$ is not preserved, but writing $J = iG'/G + S/T$ at each step shows that the degree in n still increases by $[1]/[1]$ only. For the degree in x , the bound is obtained by bounding each term in the expression of R that is used in the algorithm. $\square$

Lemma 12 *With the same notation, write the denominator g of H as $g = e f h$, where*

$$e = \gcd(g, T, T'), \quad f = \gcd(g/e, T).$$

Also denote m the highest multiplicity of the roots of g , and let $e = e_1 e_2^2 \dots e_m^m$ and $h = h_1 h_2^2 \dots h_m^m$ be the square-free decompositions of e and h .

Then, the result of Algorithm HermiteReduction($P, H, S/T$) is a polynomial in x of degree at most

$$\max(\deg_x P + \deg_x^\infty H, \deg_x P - 1, \deg_x A - 1, \deg_x B - 2).$$

Seen as a rational function in n , it has degree at most

$$\text{Rdeg}_n P + \frac{[\max_{e_k \neq 1} k + \sum_{h_k \neq 1} k + \deg_x f]}{[\sum_{h_k \neq 1} k + \deg_x f]}.$$

Moreover, the certificate Q in the algorithm satisfies $Q = qB/(gH)$ for some polynomial q such that $\deg_x q < \deg_x g$.

PROOF. Following the notation of Algorithm **HermiteReduction**, the partial fraction decomposition of H produces U with $\deg_x U \leq \deg_x^\infty H$ and U_k with $\deg_x U_k < \deg_x g_k$. PU obviously satisfies the bounds. Now write the square-free decomposition $f = f_1 f_2^2 \cdots f_m^m$. By Lemma 11, $\text{Rdeg}_n r_k \leq \text{Rdeg}_n P + [k(\mathbf{1}_{e_k \neq 1} + \mathbf{1}_{h_k \neq 1} + \deg_x f_k)] / [k(\mathbf{1}_{h_k \neq 1} + \deg_x f_k)]$. Normalizing $R + r_1 + r_2 + \cdots + r_m$ then yields the result. The bound for the degree in x is obtained by bounding separately each term using Lemma 11. The form of Q follows from the fact that the certificate of the k -th call to **BasicReduction** has the form $q_k B / (g_k^k H)$ with $\deg_x q_k < \deg_x (g_k^k)$. $\square$

3.3 Mixed Creative Telescoping

Lemma 13 *With the notation of Eq. (★), Eq. (4), Algorithm MixedCT, and $d_H = \deg_x^\infty H$, for all i in $\{1, 2, \dots, \delta\}$ we have*

$$\text{Rdeg}_n R_i \leq \deg_n P \cdot \frac{[1]}{[0]} + \alpha + i(\beta + \gamma)$$

where

$$\alpha = \begin{cases} \max(\deg_x P - \delta + 1, 0) \cdot \frac{[1]}{[1]} & \text{if } \delta = \deg_x A, \\ \left[\max \left(\left\lfloor \frac{\deg_x P - \delta}{\delta - \deg_x A} \right\rfloor + 1, 0 \right) \right] / [0] & \text{otherwise.} \end{cases}$$

$$\beta = \frac{[\max_{e_k \neq 1} k + \sum_{h_k \neq 1} k + \deg_x f]}{[\sum_{h_k \neq 1} k + \deg_x f]}$$

$$\gamma = \begin{cases} (d_H + 1) \cdot \frac{[1]}{[1]} & \text{if } d_H \geq 0 \text{ and } \delta = \deg_x A, \\ \left[\left\lfloor \frac{d_H}{\delta - \deg_x A} \right\rfloor + 1 \right] / [0] & \text{if } d_H \geq 0 \text{ and } \delta = \deg_x B - 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. By Lemma 8, the initial confinement increases $\text{Rdeg}_n P$ by α . **HermiteReduction** is then always used with an input polynomial of degree less than δ . By Lemma 12 each call to **HermiteReduction** increases $\text{Rdeg}_n P$ by at most β , and produces an output of degree at most δ if $d_H < 0$ or $\delta + d_H$ if $d_H \geq 0$. Thus the confinement is only necessary in the latter case. Plugging this bound into Lemma 8 shows that each call to **Confinement** increases $\text{Rdeg}_n P$ by at most γ . $\square$

3.4 Degree bound on the telescoper

Lemma 14 *With the notation of (★), let $(c_0, \dots, c_{r-1})$ be the output of MixedCT($P, H, S/T$) and write $H = f/g$ with $\gcd(f, g) = 1$.*

Then there exists a polynomial $Q \in \mathbf{k}(n)[x]$ such that

$$\left(S_n^r - \sum_{i=0}^{r-1} c_i S_n^i \right) (P\Phi) = \left(\frac{Q}{g^r} B\Phi \right)',$$

with $\deg_x Q \leq r \deg_x H + \max(\deg_x P - \delta, 0) - 1$.

PROOF. Tracking the certificates of the various rewritings in MixedCT, it suffices to show that for all $i \in \{0, \dots, r\}$

$$P(n+i-1, x) H^i \frac{\Phi}{H} = R_i \frac{\Phi}{H} + \left(\frac{Q_i}{g^i} B \frac{\Phi}{H} \right)'$$

for some polynomial Q_i such that

$$\deg_x Q_i \leq \max(\deg_x P - \delta, 0) + i \deg_x H - 1.$$

This is obvious for $i = 0$ (initial confinement). Assume this is true for $i - 1$, then the next Hermite reduction writes

$$R_{i-1}(n+1, x)\Phi = \tilde{R}_i \frac{\Phi}{H} + \left(\frac{\tilde{Q}_i}{g} B \frac{\Phi}{H} \right)'$$

with $\deg_x \tilde{Q}_i < \deg_x g$ and $\deg_x \tilde{R}_i \leq \delta - 1 + \delta_H$, where $\delta_H = \max(\deg_x^\infty H, 0)$ (see Lemma 12). As for the confinement,

$$\tilde{R}_i \frac{\Phi}{H} = R_i \frac{\Phi}{H} + \left(\bar{Q}_i B \frac{\Phi}{H} \right)',$$

with $\deg_x \bar{Q}_i \leq \delta_H - 1$ by Lemma 2. Thus, the property is satisfied for i with

$$Q_i = f Q_{i-1}(n+1, x) + \tilde{Q}_i g^{i-1} + \bar{Q}_i g^i,$$

from which follows the bound on the degree of Q_i . $\square$

Theorem 15 *With the notation of (★), the telescoper L produced by Algorithm MixedCT($P, H, S/T$) satisfies*

$$\deg_n L \leq r(\deg_n P + \deg_x H) + \max(\deg_x P - \delta, 0),$$

where r is the order of L .

PROOF. Write $H = f/g$ with $\gcd(f, g) = 1$. Rewriting Lemma 14 in terms of polynomials yields

$$P(n+r)f^r - \sum_{i=0}^{r-1} c_i P(n+i)f^i g^{r-i} = (QB)' - rQ \frac{Bg'}{g} + QA$$

for some $Q = \sum_{i=0}^s q_i x^i$ with $s = \max(\deg_x P - \delta, 0) + r \deg_x H - 1$. This equation is a linear system with two blocks of unknowns: $c_0, \dots, c_{r-1}$ with coefficients of degree bounded by $\deg_n P$ and $q_0, \dots, q_s$ with coefficients of degree 1, which by Hadamard's bound yields $\deg_n c_i \leq r \deg_n P + s + 1$, whence the theorem. $\square$

4. COMPLEXITY

We will rely on some classical complexity results for the basic operations on polynomials and rational functions. Standard references for these questions are the books [13] and [6]. We will also use the fact that linear systems with polynomial coefficients can be solved efficiently using Storjohann and Villard's algorithm [19]. The needed results are summarized in the following lemma.

Lemma 16 *Addition, product and differentiation of rational functions in $\mathbb{K}(x)$ of regular degree less than d , as well as extended gcd and square-free decomposition in $\mathbb{K}[x]_d$ can be performed using $\tilde{O}(d)$ operations in $\mathbb{K}$.*

The kernel of a $s \times (s+1)$ matrix with polynomial entries in $\mathbf{k}[x]_d$ can be solved using $\tilde{O}(s^d)$ operations in $\mathbf{k}$.

4.1 Confinement

Lemma 17 *With $P \in \mathbb{K}[x]$, $A/B \in \mathbb{K}(x)$ as input, and $\delta = \max(\deg_x A, \deg_x B - 1)$, Confinement performs at most*

$$O(\delta \deg_x P)$$

operations in $\mathbb{K}$.

PROOF. Each iteration of the loop performs $O(\delta)$ operations in $\mathbb{K}$ and the loop is executed $\deg_x P - \delta + 1$ times. The computation of R then needs $\tilde{O}(\deg_x P)$ operations in $\mathbb{K}$. $\square$

4.2 Hermite Reduction

Lemma 18 *With the same notation as in the preceding lemma, G a square-free factor of B and k a positive integer, $\text{BasicReduction}(P, A/B, G, k)$ performs at most*

$$\tilde{O}(k(\deg_x P + \delta))$$

operations in $\mathbb{K}$.

PROOF. The costly steps are the gcd computations, which according to Lemma 16 can be performed using $\tilde{O}(\deg_x P + \delta)$ operations in $\mathbb{K}$. The result then follows since there are k gcd computations. $\square$

Lemma 19 *With the same notation as in Lemma 12, set $\epsilon = \sum_{g_k \neq 1} k$. Then $\text{HermiteReduction}(P, H, S/T)$ performs at most*

$$\tilde{O}(\max(\epsilon \deg_x P, d_H) + \deg_x g + \epsilon \delta)$$

operations in $\mathbb{K} = \mathbf{k}(n)$.

PROOF. By Lemma 16, the square-free decomposition of g can be computed in $\tilde{O}(\deg_x g)$ operations and the product PU is computed at a cost $\tilde{O}(\max(\deg_x P, \deg_x^\infty H))$. By the preceding lemma, the k -th call to BasicReduction uses $\tilde{O}(k(\deg_x P + \deg_x g_k + \delta))$ operations. The announced bound is then obtained by summation. $\square$

4.3 Mixed creative telescoping

For the sake of simplicity, Algorithm `MixedCT` searches for telescopers for all the possible orders, starting from 0. In practice, a more efficient variant consists in carrying a dichotomic search of the order between 0 and δ . This way the complexity is that of the last step up to a logarithmic factor. Here, we analyze this variant.

Lemma 20 *With the same notation as in Section 3.3, and $\epsilon = \sum_{g_k \neq 1} k$, the number of operations in $\mathbb{K} = \mathbf{k}(n)$ performed by $\text{MixedCT}(P, H, S/T)$ is*

1. $\tilde{O}(\delta \max(\epsilon \deg_x P, d_H) + \delta \deg_x g + \epsilon \delta^2 + \delta^\omega)$ if $d_H < 0$;
2. $\tilde{O}(\delta \max(\epsilon \deg_x P, \delta d_H) + \delta \deg_x g + \epsilon \delta^2 + \delta^3)$ if $d_H \geq 0$.

PROOF. When $d_H < 0$, by Lemma 12 there are no confinement steps and the construction of the system to solve amounts to δ Hermite reductions. The algorithm then computes a vector in the kernel of a $\delta \times (\delta + 1)$ matrix, which by Lemma 16 can be performed in $\tilde{O}(\delta^\omega)$ operations in $\mathbb{K}$, hence the complexity.

When $d_H \geq 0$, we have to add the cost of the confinement steps, which by Lemma 12 are performed on polynomials of degree at most $\delta + d_H$. There are at most $\delta + 1$ calls to confinement, so the result follows from Lemma 17. $\square$

Theorem 21 *With the same notation, set $\mu = \max(a, b)$ where $\deg_n P \cdot [1]/[0] + \alpha + \delta(\beta + \gamma) = [a]/[b]$, the number of operations in $\mathbf{k}$ performed by Algorithm `MixedCT` is*

$$\tilde{O}(\mu(\epsilon \delta \deg_x P + \delta \deg_x g + \epsilon \delta^2 + \delta^\omega)),$$

if $d_H < 0$, or

$$\tilde{O}(\mu(\epsilon \delta \deg_x P + \delta \deg_x g + \epsilon \delta^2 + \delta^3 + \delta^2 d_H))$$

if $d_H \geq 0$.

PROOF. The result follows directly from the preceding lemma and the fact that all the elements of $\mathbb{K} = \mathbf{k}(n)$ appearing in the construction of the linear system have numerator and denominator of degree in n bounded by μ . The cost of solving is then $\tilde{O}(\delta^\omega \mu)$ operations in $\mathbf{k}$ by Lemma 16. $\square$

The complexity result from Theorem 1 follows directly.

5. EXPERIMENTS AND APPLICATIONS

5.1 Various Integrals

Example 1. The Jacobi polynomials have the following integral representation, up to a factor that does not depend on n :

$$\oint \left(\frac{z^2 - 1}{2(z - x)} \right)^n (1 - z)^\alpha (1 + z)^\beta \frac{dz}{z - x},$$

with a contour enclosing $z = x$ once in the positive sense [18, 18.10.8]. It is well-known that the Jacobi polynomials satisfy a recurrence of order 2. Theorem 6 is tight in that case: it predicts a bound 2 on the order of the telescoper, since the logarithmic derivative of the integrand has numerator of degree 2 and denominator of degree 3.

Note however that in such an example, a more direct and efficient way to obtain a recurrence is to change the variable z into $x + u$ making the integral that of the extraction of the n -th coefficient in a hyperexponential term. That is achieved easily by translating the first order linear differential equation it satisfies into a linear recurrence. The only difference is that our method guarantees the minimality of the telescoper.

Example 2. A typical example where several of the difficulties are met at once is the term

$$\left(1 + \frac{x}{n^2 + 1}\right) \left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3}\right)^n \sqrt{x^2 - 5} e^{\frac{x^3+1}{x(x-3)(x-4)^2}}.$$

Our code¹ finds a telescoper of order 9 and degree 90 in 1.4 sec. The only other code we are aware of that can perform this computation is Koutschan's `HolonomicFunctions` package [16], which takes more than 3 min. This is by no means a criticism of this excellent and very versatile package but rather to indicate the advantage of implementing specialized algorithms like ours for this class.

5.2 Inversion of rational functions

Here is a generalization of Manivel's lemma [12], which led us to this work. It gives an efficient way to compute the recurrence satisfied by the coefficients of the compositional inverse of a rational function. The starting point is Lagrange inversion. Let $f \in \mathbb{Q}(x)$ be a rational function such that $f(0) = 0$, and denote by $f^{(-1)}$ its compositional inverse. By Cauchy's formula, the n -th coefficient u_n of $f^{(-1)}$ is given by

$$u_n = \frac{1}{2\pi i} \oint f^{(-1)}(x) \frac{dx}{x^{n+1}},$$

where the contour is a small circle around the origin. Integrating by parts and then using the change of variables

¹The code and Maple worksheet for these examples can be found at <http://mixedct.gforge.inria.fr>.

$x = f(u)$ yields

$$u_n = \frac{1}{2\pi i n} \oint \frac{f^{(-1)'}(x) dx}{x^n} = \frac{1}{2\pi i n} \oint \frac{du}{f(u)^n}.$$

Thus a recurrence for u_n can be computed with Algorithm MixedCT. Theorem 6 and Theorem 15 then provide bounds for the order and degree of this recurrence.

Theorem 22 *Let $f \in \mathbb{Q}(x)$ be a rational function such that $f(0) = 0$. Write $f = P/Q$ and denote $P = P_1 P_2^2 \cdots P_m^m$ the square-free decomposition of P . Also denote p, p^*, q, q^* the degrees of P, P^*, Q, Q^* respectively. Then the Taylor coefficients of $f^{(-1)}$ satisfy a recurrence of order at most*

$$q^* + p^* - 1$$

and of rational degree in n at most

$$\frac{(q^* + p^*)(q^* + p^* + 1)}{2} (\max(q - p, 0) + \sum_{P_k \neq 1} k) \cdot \frac{[1]}{[1]}.$$

k	order	degree	coeffs	time
5	10	61	1759	2.49
6	12	88	2440	4.64
7	14	120	3778	13.36
8	16	157	4666	33.89
9	18	199	6192	88.34
10	20	246	8364	260.59
11	22	298	10146	628.21
12	24	355	11802	1451.54

Table 1. Order, degree, bit size and timings for Example 3

Example 3. Experimental results on the family of rational functions $f_k = x P_k(x)^2 / Q_k(x)$ with P_k and Q_k two dense polynomials of degree k and integer coefficients of absolute value bounded by 100 are presented in Table 1. The first column gives the index k . The second one is the order of the minimal-order telescopers, which is as predicted by Theorem 22. The next one gives the degree of the telescoper; it displays a quadratic growth, as predicted by Theorem 22. The column “coeffs” gives the bit size of the largest coefficient of the telescoper, whose growth seems slightly more than quadratic. Finally, the time (in seconds) taken by our implementation is given in the last column.

Acknowledgements. We are grateful to the referees for their thorough work and helpful comments. This work has been supported in part by FastRelax ANR-14-CE25-0018-01.

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