

Isoperimetric ratios of toroidal Dupin cyclides

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ABSTRACT

The combination of recent results due to Yu and Chen (2022) [17] and to Bostan and Yurkevich (2022) [2] shows that the 3-D Euclidean shape of the square Clifford torus is uniquely determined by its isoperimetric ratio. This solves part of the still open uniqueness problem of the Canham model for biomembranes. In this work we investigate a generalization of the aforementioned results to the case of a *rectangular* Clifford torus. As in the square case, we find closed-form formulas in terms of hypergeometric functions for the isoperimetric ratio of its stereographic projection to $\mathbb{R}^3$ and show that the corresponding function is strictly increasing. But unlike the square case, we show that the isoperimetric ratio *does not* uniquely determine the Euclidean shape of a rectangular Clifford torus. These results are obtained using a mixture of differential geometry and computer algebra. Moreover, the computer algebra technique in this work is more streamlined and more direct than in the aforementioned approaches, even for the most important case of the square Clifford torus.

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1. Introduction

In differential geometry, the *Willmore energy* of a given (smooth and connected) closed surface S embedded in the 3-D Euclidean space is defined as the integral, taken over S , of the square of the mean curvature. It measures to which extent such a surface deviates from a round sphere. Willmore [16, Thm. 1] proved that among all closed surfaces in $\mathbb{R}^3$ the Willmore energy is at least equal to 4π , and that it is minimized exactly by the round sphere. However, for surfaces with positive genus a long-standing conjecture due to Willmore [16] predicts a better lower bound, namely $2\pi^2$. In 2014 Marques and Neves [13] proved Willmore's conjecture, by showing that among all closed surfaces in $\mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimized exactly for the class of projections to $\mathbb{R}^3$ of the *square Clifford torus* $C_{\pi/4}$ defined by

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E-mail addresses: alin.bostan@inria.fr (A. Bostan), yut@drexel.edu (T. Yu), sergey.yurkevich@univie.ac.at (S. Yurkevich).¹ We deeply regret the much too early passing of Jingmin Chen (August 27, 1989 – July 30, 2024).

$$C_{\pi/4} = \{(\cos u, \sin u, \cos v, \sin v)/\sqrt{2} : u, v \in [0, 2\pi]\} \subseteq \mathbb{S}^3 \subseteq \mathbb{R}^4.$$

Closely connected is the classical *Canham model for biomembranes* [3] which asks for a minimizer of the Willmore energy for a closed surface S in $\mathbb{R}^3$ with given genus, surface area and volume. By the scale invariance of Willmore energy, prescribing the surface area and volume of S is equivalent to fixing the surface's *isoperimetric ratio*, defined as

$$v(S) := 6\sqrt{\pi} \cdot \frac{\text{Vol}(S)}{\text{Area}(S)^{3/2}}, \quad (1.1)$$

which is normalized so that it equals 1 for the sphere.

Motivated by the open *uniqueness problem* of Canham's model (discussed in [12,18]) the following result was established as a combination of several works by the authors of the present paper [2,17], as well as Melczer and Mezzarobba [15]:

Theorem 1.1. *The 3-D Euclidean shape of $C_{\pi/4}$ is uniquely determined by its isoperimetric ratio.*

By “3-D Euclidean shapes” we refer to the different stereographic projections of $C_{\pi/4}$ into $\mathbb{R}^3$ for which the center of projection does not lie on the torus. Two subsets of $\mathbb{R}^3$ are considered to have the same Euclidean shape (or simply the same *shape*) if they can be transformed from one to another by a similarity transformation; in the latter case, we call two such subsets *homothetic*. We refer to [17, Figure 1] for a depiction of various Euclidean shapes of the square Clifford torus.

The stereographic images in $\mathbb{R}^3$ of $C_{\pi/4} \subset \mathbb{S}^3$ are homothetic to the sphere inversions of the torus of revolution $T_{\sqrt{2}}$ with major radius $\sqrt{2}$ and minor radius 1. More generally, the stereographic images of the *rectangular Clifford torus*

$$C_\alpha = \left\{ (\cos \alpha \cos u, \cos \alpha \sin u, \sin \alpha \cos v, \sin \alpha \sin v) : u, v \in [0, 2\pi] \right\}, \quad \alpha \in (0, \pi/2), \quad (1.2)$$

are homothetic to the sphere inversions of $T_{\csc \alpha}$, where

$$T_R := \left\{ ((R + \sin v) \cos u, (R + \sin v) \sin u, \cos v) : u, v \in [0, 2\pi] \right\}$$

is the torus of revolution with major radius R and minor radius 1.

Denote by $i_{\mathbf{x}}$ the sphere inversion map about a unit sphere centered at $\mathbf{x} \in \mathbb{R}^3$. If $R \in (1, \infty)$ and $\mathbf{x} \notin T_R$, then $i_{\mathbf{x}}(T_R)$ is called a *toroidal Dupin cyclide*. Theorem 1.1 is equivalent to saying that the Euclidean shapes of $\{i_{\mathbf{x}}(T_{\sqrt{2}}) : \mathbf{x} \in \mathbb{R}^3 \setminus T_{\sqrt{2}}\}$ are in one-to-one correspondence with their isoperimetric ratios (1.1). Note that for notational convenience, if $\mathbf{x} \in T_R$, we define $i_{\mathbf{x}}(T_R)$ to be an arbitrary round sphere.

The proof of Theorem 1.1 crucially relies on the following fact:

Theorem 1.2. *The isoperimetric ratio of $i_{[\varrho, 0, 0]}(T_{\sqrt{2}})$ increases monotonically for $\varrho \in [0, \sqrt{2} - 1)$.*

This statement was conjectured in [17] and independently proved in [2] and [15]. In [2] Theorem 1.2 is proved based on hypergeometric representations of the area and volume of $i_{[\varrho, 0, 0]}(T_{\sqrt{2}})$. These hypergeometric representations could be found using P-recursive sequences given in [17, Proposition 4.1]. In [15], the result is proved with rigorous asymptotic analysis of another but related P-recurrence associated with the derivative of the isoperimetric ratio of $i_{[\varrho, 0, 0]}(T_{\sqrt{2}})$ with respect to ϱ (also derived in [17, Prop. 4.1]). The approach from [15] was later generalized and automated in [7].

Having established the case of the *square Clifford torus* $C_{\pi/4}$ in Theorem 1.1 and Theorem 1.2, in this work we investigate the case of a general *rectangular Clifford torus* C_α for $\alpha \in (0, \pi/2)$. Along the way, we streamline the analysis in the case of $C_{\pi/4}$ and also prove the following curious fact:

Proposition 1.3. *Theorem 1.1 does **not** hold for C_α with $\alpha \neq \pi/4$.*

Phrased differently, this proposition says that for any $R > 1$ such that $R \neq \sqrt{2}$, the isoperimetric ratio of $i_{\mathbf{x}}(T_R)$ does *not* uniquely determine its Euclidean shape. While Theorem 1.1 does not generalize to an arbitrary α , Theorem 1.2 does generalize to a general $R > 1$:

Theorem 1.4. *For any $R \in (1, \infty)$, when ϱ increases from 0 to $R - 1$, the isoperimetric ratio of $i_{[\varrho, 0, 0]}(T_R)$ increases monotonically from $3/(2\sqrt{\pi R})$ to 1.*

The proof of Theorem 1.4 appeared in a slightly modified version in the PhD thesis [19, §4.3] of the fourth author and is reproduced with some adaptations in this paper. Similarly to the approach in [2] for the case $R = \sqrt{2}$, the proof technique presented here is based on hypergeometric representations of the area and volume of $i_{[\varrho, 0, 0]}(T_R)$. Like in [19, §4.3], but contrary to [2, 17], we explain here in more detail how the hypergeometric representations could be found and proved using the technique of creative telescoping. Moreover, it turns out that when $R \neq \sqrt{2}$ the representations involve the generalized hypergeometric function ${}_3F_2$, instead of just the Gaussian hypergeometric function ${}_2F_1$ as seen in [17]. Finally, as evident in Section 4.2 the proof of monotonicity becomes more involved in the $R \neq \sqrt{2}$ case.

Contribution of the paper: This paper combines two fields: differential geometry and computer algebra. On the geometrical side, we extend results of [17] from $C_{\pi/4}$ to C_α for any $\alpha \in (0, \pi/2)$. We also prove Theorem 2.2 and Theorem 2.3 – results that were previously stated without a proof [5, 17], and we provide geometric intuition for Theorem 1.4 and Proposition 1.3. On the side of computer algebra, we generalize but also streamline and simplify the method for obtaining the hypergeometric expressions for A_R and V_R below. In [17] getting the simpler expressions for $A = A_{\sqrt{2}}$ and $V = V_{\sqrt{2}}$ was a tedious procedure which required the evaluation of binomial sums, which we now completely avoid. Finally, the proof of monotonicity of Iso_R for any $R > 1$ had to be non-trivially generalized relative to the approach in [2] for $R = \sqrt{2}$. We manage to exploit the hypergeometric nature of the problem in order to deal with the new parameter R , whereas it is not clear how to adapt automatic methods like those in [7, 9] in such parametric cases.

Historical remarks: A similar formula to (4.9) for the reduced volume is stated in [8, §4.1] but to the best of our knowledge no proof is given there. In the same section, Fourcade also mentions that “*It appears that the lowest reduced volume torus corresponds to the axisymmetric case*”; his wording indicates that the claim – proven in our work in Theorem 1.4 – was unproven at the time of writing of [8]. We would also like to point out the sentence in Jülicher’s thesis [10]: *Variiert man den Betrag λ dieses Vektors, so erzeugt die konforme Abbildung unabhängig von der Richtung ϕ eine einparametrische Familie nichtaxialsymmetrischer Formen mit wachsendem reduzierten Volumen.*² In this sentence, Jülicher states as a fact that the isoperimetric ratio is increasing. While he cites Fourcade for the analytic solution, he does not explain the statement about the increasing reduced volume. Our speculation is that the word “appears” got lost between the works of Fourcade and Jülicher.

Finally, for a pedagogical introduction to the isoperimetric-constrained Willmore problem in differential geometry and the Canham-Helfrich shape problem in cellular biology, we refer to the expository article [12].

Structure of the paper: In Section 2, we state three properties of the sphere inversions of T_R . From these properties, we also see

- (i) how Theorem 1.1 follows from Theorem 1.2, whereas

² In English: *Varying the magnitude λ of this vector, the conformal mapping generates a one-parameter family of nonaxially symmetric forms with increasing reduced volume, independent of the direction ϕ .*

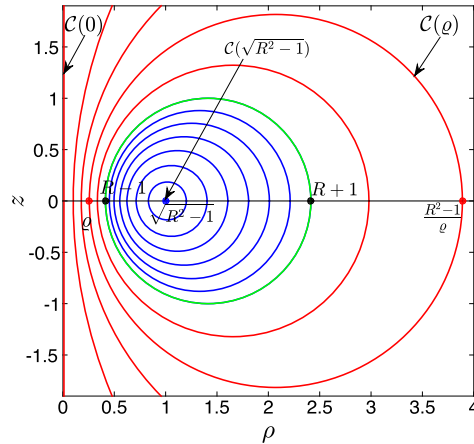


Fig. 1. The circles $\mathcal{C}(\varrho) := \mathcal{C}(\varrho; R)$ for a fixed $R \in (1, \infty)$. Note that $\mathcal{C}(\varrho; R) = \mathcal{C}((R^2 - 1)/\varrho; R)$ and $\{\mathcal{C}(\varrho; R) : \varrho \in [0, \sqrt{R^2 - 1}]\}$ partitions the half- ρ - z -plane $\rho \geq 0$, so $\mathcal{T}(\varrho; R) = \mathcal{T}((R^2 - 1)/\varrho; R)$, and $\{\mathcal{T}(\varrho; R) : \varrho \in [0, \sqrt{R^2 - 1}]\}$ partitions $\mathbb{R}^3$. Notice also that $\mathcal{T}(R \pm 1; R) = T_R$ = the torus of revolution generated by revolving the green circle about the z -axis. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

(ii) the natural generalization of Theorem 1.2, namely, Theorem 1.4 does *not* lead to a generalization of Theorem 1.1, hence proving Proposition 1.3.

The first of the aforementioned properties is proven in [17, Section 2]; the latter two will be proved in Section 3. Finally, Section 4 contains the proof of our main Theorem 1.4.

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2. Properties of Toroidal Dupin cyclides and Proposition 1.3

In this section, we first state three results, Theorems 2.1–2.3, about toroidal Dupin cyclides. Theorem 2.1 is proved in detail in [17, Section 2], however Theorems 2.2 and 2.3 are stated without proof in [17, Section 2]. We shall fill in the proofs of the latter two results in Section 3.

Denote by $\mathcal{C}(\varrho; R)$ the circle in the ρ - z plane with a diameter connecting $(\varrho, 0)$ and $((R^2 - 1)/\varrho, 0)$ (see Fig. 1), and let

$$\mathcal{T}(\varrho; R) := \{(\rho \cos(\theta), \rho \sin(\theta), z) : (\rho, z) \in \mathcal{C}(\varrho; R), \theta \in [0, 2\pi]\}$$

be the torus defined by revolving $\mathcal{C}(\varrho; R)$ about the z -axis. By convention, $\mathcal{C}(0; R) = \mathcal{C}(\infty; R)$ is the z -axis (of the ρ - z plane), so $\mathcal{T}(0; R) = \mathcal{T}(\infty; R)$ is the z -axis of the x - y - z space.

Theorem 2.1 (Proven in [17]). *For any fixed $R \in (1, \infty)$ and $\varrho \in [0, \sqrt{R^2 - 1}]$, all the cyclides in*

$$\{i_{\mathbf{x}}(T_R) : \mathbf{x} \in \mathcal{T}(\varrho; R)\} \quad (2.1)$$

are homothetic in $\mathbb{R}^3$.

By virtue of this result, for $\varrho \in [0, \sqrt{R^2 - 1}]$ we use the shorthand notation $i_\varrho(T_R)$ to represent the (equivalence class of) common Euclidean shape of the cyclides in (2.1). Note that $i_{R-1}(T_R)$ is the round sphere.

Theorem 2.2 (Stated in [5,17], proven in Section 3). For any $R \in (1, \infty)$, $\varrho \in [0, \sqrt{R^2 - 1}]$,

$$i_\varrho(T_R) = i_{\varrho'}(T_{R'}) \quad \text{where} \quad (R', \varrho') = \frac{1}{\sqrt{R^2 - 1}} \left(R, \frac{\sqrt{R^2 - 1} - \varrho}{\sqrt{R^2 - 1} + \varrho} \right). \quad (2.2)$$

Theorem 2.3 (Stated in [17], proven in Section 3). The shapes in

$$\left\{ i_\varrho(T_R) : R \in (1, \infty), \varrho \in [0, R - 1) \right\} \quad (2.3)$$

are all distinct from each other. Moreover, any toroidal Dupin cyclide is homothetic to exactly one of the shapes in (2.3).

Note that for any fixed $R \in (1, \infty)$, the mapping $\varrho \mapsto \varrho'$ in (2.2) maps $(R - 1, \sqrt{R^2 - 1}]$ bijectively (and decreasingly) to $[0, R' - 1)$, where $R' = \frac{R}{\sqrt{R^2 - 1}}$ (as in (2.2)). By Theorem 2.2,

$$\left\{ i_\varrho(T_R) : \varrho \in (R - 1, \sqrt{R^2 - 1}] \right\} = \left\{ i_{\varrho'}(T_{R'}) : \varrho' \in [0, R' - 1) \right\}. \quad (2.4)$$

The special role of the square Clifford torus manifests itself in the $R \leftrightarrow R'$ correspondence: note that $R' = \sqrt{2}$ if and only if $R = \sqrt{2}$.

Combining this with Theorems 2.1 and 2.3, we can conclude:

Corollary 2.4. When $R \neq \sqrt{2}$, the distinct shapes in $\{i_{\mathbf{x}}(T_R) : \mathbf{x} \in \mathbb{R}^3 \setminus T_R\}$ are those in

$$\{i_\varrho(T_R) : \varrho \in [0, \sqrt{R^2 - 1}] \setminus \{R - 1\}\}, \quad (2.5)$$

or equivalently, those in

$$\left\{ i_\varrho(T_R) : \varrho \in [0, R - 1) \right\} \cup \left\{ i_{\varrho'}(T_{R'}) : \varrho' \in [0, R' - 1) \right\}, \quad R' = \frac{R}{\sqrt{R^2 - 1}}. \quad (2.6)$$

When $R = \sqrt{2}$, the distinct shapes in $\{i_{\mathbf{x}}(T_{\sqrt{2}}) : \mathbf{x} \in \mathbb{R}^3 \setminus T_{\sqrt{2}}\}$ are those in $\{i_\varrho(T_{\sqrt{2}}) : \varrho \in [0, \sqrt{2} - 1)\}$.

Now Theorem 1.1 follows by combining the second part of Corollary 2.4 with Theorem 1.2. Moreover, we can now show that Theorem 1.4 implies Proposition 1.3:

Proof of Proposition 1.3. By Corollary 2.4, when $R \neq \sqrt{2}$, the distinct shapes of $\{i_{\mathbf{x}}(T_R) : \mathbf{x} \in \mathbb{R}^3 \setminus T_R\}$ are those in (2.5). As ϱ increases from 0 to $R - 1$, the isoperimetric ratio of $i_\varrho(T_R)$ increases from $3/(2\sqrt{\pi R})$ to 1, by Theorem 1.4. But when ϱ further increases from $R - 1$ to $\sqrt{R^2 - 1}$, the isoperimetric ratio of $i_\varrho(T_R)$ is that of $i_{\varrho'}(T_{R'})$, where (R', ϱ') is given by (2.2), which decreases from 1 to $3/(2\sqrt{\pi R'})$. So for any $v \in [\max(\frac{3}{2\sqrt{\pi R}}, \frac{3}{2\sqrt{\pi R'}}), 1)$, there are two distinct $i_\varrho(T_R)$ sharing the same isoperimetric ratio v . $\square$

For a visualization of the proof above we refer to Fig. 2; the isoperimetric ratios can be computed either using numerical integration or the hypergeometric representations proved in Propositions 4.1 and 4.2.

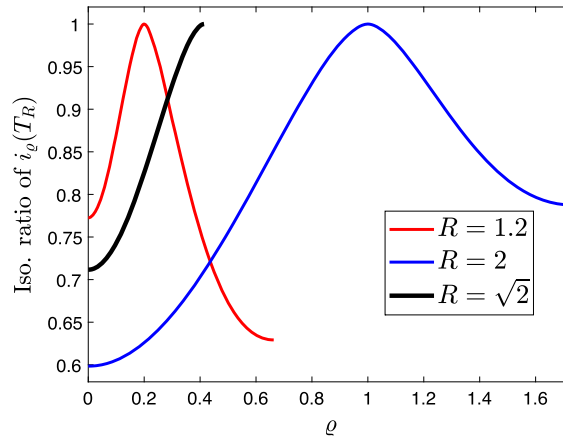


Fig. 2. Isoperimetric ratio of $i_\varrho(T_R)$. When $R \neq \sqrt{2}$, all the shapes in $\{i_\varrho(T_R) : \varrho \in [0, \sqrt{R^2 - 1}]\}$ are distinct. When $R = \sqrt{2}$, only the shapes in $\{i_\varrho(T_{\sqrt{2}}) : \varrho \in [0, \sqrt{2} - 1]\}$ are distinct.

3. Maxwell ratios and proofs of Theorems 2.2 and 2.3

The proofs of the Theorems 2.2 and 2.3 boil down to an elementary calculation involving (circle inversions of) circle pairs once we accept the following facts, which follow from Maxwell's ellipse-hyperbola characterization of Dupin cyclides [1,4,14].

Every toroidal Dupin cyclide $\mathfrak{C}$ has two mutually orthogonal symmetry planes, at one of which the cross-section of $\mathfrak{C}$ consists of *two circles exterior to each other*, and the other consisting of *two circles with one lying inside another*. We call the former symmetry plane P_1 , and the latter P_2 . Denote by $r_1, r_2, r_1 \geq r_2$, the radii of the two circles on the P_1 cross-section, and by d the distance of their centers. Then the ratio

$$r_1 : r_2 : d \quad (3.1)$$

uniquely determines the Euclidean shape of $\mathfrak{C}$; we refer to (3.1) as the P_1 -ratio of $\mathfrak{C}$. Similarly, denote by $\tilde{r}_1, \tilde{r}_2, \tilde{r}_1 \geq \tilde{r}_2$, the radii of the two circles on the P_2 cross-section, and $\tilde{d}$ the distance of their centers. Then the P_2 -ratio

$$\tilde{r}_1 : \tilde{r}_2 : \tilde{d} \quad (3.2)$$

also uniquely determines the Euclidean shape of $\mathfrak{C}$. Using Maxwell's property (see [17, Section 2] for details), (r_1, r_2, d) and $(\tilde{r}_1, \tilde{r}_2, \tilde{d})$ are related by the linear isomorphism $\tilde{r}_1 = \frac{d+(r_1+r_2)}{2}$, $\tilde{r}_2 = \frac{d-(r_1+r_2)}{2}$, $\tilde{d} = r_1 - r_2$, so each of the two ratios in (3.1)-(3.2) determines the other. For the proof of Theorem 2.3, it is slightly more convenient to use the ratio

$$a : f : (L - a), \text{ where } a = d/2, f = (r_1 - r_2)/2, L = (r_1 + r_2 + d)/2, \quad (3.3)$$

to determine the Euclidean shape of $\mathfrak{C}$. It so happens that a, f and L are the relevant measurements in Maxwell's characterization of $\mathfrak{C}$ [17, Section 2]; we therefore refer to (3.3) as the *Maxwell ratio* of $\mathfrak{C}$.

Lemma 3.1 (Equivalent to Lemma 2.3 in [17]). *Let $R \in (1, \infty)$ and $\mathfrak{C} = i_{[\varrho, 0, 0]}(T_R)$.*

(i) *If $\varrho \in [0, R - 1)$, the P_1 -ratio of $\mathfrak{C}$ is*

$$r_1 : r_2 : d = (R + \varrho)^2 - 1 : (R - \varrho)^2 - 1 : 2R(R^2 - \varrho^2 - 1). \quad (3.4)$$

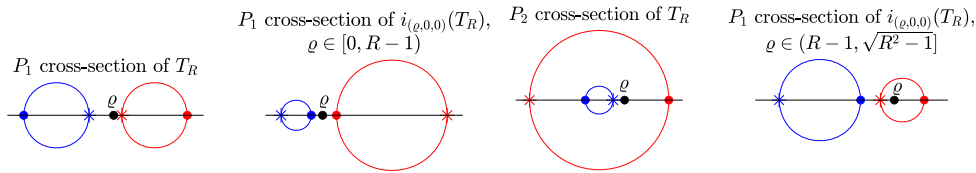


Fig. 3. The shape space of 3-D toroidal Dupin cyclides reduces to 2-D (circle) inversions of circle pairs (which are cross-sections of the cyclides), which further reduces to calculations involving 1-D inversions.

(ii) If $\varrho \in (R - 1, \sqrt{R^2 - 1}]$, the P_1 -ratio of $\mathfrak{C}$ is

$$r_1 : r_2 : d = (R - 1)((R + 1)^2 - \varrho^2) : (R + 1)(\varrho^2 - (R - 1)^2) : 4R\varrho. \quad (3.5)$$

This result is equivalent to [17, Lemma 2.3], except that the P_1 -ratios are presented differently. We recall its proof here with the help of a graphical illustration. The key point is that, thanks to Theorem 2.1, we only need to consider sphere inversions centered at the x -axis. This greatly simplifies the computation of the P_1 -ratio of $\mathfrak{C}$.

Proof. In case (i) of the lemma, the P_1 symmetry plane of both $\mathfrak{C}$ and T_R is the x - z plane, so the P_1 cross-section of $\mathfrak{C}$ is the circle inversion of the P_1 cross-section of T_R about the unit circle centered at $(\varrho, 0)$ on the x - z plane, see the first two panels of Fig. 3. In case (ii), the P_1 symmetry plane of $\mathfrak{C}$ is the x - y plane, which is the P_2 symmetry plane of T_R , so the P_1 cross-section of $\mathfrak{C}$ is the circle inversion of the P_2 cross-section of T_R about the unit circle centered at $(\varrho, 0)$ on the x - y plane, as in the next two panels of Fig. 3.

Furthermore, the circle pairs in these cross-sections all share the same symmetry line (the middle black lines in Fig. 3), so their radii and centers can be calculated easily by 1-D inversions, defined by $i(x, \varrho) := \varrho + 1/(x - \varrho)$.

(i) When $\varrho \in [0, R - 1)$, we have

$$\begin{aligned} r_1 &= \frac{i(R - 1, \varrho) - i(R + 1, \varrho)}{2}, & r_2 &= \frac{i(-(R + 1), \varrho) - i(-(R - 1), \varrho)}{2}, \\ d &= \frac{i(R - 1, \varrho) + i(R + 1, \varrho)}{2} - \frac{i(-(R + 1), \varrho) + i(-(R - 1), \varrho)}{2}. \end{aligned}$$

(ii) When $\varrho \in (R - 1, \sqrt{R^2 - 1}]$, we have

$$\begin{aligned} r_1 &= \frac{i(-(R - 1), \varrho) - i(R - 1, \varrho)}{2}, & r_2 &= \frac{i(R + 1, \varrho) - i(-(R + 1), \varrho)}{2}, \\ d &= \frac{i(R + 1, \varrho) + i(-(R + 1), \varrho)}{2} - \frac{i(-(R - 1), \varrho) + i(R - 1, \varrho)}{2}. \end{aligned}$$

The P_1 ratios then follow by routine calculations. $\square$

Proof of Theorem 2.2. The proof of Theorem 2.2 follows by checking that the ratio in (3.5) with R replaced by $R/\sqrt{R^2 - 1}$ and ϱ replaced by $\frac{1}{\sqrt{R^2 - 1}} \frac{\sqrt{R^2 - 1} - \varrho}{\sqrt{R^2 - 1} + \varrho}$ is the same as the ratio in (3.4). $\square$

Proof of Theorem 2.3. The second claim in the theorem follows from the first claim and Theorem 2.1. The first claim is equivalent to saying that the P_1 -ratio in (3.4) is distinct for distinct (R, ϱ) in $\mathfrak{T} = \{(R, \varrho) : R \in (1, \infty), \varrho \in [0, R - 1)\}$, which is equivalent to having distinct Maxwell ratio (3.3). The latter can be expressed as

$$\begin{aligned}
a : f : (L - a) = d : (r_1 - r_2) : (r_1 + r_2) = 1 : \frac{(R + \varrho)^2 - (R - \varrho)^2}{2R(R^2 - \varrho^2 - 1)} : \frac{(R + \varrho)^2 + (R - \varrho)^2 - 2}{2R(R^2 - \varrho^2 - 1)} \\
= 1 : \frac{2\varrho}{R^2 - \varrho^2 - 1} : \frac{R^2 + \varrho^2 - 1}{R(R^2 - \varrho^2 - 1)}.
\end{aligned}$$

The claim then follows because the map $(R, \varrho) \mapsto \left(\frac{2\varrho}{R^2 - \varrho^2 - 1}, \frac{R^2 + \varrho^2 - 1}{R(R^2 - \varrho^2 - 1)} \right)$ is injective on $\mathfrak{T}$. It is easy to check that $\phi(R, \varrho) \in \mathfrak{T}' := \{(a, b) : a \in [0, 1], b \in (a, 1)\}$ for $(R, \varrho) \in \mathfrak{T}$. One can then see that $\phi : \mathfrak{T} \rightarrow \mathfrak{T}'$ is bijective by verifying that $\phi^{-1}(a, b) = (\sqrt{1 - a^2}, (b\sqrt{1 - a^2} - \sqrt{b^2 - a^2})/a)/\sqrt{b^2 - a^2}$. $\square$

4. Hypergeometric representations and proof of Theorem 1.4

To prove Theorem 1.4, we first show that the surface area and enclosing volume of $i_{[\varrho, 0, 0]}(T_R)$, for $\varrho \in [0, R - 1]$, can be expressed as integrals of trigonometric functions. Write

$$\mathbf{x}(u, v, r) := [(R + r \sin(v)) \cos(u), (R + r \sin(v)) \sin(u), r \cos(v)].$$

Since the conformal factor of $i_{\mathbf{a}}$ at $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{a}\}$ is

$$\lambda^2(\mathbf{a}, \mathbf{x}) = \frac{1}{(\langle \mathbf{a}, \mathbf{a} \rangle - 2\langle \mathbf{a}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{x} \rangle)^2},$$

i.e. $\langle di_{\mathbf{a}}|_{\mathbf{x}} v, di_{\mathbf{a}}|_{\mathbf{x}} w \rangle = \lambda^2(\mathbf{a}, \mathbf{x}) \langle v, w \rangle$, the area and enclosing volume of $i_{[\varrho, 0, 0]}(T_R)$ are given by

$$A_R(\varrho) = \int_0^{2\pi} \int_0^{2\pi} \frac{d\text{Area}(u, v)}{Q(\varrho; \mathbf{x}(u, v, 1))^2}, \quad V_R(\varrho) = \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{d\text{Vol}(u, v, r)}{Q(\varrho; \mathbf{x}(u, v, r))^3}, \quad (4.1)$$

where

$$\begin{aligned}
Q(\varrho; \mathbf{x}) &:= \frac{1}{\lambda([\varrho, 0, 0], \mathbf{x})} = \varrho^2 - 2\mathbf{x}_1 \varrho + \|\mathbf{x}\|^2, \\
d\text{Area}(u, v) &= \|\mathbf{x}_u(u, v, 1) \times \mathbf{x}_v(u, v, 1)\| du dv = (R + \sin(v)) du dv, \\
d\text{Vol}(u, v, r) &= |\det[\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_r]| du dv dr = r(R + r \sin(v)) du dv dr.
\end{aligned}$$

Notice also that

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 = R^2 + r^2 + 2Rr \sin(v).$$

It can be shown that A_R and V_R (as in [17, §4.2]) extend to holomorphic functions on the disk

$$D := \{z \in \mathbb{C} : |z| < R - 1\}.$$

Theorem 1.4 is equivalent to showing that the isoperimetric ratio of $i_{[\varrho, 0, 0]}(T_R)$, which equals

$$6\sqrt{\pi} \cdot \frac{V_R(\varrho)}{A_R(\varrho)^{3/2}} =: \text{Iso}_R(\varrho), \quad (4.2)$$

defines a monotonically increasing function in ϱ from $[0, R - 1]$ to $[3/\sqrt{4\pi R}, 1)$. This will be proven in Section 4.2, after we establish the hypergeometric expressions for $A_R(\varrho)$ and $V_R(\varrho)$ in Section 4.1.

4.1. Hypergeometric representation of A_R and V_R

The following section is analogous to a part of the PhD thesis of the fourth author [19, §4.3], however the defining integrals in (4.3) and (4.4) are slightly different from [19, (4.8) & (4.9)]. In practice this amounts to little more than the substitution $z \mapsto z(R^2 - 1)^2$, however for sake of completeness we redo all proofs in the new setting. For more details than provided here as well as more explanations on how the expressions were actually *found* (not only proven) we refer to [19, §4.3].

Let us first write out the integrals for A_R and V_R explicitly:

$$A_R(z) = \int_0^{2\pi} \int_0^{2\pi} \frac{(R + \sin(v)) du dv}{(z^2 - 2(R + \sin(v)) \cos(u)z + R^2 + 1 + 2R \sin(v))^2}, \quad (4.3)$$

$$V_R(z) = \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r(R + r \sin(v)) du dv dr}{(z^2 - 2z(R + r \sin(v)) \cos(u) + R^2 + r^2 + 2Rr \sin(v))^3}. \quad (4.4)$$

Proposition 4.1. *For $R > 1$ and $z \in [0, R - 1)$ it holds that*

$$\begin{aligned} A_R(z) &= \frac{4\pi^2 R ((R^2 - 1)^2 - z^4)}{((R - 1)^2 - z^2)^2 ((R + 1)^2 - z^2)^2} \cdot {}_2F_1 \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix}; \frac{4z^2}{(R^2 - 1 - z^2)^2} \right] \\ &= \frac{4\pi^2 R}{(R^2 - 1)^2} + \frac{4\pi^2 R (4R^2 + 5)}{(R^2 - 1)^4} z^2 + \frac{9\pi^2 R (4R^4 + 16R^2 + 5)}{(R^2 - 1)^6} z^4 + \dots \end{aligned} \quad (4.5)$$

Proof. With a Weierstrass substitution we rewrite the integral representation of $A_R(z)$ as an integral of a rational function over the closed surface $\gamma = \{(x, y) \in \mathbb{C}^2 : |x| = |y| = 1\} \subseteq \mathbb{C}^2$:

$$A_R(z) = -2 \int_{|x|=|y|=1} \frac{(2Ry - y^2 - 1) x dx dy}{(2z^2 yx - (2Ry - y^2 - 1)(x^2 + 1)z + 2x(Ry - 1)(R - y))^2}.$$

Let $a \in \mathbb{Q}(x, y, z, R)$ be this integrand. Koutschan's improved and implemented version [11] of Chyzak's algorithm for creative telescoping [6] finds a second-order differential operator $L_a \in \mathbb{Q}(z, R)\langle \partial_z \rangle$ and (explicit but huge) rational functions $C_1, C_2 \in \mathbb{Q}(z, R, x, y)$, such that

$$L_a \cdot a = \partial_x C_1 + \partial_y C_2.$$

More precisely,

```
<< RISC'HolonomicFunctions'
FindCreativeTelescoping[a, {Der[x], Der[y]}, Der[z]]
```

takes a few seconds on a regular laptop and outputs $\{\{L_a\}, \{C_1, C_2\}\}$. The operator $L_a \in \mathbb{Q}(z, R)\langle \partial_z \rangle$ has leading coefficient

$$z((R - 1)^2 - z^2)((R + 1)^2 - z^2)(R^2 - 1 - z^2)(R^2 - 1 + z^2)^2,$$

and degrees of 11 in z and 10 in the parameter R . The certificates are considerably larger: their numerators have degrees 6, 5, 16, 20 and 6, 6, 15, 18 in x, y, z, R respectively. The denominators of C_1, C_2 have degrees 3, 2, 8, 9 and 2, 4, 8, 10 in these variables respectively. We check that the denominators are factors of

$$\text{denom}(a) \cdot x \cdot y \cdot (1 - 2Ry + y^2) \cdot H(z, R),$$

for some polynomial $H(z, R) \in \mathbb{Q}[z, R]$, i.e. C_1, C_2 are well-defined on $\gamma \subseteq \mathbb{C}^2$. It follows that

$$L_a \cdot A_R(z) = -2 \int_{|x|=|y|=1} L_a \cdot a \, dx dy = 0,$$

because L_a commutes with $\oint_\gamma \cdot dx dy$ and $\oint_\gamma \partial_x C_1 dx dy = \oint_\gamma \partial_y C_2 dx dy = 0$.

Now that we have found a linear differential operator that provably annihilates $A_R(z)$, it is easy to conclude the proof. First, we observe using the defining differential equation of the Gaussian hypergeometric function and closure properties of linear ODEs implemented in any computer algebra software that L_a also annihilates the right-hand side of (4.5). Moreover, we compute the linear recurrence satisfied by the coefficient sequence $(u_n(R))_{n \geq 0}$ of any power series solution of $L_a \cdot y = 0$:

$$p_{10}(n, R)u_{n+10}(R) + \cdots + p_0(n, R)u_n(R) = 0, \quad \text{for some } p_0(n, R), \dots, p_{10}(n, R) \in \mathbb{Z}[n, R],$$

and observe that $p_{10}(n, R) = -(R^2 - 1)^5(n + 10)^2 \neq 0$ for $n \in \mathbb{N}$. It follows that any power series solution of $L_a \cdot y = 0$ is uniquely determined by its first 10 coefficients. Finally, we check that the first 10 coefficients of both sides of (4.5) are identical. $\square$

Proposition 4.2. *For $R > 1$ and $z \in [0, R - 1)$ it holds that*

$$\begin{aligned} V_R(z) &= \frac{2R\pi^2(R^2 - 1 - z^2)^3}{((R - 1)^2 - z^2)^3((R + 1)^2 - z^2)^3} \cdot {}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2R^2 - 4} + 1 \\ 1, \frac{3}{2R^2 - 4} \end{matrix}; \frac{4z^2}{(R^2 - 1 - z^2)^2} \right] \\ &= \frac{2R\pi^2}{(R^2 - 1)^3} + \frac{6R\pi^2(3R^2 + 2)}{(R^2 - 1)^5}z^2 + \frac{3R\pi^2(48R^4 + 104R^2 + 23)}{2(R^2 - 1)^7}z^4 + \cdots \end{aligned} \quad (4.6)$$

Proof. The proof works analogously to the proof of Proposition 4.1 with one difference. When creative telescoping is applied to the rational function $v \in \mathbb{Q}(x, y, z, r, R)$ whose triple integral defines $V_R(z)$, the algorithm finds some rational functions $C_1, C_2, C_3 \in \mathbb{Q}(z, r, R, x, y)$ such that

$$v = \partial_x C_1 + \partial_y C_2 + \partial_r C_3.$$

However, in this case one cannot conclude that $V_R(z) = \int_0^1 \int_\gamma v dx dy dr = C_3|_{r=1} - C_3|_{r=0}$ because the integration contour intersects the common denominator of C_1, C_2, C_3 , and therefore integration and differentiation cannot be exchanged. This issue is known as an “irregular certificate” in the field of creative telescoping.

We overcome this problem by first proving that

$$\begin{aligned} V_R^r(z) &:= \int_0^{2\pi} \int_0^{2\pi} \frac{r(R + r \sin(v)) \, du \, dv}{(z^2 - 2z(R + r \sin(v)) \cos(u) + R^2 + r^2 + 2Rr \sin(v))^3} = \\ &= Q_1(z, R, r) {}_2F_1 \left[\begin{matrix} 1/2, 1/2 \\ 2 \end{matrix}; Z \right] + Q_2(z, R, r) {}_2F_1 \left[\begin{matrix} 3/2, 3/2 \\ 3 \end{matrix}; Z \right], \end{aligned} \quad (4.7)$$

for $Z = 4r^2 z^2 / ((R - r + z)(R + r - z)(R + r + z)(R - r - z))$ and explicit rational functions $Q_1, Q_2 \in \mathbb{Q}(z, R, r)$. Then one can find an explicit antiderivative of the right-hand side in terms of a hypergeometric function and simply evaluate it at $r = 1$ and $r = 0$:

$$V_R(z) = \int_0^1 V_R^r(z) dr = \frac{2r^2 R \pi^2 (R^2 - r^2 - z^2)^3}{((R-r)^2 - z^2)^3 ((R+r)^2 - z^2)^3} \cdot {}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3r^2}{2R^2-4r^2} + 1 \\ 1, \frac{3r^2}{2R^2-4r^2} \end{matrix}; \frac{4r^2 z^2}{(R^2 - r^2 - z^2)^2} \right] \Big|_0^1. \quad (4.8)$$

Once the expression in (4.8) is found, the verification of the proof is easy: Like in the proof of Proposition 4.1 compute with creative telescoping the annihilator for $V_R^r(z)$ (check that its poles have no intersection with γ) and verify that the derivative of the right-hand side of (4.7) satisfies the same differential equation. Here the computation of L_v, C_1, C_2 with `FindCreativeTelescoping` such that $L_v \cdot v = \partial_x C_1 + \partial_y C_2$ takes roughly three times longer than for $A_R(z)$ on the same laptop but is still less than 1 minute. The operator L_v has order three, and degrees 16 in z and 14 in both r and R . The certificates are again considerably larger: their numerators have degrees 5, 5, 24, 26, 26 and 5, 8, 23, 25, 26 in x, y, z, r, R , and their denominators are of degrees 4, 4, 12, 13, 12 and 4, 5, 12, 14, 12 respectively. We refer to [19, §4.3.2] for the explanation on how the expression in (4.8) can be first guessed before it is proven. Finally, by uniqueness (after checking enough initial conditions) one concludes (4.7) and consequently (4.8). $\square$

4.2. Iso_R is increasing

Combining the definition (4.2) of Iso_R with the propositions in Section 4.1 we arrive at

$$\text{Iso}_R^2(z) = 36\pi \cdot \frac{V_R(z)^2}{A_R(z)^3} = \frac{9}{4\pi R} \frac{{}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2R^2-4} + 1 \\ 1, \frac{3}{2R^2-4} \end{matrix}; \frac{4z^2}{(R^2-1-z^2)^2} \right]^2}{{}_2F_1 \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix}; \frac{4z^2}{(R^2-1-z^2)^2} \right]^3} \left(\frac{R^2 - z^2 - 1}{R^2 + z^2 - 1} \right)^3. \quad (4.9)$$

Note that

$${}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2(R^2-2)} + 1 \\ 1, \frac{3}{2(R^2-2)} \end{matrix}; 1 \right] = \frac{16R^2}{3\pi} \quad \text{and} \quad {}_2F_1 \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix}; 1 \right] = \frac{4}{\pi},$$

where the first identity is due to

$${}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2(R^2-2)} + 1 \\ 1, \frac{3}{2(R^2-2)} \end{matrix}; x \right] = {}_2F_1 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2} \\ 1 \end{matrix}; x \right] + \frac{3}{2} (R^2 - 2) \cdot x \cdot {}_2F_1 \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 2 \end{matrix}; x \right]. \quad (4.10)$$

We mention that this identity (4.10) also highlights the special role of $R = \sqrt{2}$ covered in [2,17], as in this case the second summand vanishes.

From the formulas above we immediately obtain that $\lim_{z \rightarrow R-1} \text{Iso}_R(z) = 1$. We are now in a position to finish the proof of Theorem 1.4. After the substitution

$$x = \frac{4z^2}{(R^2 - 1 - z^2)^2},$$

it is enough to show that the function

$$h_R(x) := \frac{{}_3F_2 \left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2(R^2-2)} + 1 \\ 1, \frac{3}{2(R^2-2)} \end{matrix}; x \right]^2}{{}_2F_1 \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix}; x \right]^3} \cdot (1 + (R^2 - 1) \cdot x)^{-3/2}$$

is increasing on $x \in (0, 1)$ for all $R > 1$. Now define the two functions

$$f(x) := \frac{(x+1)^{1/2}}{{}_2F_1\left[-\frac{1}{2}, -\frac{1}{2}; x\right]} \quad \text{and} \quad g_R(x) := \frac{{}_3F_2\left[\begin{matrix} -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2(R^2-2)}+1 \\ 1, \frac{3}{2(R^2-2)} \end{matrix}; x\right]}{(1+x)^{3/4} \cdot (1+(R^2-1) \cdot x)^{3/4}}$$

and note that clearly $h_R = f^3 \cdot g_R^2$. We will show that both f and g_R increase on $x \in (0, 1)$ for any $R > 1$. Since f and g_R are positive on this interval, this will be enough to conclude Theorem 1.4.

The fact that $f(x)$ is increasing on $(0, 1)$ is already proved in [2, Proposition 1]. For $g_R(x)$ we have to adapt the argument slightly, since the parameter R indeed complicates matters. Let

$$\frac{4 \cdot g'_R(x) \cdot (1+x)^{7/4} \cdot (1+(R^2-1) \cdot x)^{7/4}}{3 \cdot (1-x)^2 \cdot (R^2-1)} =: \sum_{n \geq 0} u_n(R) x^n,$$

for some rational functions $u_n(R) \in \mathbb{Q}(R)$, $n \geq 1$ and $u_0(R) = 1$. It turns out that the sequence $(u_n(R))_{n \geq 0}$ is hypergeometric, and this will allow us to prove that $u_n(R) > 0$ for $R > 1$. Clearly, this implies that $g'_R(x) > 0$, i.e. $g_R(x)$ is increasing. For all $n \geq 0$ define the sequence of polynomials

$$p_n(R) := 4(R^4 + 4R^2 - 4)n^3 + 6(R^4 + R^2 - 2)n^2 + (2R^4 - 13R^2 + 10)n - 3R^2 + 3.$$

Then it is not difficult to check with a computer that for $n \geq 0$ we have

$$\frac{u_{n+1}(R)}{u_n(R)} = \frac{(2n-1)(2n+1) \cdot p_{n+1}(R)}{4(n+2)(n+1) \cdot p_n(R)}.$$

Hence, it holds that $u_n(R) > 0$ for $R > 1$ if we can prove that $p_n(R) > 0$ for all $n \geq 1$. Observe that for $n \geq 1$ and $R > 1$:

$$p_n(R) > 4R^4 n^3 + (10 - 11R^2)n + 3 - 3R^2 =: q(n).$$

The polynomial $q(n)$ is strictly increasing in n for $n \geq 1$ since the larger root of its derivative evaluates to $\sqrt{33R^2 - 30}/(6R^2) < 1$. Therefore we can set $n = 1$ and obtain

$$p_n(R) > q(1) = (2R^2 - 7/2)^2 + 3/4 > 0.$$

The proof of Theorem 1.4 is now completed. $\square$

Data availability

No data was used for the research described in the article.

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