
On an Integral Identity

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Abstract. We give three elementary proofs of a nice equality of definite integrals, recently proven by Ekhad, Zeilberger, and Zudilin. The equality arises in the theory of bivariate hypergeometric functions, and has connections with irrationality proofs in number theory. We furthermore provide a generalization together with an equally elementary proof and discuss some consequences.

1. INTRODUCTION. The following infinite family of equalities between definite integrals was proven by Ekhad, Zeilberger, and Zudilin [5], using the Almkvist–Zeilberger *creative telescoping algorithm* [2] for symbolic integration:

$$\int_0^1 \frac{x^n (1-x)^n}{((x+a)(x+b))^{n+1}} dx = \int_0^1 \frac{x^n (1-x)^n}{((a-b)x + (a+1)b)^{n+1}} dx, \quad (1)$$

for any reals $a > b > 0$ and any nonnegative integer n .

As pointed out in [5], these integrals “are not taken from a pool of no-one-cares analytic creatures”: they are related to rational approximations to some logarithmic values [1] and a trained eye could recognize in identity (1) a particular case of a known relation for Appell’s bivariate hypergeometric function (see Section 5).

The proof provided in [5] is, without any doubt, elementary. It requires some clever (and at first sight, magic) auxiliary rational functions coming from the silicon guts of the first author of [5]. Of course, this is not objectable at all, and we are ourselves convinced that computer-assisted proofs are an increasingly important trend in mathematics. At the same time we think that, *pour l’honneur de l’esprit humain*, it is of some interest to offer a proof that a first-year undergraduate could not only follow but also create by hand.

In Sections 2–4 of this article, we provide three elementary proofs of the identity (1). We also generalize the identity. In Section 5, we discuss some direct consequences, emphasizing the relation of the identities with known identities for hypergeometric functions. We also give a couple of combinatorial identities, and finally conclude that the Legendre polynomials are eigenfunctions of a certain differential operator.

2. FIRST PROOF: USING A RATIONAL CHANGE OF VARIABLES. Note that the natural range to assure the convergence of the integrals is $a, b > 0$ and $n \in \mathbb{R}_{>-1}$. Let us first demonstrate that an utterly simple change of variables proves the equality in this extended range and in fact an even more general equality:

Theorem 1. *If $a, b > 0$, $k, n \in \mathbb{R}$, and $s, \ell \in \mathbb{R}_{>-1}$, then*

$$\int_0^1 \frac{x^\ell (1-x)^s}{(x+a)^{k+1} (x+b)^{n+1}} dx = \frac{(b+1)^{s-n}}{b^{n-\ell}} \int_0^1 \frac{x^s (1-x)^\ell (x+b)^{n+k-\ell-s}}{((a-b)x + (a+1)b)^{k+1}} dx. \quad (2)$$

Proof. Consider the rational change of variables

$$x = \frac{b(1-u)}{b+u}, \quad \text{and consequently} \quad dx = -\frac{b(1+b)}{(b+u)^2} du.$$

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Note that it is a promising change because it takes 0, 1, $b(a+1)/(b-a)$, and ∞ , which are the singularities of the second integrand, into 1, 0, $-a$, and $-b$, respectively, which are singularities of the first integrand. The interval $[0, 1]$ is preserved. Thus, a direct calculation shows that the integral in the left-hand side is

$$\begin{aligned} I &= \int_0^1 \frac{\left(\frac{b(1-u)}{b+u}\right)^\ell \left(1 - \frac{b(1-u)}{b+u}\right)^s}{\left(\frac{b(1-u)}{b+u} + a\right)^{k+1} \left(\frac{b(1-u)}{b+u} + b\right)^{n+1}} \frac{b(1+b)}{(b+u)^2} du \\ &= \int_0^1 \frac{(1-u)^\ell u^s b^{\ell+1} (1+b)^{s+1} (u+b)^{k+n-\ell-s}}{(b(1-u) + a(b+u))^{k+1} (b(1-u) + b(b+u))^{n+1}} du \\ &= \frac{(b+1)^{s-n}}{b^{n-\ell}} \int_0^1 \frac{u^s (1-u)^\ell (b+u)^{k+n-\ell-s}}{((a-b)u + (a+1)b)^{k+1}} du, \end{aligned}$$

which proves the result. $\blacksquare$

We will come back to equation (2) and draw some conclusions from this more general identity in Section 5.

3. SECOND PROOF: USING INDEFINITE INTEGRATION. We still consider the extended convergence range $a, b > 0$, but now n is a nonnegative integer. The following proof is based on the generating functions [11] of the sequence of integrals when n varies.

Proof of (1) for $a, b > 0$ and $n \in \mathbb{Z}_{\geq 0}$. Let I_1 and I_2 be the generating functions of each side of (1), i.e., multiplying by t^n and summing from $n = 0$ to ∞ . They clearly converge uniformly for small values of t and we have $I_j(t) = \int_0^1 dx/P_j(x, t)$ where

$$\begin{aligned} P_1(x, t) &= (x+a)(x+b) - tx(1-x) \quad \text{and} \\ P_2(x, t) &= (a-b)x + (a+1)b - tx(1-x). \end{aligned} \tag{3}$$

If $P(x) = Ax^2 + Bx + C$ has no zero in $[0, 1]$, and $\Delta = B^2 - 4AC > 0$, then, using standard integration techniques, we obtain

$$\int_0^1 \frac{dx}{P(x)} = \frac{1}{r} \log \left(\frac{B+2C+r}{B+2C-r} \right) \quad \text{with } r = \sqrt{\Delta}.$$

Note that for t small enough, P_1 and P_2 fulfill these conditions. A calculation shows that both polynomials have the same discriminant Δ and the same values of $B+2C$. Hence, $I_1(t) = I_2(t)$ in some interval containing the origin and then the integrals in equation (1), which are their Taylor coefficients at $t = 0$, are equal. $\blacksquare$

4. THIRD PROOF: CREATIVE TELESOPING. We cannot resist the temptation to offer a third proof, in the spirit of the one in [5], but based on a different kind of “creative telescoping.” The starting point is the same as in Section 3, namely that the family of identities (1) is equivalent to the fact that the two integrals $I_1(t)$ and $I_2(t)$ between $x = 0$ and $x = 1$ of the rational functions $F_1 = 1/P_1$ and $F_2 = 1/P_2$, with P_j as in (3), are equal.

Creative telescoping [2], this time in the classical “differential-differential” setting, shows that F_1 and F_2 satisfy the equalities¹

$$(t - 2ab - a - b)F_j + (t^2 - 2t(2ab + a + b) + (a - b)^2)\frac{\partial F_j}{\partial t} + \frac{\partial}{\partial x}(F_j R_j) = 0,$$

where $R_1(t, x)$ and $R_2(t, x)$ are the rational functions

$$R_1(t, x) = ((a + b + t + 2)x + 2ab + a + b - t)x$$

and

$$R_2(t, x) = \frac{((a - b)^2 - (2ab + a + b)t)x + b(a + 1)(a - b + t)}{t}.$$

Hence, by integration between $x = 0$ and $x = 1$, one obtains that both I_1 and I_2 are solutions of the differential equation

$$(t - 2ab - a - b)I(t) + (t^2 - 2t(2ab + a + b) + (a - b)^2)I'(t) + 2 = 0.$$

Therefore, $I_1 = I_2$ by Cauchy’s theorem, since $I_1 - I_2$ is the solution of a differential equation of first order with leading term nonvanishing at $t = 0$, and its evaluation at $t = 0$ is zero, as $I_1(0) = I_2(0) = \frac{1}{a-b} \ln\left(\frac{a(b+1)}{(a+1)b}\right)$. $\square$

5. SOME CONSEQUENCES.

Appell’s identity. We will show that identity (2) from our Theorem 1 contains, as a particular case, a classical hypergeometric identity due to Appell. The relevant definitions needed to state this identity are the classical Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; t)$ and the Appell bivariate hypergeometric function $F_1(\alpha; \beta, \beta'; \gamma; x, y)$, given respectively by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{t^n}{n!} \quad \text{and} \quad \sum_{m,n \geq 0} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!},$$

where $(a)_n$ denotes the rising factorial $a(a + 1) \cdots (a + n - 1)$ for $n \in \mathbb{N}$ and it is assumed $|t|, |x|, |y| < 1$ to assure the convergence.

It is very classical that these hypergeometric functions admit the following integral representations, which hold as soon as $\beta, \beta' > 0$ and $\gamma := \beta + \beta' > \alpha > 0$:

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \frac{t^{\beta-1}(1-t)^{\beta'-1}}{(1-tz)^\alpha} dt \quad (4)$$

and

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{(1-tx)^\beta (1-ty)^{\beta'}} dt. \quad (5)$$

Equation (4) is due to Euler [3, Theorem 2.2.1], and (5) to Picard [7]; see also [4, equation (9)].

With these notations, we are able to state the following hypergeometric function identity, which appears on page 8 of Appell’s classical memoir [4]:

¹We have obtained these equalities with the help of the commands `Zeilberger` and `creative.telescoping` available in the Maple packages `DEtools` and `Mgfun`.

Corollary 2 (Appell's identity). If $\alpha, \beta, \beta' > 0$, $\beta + \beta' > \alpha$, $|x| < 1$, and $|y| < 1$ then

$${}_2F_1\left(\alpha, \beta; \beta + \beta'; \frac{y-x}{y-1}\right) = (1-y)^\alpha F_1(\alpha; \beta, \beta'; \beta + \beta'; x, y). \quad (6)$$

Equivalently, in terms of integrals:

$$\int_0^1 \frac{t^{\beta-1} (1-t)^{\beta'-1}}{((y-x)t + 1-y)^\alpha} dt = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\beta + \beta' - \alpha)} \int_0^1 \frac{t^{\alpha-1} (1-t)^{\beta+\beta'-\alpha-1}}{(1-tx)^\beta (1-ty)^{\beta'}} dt. \quad (7)$$

Remark 3. Identity (1) is a particular case of (7), with $\alpha = \beta = \beta' = n + 1$.

Proof of Corollary 2. If we set $n = \ell + s - k$ in (2), then the identity becomes

$$\begin{aligned} & \int_0^1 \frac{x^\ell (1-x)^s}{(x+a)^{k+1} (x+b)^{\ell+s-k+1}} dx \\ &= \frac{(b+1)^{k-\ell}}{b^{s-k}} \int_0^1 \frac{x^s (1-x)^\ell}{((a-b)x + (a+1)b)^{k+1}} dx. \end{aligned} \quad (8)$$

Next we replace the integration variable x with $1-t$ in the first integral and with t in the second integral, then a with $-1/x$ and b with $-1/y$, to deduce the following equivalent form for $x, y < 0$:

$$\int_0^1 \frac{t^\ell (1-t)^s}{\left(1 - \frac{y-x}{y-1}t\right)^{k+1}} dt = (1-y)^{\ell+1} \int_0^1 \frac{t^\ell (1-t)^s}{(1-tx)^{k+1} (1-ty)^{\ell+s-k+1}} dt.$$

This last integral can be analytically continued to the values of x and y as in the statement. Setting $\ell = \alpha - 1$, $k = \beta - 1$, and $s = \beta + \beta' - \alpha - 1$, using the integral representations (4) and (5) and the obvious symmetry ${}_2F_1(\alpha, \beta; \gamma; t) = {}_2F_1(\beta, \alpha; \gamma; t)$, we deduce (6). ■

The close relationship between equation (8) and the univariate and bivariate hypergeometric functions is shown in the previous proof of Corollary 2. Taking into account that equation (7) is a formulation of this result not involving any hypergeometric function, it appears as a natural problem to provide a more direct proof of equation (7). We present an independent proof involving basic real and complex analysis.

Alternative proof of equation (7). We write $\gamma = \beta + \beta'$ as before. Changing $t \mapsto 1-t$ in the first integral of equation (7) and multiplying the identity by $\Gamma(\alpha)\Gamma(\gamma - \alpha)$, we want to prove that the functions

$$G_1(x, y) = \Gamma(\alpha)\Gamma(\gamma - \alpha) \int_0^1 \frac{t^{\gamma-\beta-1} (1-t)^{\beta-1}}{(1-ty - (1-t)x)^\alpha} dt$$

and

$$G_2(x, y) = \Gamma(\beta)\Gamma(\gamma - \beta) \int_0^1 \frac{t^{\alpha-1} (1-t)^{\gamma-\alpha-1}}{(1-tx)^\beta (1-ty)^{\gamma-\beta}} dt$$

coincide. Both are analytic functions of x and y on $\text{Re}(x), \text{Re}(y) < 1/2$ (for instance by Morera's theorem applied separately in x and y). Then it is enough to prove

$$\frac{\partial^{n+m} G_1}{\partial x^n \partial y^m}(0, 0) = \frac{\partial^{n+m} G_2}{\partial x^n \partial y^m}(0, 0) \quad \text{for every } m, n \in \mathbb{Z}_{\geq 0},$$

because in this case their Taylor coefficients coincide. Noting that the k th derivative of $(1-x)^{-\delta}$ is $\frac{\Gamma(\delta+k)}{\Gamma(\delta)}(1-x)^{-\delta-k}$, we see that this is the same as

$$\begin{aligned} & \Gamma(\gamma - \alpha)\Gamma(\alpha + m + n)B(\gamma + m - \beta, \beta + n) \\ &= \Gamma(\beta + n)\Gamma(\gamma + m - \beta)B(\alpha + m + n, \gamma - \alpha) \end{aligned}$$

with $B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$. Using the well-known [9, Ex. 6.3.7] elementary evaluation $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, we complete the proof. ■

Combinatorial identities and Legendre polynomials. The equality (1) and its generalizations can be interpreted in combinatorial terms. If we set $a = b > 0$, $\ell = s = n$, and replace k by $k - 1$, then we get

$$\int_0^1 \frac{x^n(1-x)^n}{(x+b)^{n+k+1}} dx = \frac{1}{b^k(b+1)^k} \int_0^1 \frac{x^n(1-x)^n}{(x+b)^{n-k+1}} dx. \quad (9)$$

Corollary 4. The functions $f_n(x) = x^n(1+x)^n$ with n a nonnegative integer satisfy

$$f_k f_n^{(n+k)} = \frac{(n+k)!}{(n-k)!} f_n^{(n-k)} \quad \text{for } 0 \leq k \leq n.$$

Proof. By Taylor expansion at $x = -b$, we have

$$x^n(1-x)^n = (-1)^n f_n(-x) = \sum_{m=0}^{2n} c_m (x+b)^m \quad \text{with} \quad c_m = (-1)^{n+m} \frac{f_n^{(m)}(b)}{m!}.$$

If we substitute this in equation (9), then a term with $\log(b+1) - \log b$ appears in the left-hand side for $m = n+k$ (the other terms are rational functions of b) and in the integral of the right-hand side for $m = n-k$. Hence $c_{n+k} = c_{n-k}/(b(b+1))^k$. ■

By essentially comparing coefficients, one gets a triple binomial identity:

Corollary 5. For each k, ℓ, n nonnegative integers with $\ell, k \leq n$,

$$\sum_{m=0}^k \binom{k}{m} \binom{n}{m+\ell} \binom{2m+2\ell}{k+n} = \binom{n}{\ell} \binom{2\ell}{n-k}.$$

Remark 6. This hypergeometric identity can be rewritten as

$$\binom{2\ell}{n+k} {}_3F_2\left(-k, \ell-n, \ell+\frac{1}{2}; \ell-\frac{k+n}{2}+1, \frac{1-n-k}{2}+\ell; 1\right) = \binom{2\ell}{n-k},$$

so it is a particular case of the Pfaff–Saalschütz identity [8, §2.3.1] for the evaluation at 1 of well-poised ${}_3F_2$'s. Alternatively, it can be automatically obtained using Zeilberger's creative telescoping algorithm [6].

Proof. In Corollary 4 change the variable x to $(x-1)/2$ to obtain, after clearing denominators,

$$(n-k)!(x^2-1)^k \frac{d^{n+k}}{dx^{n+k}}(x^2-1)^n = (n+k)! \frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n. \quad (10)$$

We rewrite the left-hand side as

$$(n-k)! \sum_{\ell_1} (-1)^{k-\ell_1} \binom{k}{k-\ell_1} x^{2\ell_1} \cdot (n+k)! \sum_{\ell_2} (-1)^{n-\ell_2} \binom{n}{\ell_2} \binom{2\ell_2}{n+k} x^{2\ell_2-n-k},$$

and the right-hand side as

$$(n+k)!(n-k)! \sum_{\ell} (-1)^{n-\ell} \binom{n}{\ell} \binom{2\ell}{n-k} x^{2\ell-n+k}.$$

Comparing coefficients and renaming $\ell_1 = k - m$, $\ell_2 = m + \ell$ yield the result. ■

Remark 7. It turns out that equation (10) is a known identity expressing a symmetry of the associated Legendre polynomials. This can be obtained from Rodrigues' formula and from the general Legendre equation; see [10] for a simple elementary proof. The one obtained here (Corollary 4) is comparatively simple.

The definition of the Legendre polynomials ensures that they are eigenfunctions of the differential operator $\frac{d}{dx}(x^2 - 1)\frac{d}{dx}$. Although $\frac{d}{dx}$ and $(x^2 - 1)$ do not commute and apparently there is not a simple formula for the commutator of their powers, Legendre polynomials are also eigenfunctions of a simple operator composed by powers of these operators.

Corollary 8. Let P_n be the n th Legendre polynomial. Then, for $0 \leq k \leq n$,

$$L[P_n] = \frac{(n+k)!}{(n-k)!} P_n \quad \text{where} \quad L = \frac{d^k}{dx^k} (x^2 - 1)^k \frac{d^k}{dx^k}.$$

Proof. Recalling that P_n is proportional to $\frac{d^n}{dx^n}(x^2 - 1)^n$ (Rodrigues' formula), the result follows from taking the k th derivative of equation (10). ■

Actually, one could prove Corollary 4 from Corollary 8 by repeated integration, and noting that both sides in Corollary 4 are divisible by x^k .

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100 Years Ago This Month in *The American Mathematical Monthly* Edited by Vadim Ponomarenko

ANNA IRWIN YOUNG, for twenty years head of the department of mathematics in Agnes Scott College, Decatur, Ga., and a charter member of our Association, died September 3, 1920.

Dr. EMILE BOREL, professor of calculus of probabilities and mathematical physics at the University of Paris, who has received the degree “doctor honoris causâ” from the University of Dublin, is the first Frenchman to be so honored. A Canadian, two Scots and two Irishmen have received similar degrees, but in each case the award was to a man of letters.

On October 24, 1921, a bill to incorporate the American Mathematical Society in the District of Columbia was introduced into the Senate of the United States by Mr. Lodge. It was twice read and referred to the committee of the judiciary.
—*Congressional Record*, October 24, 1921.

—Excerpted from “Notes and News” (1921). 28(10): 402–406.