New Algorithms for Decomposing Algebraic Sets

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joint with Christian Eder, Pierre Lairez and Mohab Safey El Din

DRN + EFI Conference

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Rheinland-Pfälzische Technische Universität Kaiserslautern Landau

Introduction

Let $\mathbb{K}$ be a field, $R := \mathbb{K}[x_1, \ldots, x_n]$, $\overline{\mathbb{K}}$ algebraic closure.
$f_1, \ldots, f_r \in R$, $X := V(f_1, \ldots, f_r) := \{f_1 = f_2 = \cdots = f_r = 0\} \subset \overline{\mathbb{K}}^n$ *algebraic set.*

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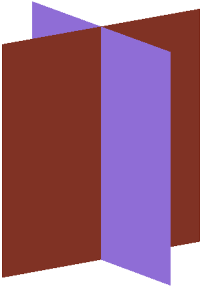
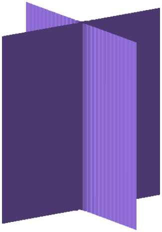


$X = V(xy) = V(x) \cup V(y)$

Definition

$X \subset \overline{\mathbb{K}}^n$ equidimensional if all irreducible components of X have the same dimension.



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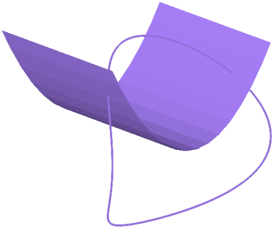

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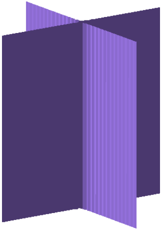

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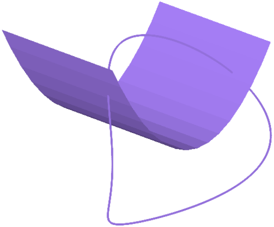

not equidimensional

Given X as $X = V(f_1, \ldots, f_r)$ with $f_1, \ldots, f_r \in \mathbb{K}[x_1, \ldots, x_n]$,

compute a representation $X = \bigcup_{i=1}^{m} X_i$, each X_i equidimensional.

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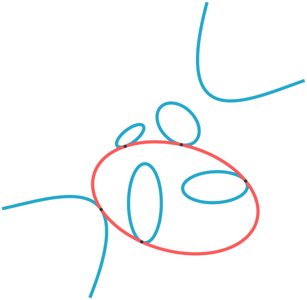

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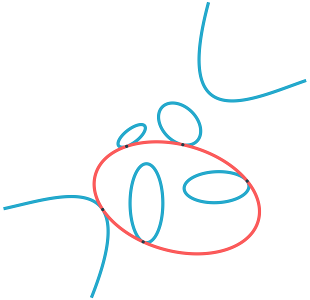

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$\Downarrow$

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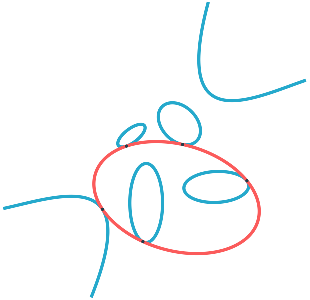

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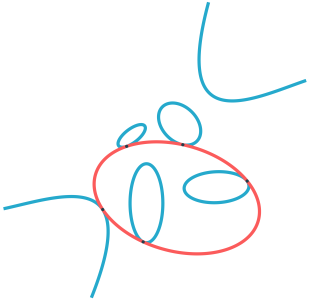

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Problem: $V(S)$ has 2-dimensional component corresponding to squares of linear forms! $\rightsquigarrow$ degenerate solutions

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Main algorithmic tool:

Definition

Gröbner basis G of ideal I w.r.t $\prec$:

Finite set $G \subset I$ s.t.

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triangular sets (e.g. Hubert 2003) and geometric resolutions (e.g. Giusti, Lecerf, and Salvy 2001)...

The Incremental Strategy

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Frequently used in other algorithms, e.g. (Lecerf 2003), (Moroz 2008), (Duff, Leykin, and Rodriguez 2022)...

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$$X \cap V(f) = \underbrace{[V(x, y) \cup V(x, z)]}_{\text{dim=1}} \cup \underbrace{[V(y, z, x(x-1))]}_{\text{dim=0}}$$

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Definition

Regular Intersection: $f|_Y \neq 0$ for every irred. component Y of X.

Regular Sequence: $f_1, \ldots, f_r$ s.t. f_i regular over $V(f_1, \ldots, f_{i-1})$ $\forall i$.

$$I(X) := \{f \in R \mid f|_X = 0\}.$$

Lemma

f intersects X regularly

$$\Updownarrow$$

For every g with $gf \in I(X)$ (syzygy cofactor) we have $g \in I(X)$

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Algorithmic Principle:

Using Gröbner basis computations,

find g with $gf \in I(X)$, $g \notin I(X)$

$$\Downarrow$$

Use g to modify X

Computing the Nondegenerate Locus

Algorithm 1

Let $f_1, \ldots, f_c \in R$ with $c \leq n$, the number of variables. Given $I := \langle f_1, \ldots, f_c \rangle$,

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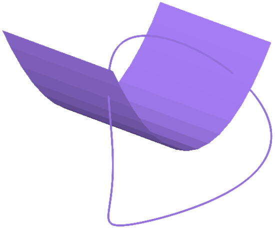

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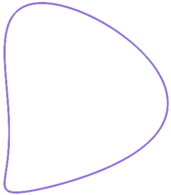
Nondegenerate Locus $\longrightarrow$



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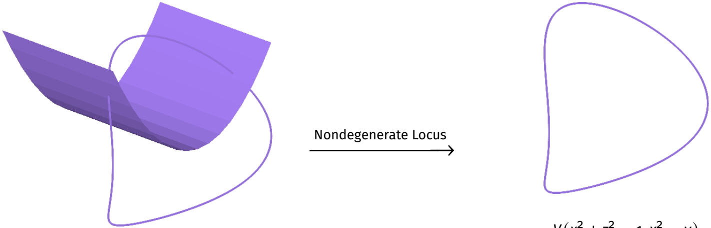

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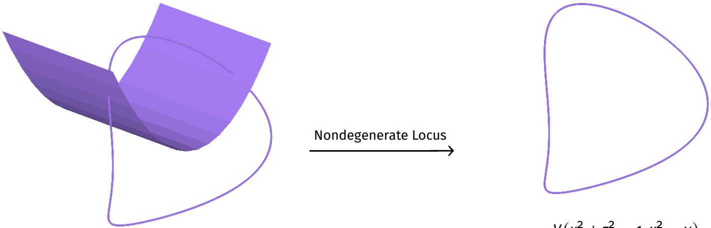

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Idea: Change X to $X \setminus V(h)$ s.t. all g with $gf \in I(X)$ lie in $I(X \setminus V(h))$.

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Compute y and x using Gröbner basis computations!

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Hot to compute Gröbner bases?

Fix: Polynomial Ring $R := \mathbb{K}[x_1, \ldots, x_n], f_1, \ldots, f_c \in R, I := \langle f_1, \ldots, f_c \rangle$.

Want: A Gröbner basis for I (w.r.t. some monomial ordering).

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Buchberger's criterion (Buchberger 1965):

Theorem

G is a Gröbner basis for I iff for every $g_1, g_2 \in G$, the *S-polynomial*

$$\frac{\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))}{\operatorname{lm}(g_1)} g_1 - \frac{\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))}{\operatorname{lm}(g_2)} g_2$$

reduces to zero by G via polynomial long division.

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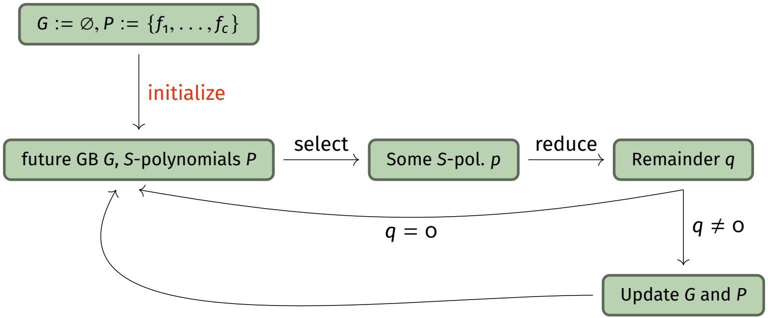

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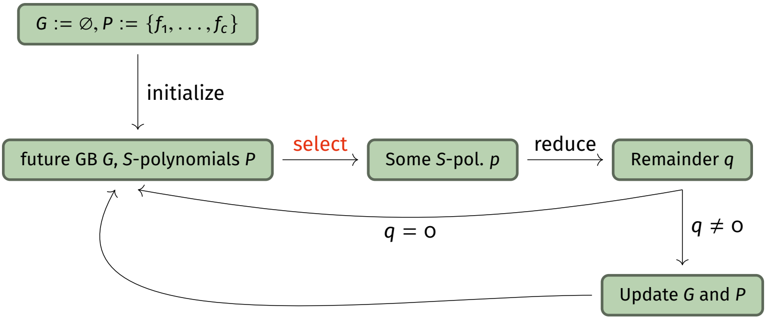

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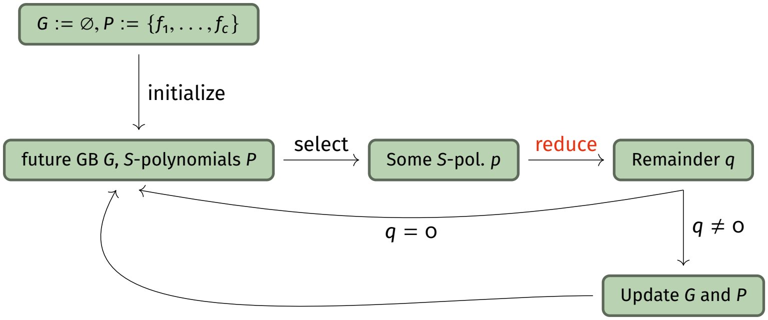

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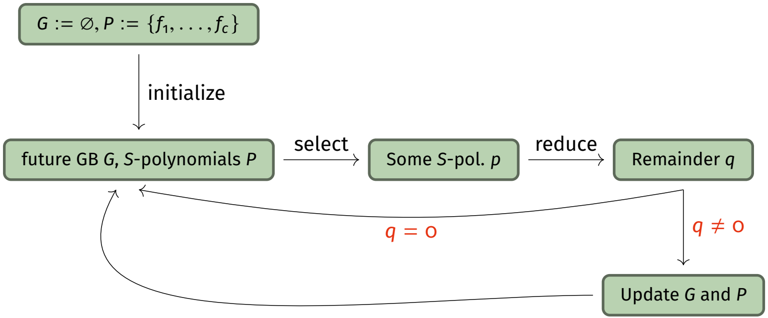

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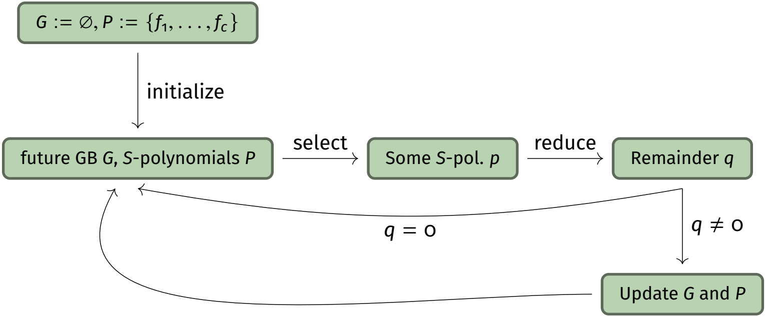
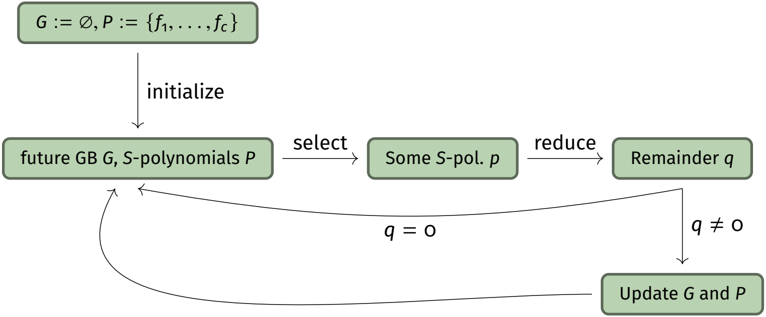


Hot to compute Gröbner bases?

Fix: Polynomial Ring $R := \mathbb{K}[x_1, \ldots, x_n]$, $f_1, \ldots, f_c \in R$, $I := \langle f_1, \ldots, f_c \rangle$.

Want: A Gröbner basis for I (w.r.t. some monomial ordering).



Think of every S-polynomial as ug_i, g_i ith element in G, u monomial

$\rightsquigarrow$ computational history

Signature-based computations à la (J. C. Faugère 2002; Gao, Guan, and Volny IV 2010; Eder and J.-C. Faugère 2017):

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Theorem

Result of reducing an S-polynomial depends *only* on its signature $(\operatorname{lm}(g), f_{k+1})$.

Koszul Criterion – An S-pol. with signature $(\mathrm{lm}(g), f_{k+1})$ reduces to zero if $\mathrm{lm}(g) \in \mathrm{lm}(I(X))$.

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in sGB algorithm:

Zero reduction with cofactor $g \Longrightarrow gf_{k+1} \in I(X)$ but $g \notin I(X)$

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Furnishes nondegenerate locus computations as before!

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Furnishes nondegenerate locus computations as before!

Algorithm with Black box GB usage $\rightsquigarrow$ Dedicated GB algorithm

	Our Algorithm	●	●	●
Sing(2,8)	0.68s	23m	∞	∞
Sing(2,10)	2.9s	∞	∞	∞
Sos(2,6,4)	169s	∞	∞	∞
Steiner	42m	∞	∞	∞

Computed in characteristic 65521 on Intel Xeon Gold 6244.
● Elimination method in Singular, see e.g. (Decker, Greuel, and Pfister 1999).
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	sGB	Our Algorithm	Ratio	GB in Maple	●	Ratio
Sing(2,10)	1.9s	2.9s	1.5	0.11s	1.642s	15
Sos(2,6,4)	148s	160s	1.08	0.172s	22.7s	132
Sos(2,7,4)	3m	30m	10	0.433s	1h	8314
Sos(2,7,5)	25m	20h	48	2.294s	>359h	>564366

Computed in characteristic 65521 on Intel Xeon Gold 6244.
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● "Naive" implementation of nondegenerate locus algorithm in Maple.

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A Signature-based Algorithm to Compute the Nondegenerate Locus of a Polynomial System

Computing Equidimensional Decompositions

Algorithm 2

This time: Compute full equidimensional decomposition of given $V(f_1, \ldots, f_r)$ with incremental strategy.

$$\Downarrow$$

Have to solve:

Given X equidimensional, $f \in \mathbb{K}[x_1, \ldots, x_n]$

Compute equidimensional decomposition of $X \cap V(f)$

After decomposing $X \cap V(f)$ another equation coming in!

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$\Longrightarrow$ Philosophy: Split $X \cap V(f)$

As much as possible, as irredundantly as possible

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$\Downarrow$

Components of lower degree while avoiding exponential blow up!

Incremental step: Again suppose we have function

$$\text{syzcof}(X, f)\text{: returns } g \text{ s.t. } gf \in I(X),\ g \notin I(X)$$

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Example

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X	$V(x,y)$	$V(x,z)$	$V(y,z)$
	$f = 0$	$f = 0$	$f \neq 0$
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Set $X_1 := \overline{X \setminus V(g)} = V(x, z)$. Then

$f = 0$ on X_1!

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$$X \cap V(f) = X_1 \cup \underbrace{[(X \setminus X_1) \cap V(f)]}_{\text{split with same approach!}}$$

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⚠ Splitting a lot via recursion

⚠ Irredundancy via disjointness (and no fake components!)

Algorithm: intersect

if $X \cap V(f)$ equidimensional **then**

  return $X \cap V(f)$

else

  $g \leftarrow \mathrm{syzcof}(X, f)$

  $X_1 \leftarrow \overline{X \setminus V(g)}$

  return $X_1 \cup \mathrm{intersect}(\mathrm{remove}(X, X_1), f)$

end if

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Decompose further $X \setminus X_1$!

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Decompose further $X \setminus X_1$!

Algorithm: remove

Write $X_1 = V(h_1, \ldots, h_r)$
$X_1' \leftarrow V(h_2, \ldots, h_r)$
return $X \setminus V(h_1) \cup \text{intersect}(\text{remove}(X, X_1'), h_1)$

Algorithm: intersect

if $X \cap V(f)$ equidimensional **then**
  return $X \cap V(f)$
else
  $g \leftarrow \underline{\mathrm{syzcof}(X, f)}$
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$$X \setminus X_1 = X \setminus V(h_1) \sqcup \left[(X \setminus X_1') \cap V(h_1)\right]$$

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    $X_1 \leftarrow X \setminus V(g)$
    return $X_1 \cup \text{intersect}(\text{remove}(X, X_1), f)$
end if

Decompose further $X \setminus X_1$!

mutually recursive algorithms for decomposition of $X \cap V(f)$ and $X \setminus X_1$!

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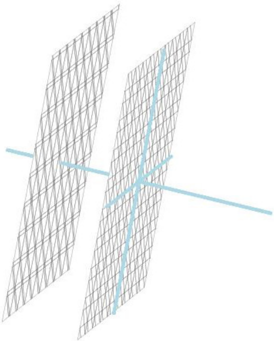
$\text{intersect}(X, f)$
$\text{syzcof}(X, f) = y$



$X = V(xy, xz, yz), f = x(x-1)$

$\text{intersect}(X, f)$
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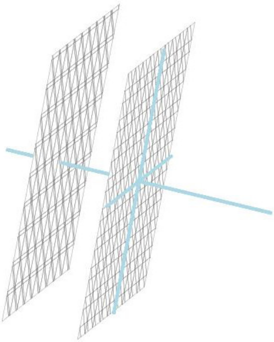
$X_1 = \overline{X \setminus V(y)}$
$= V(x, z)$



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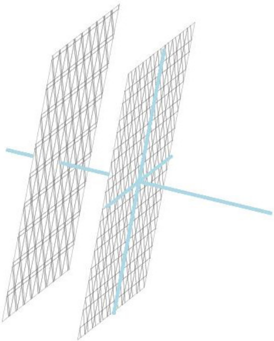

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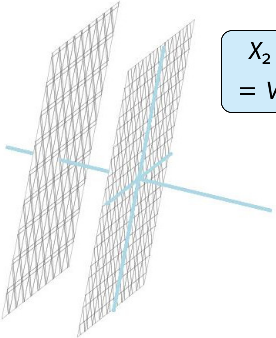
$$X_2 = X \setminus V(x) = V(y, z) \setminus V(x)$$



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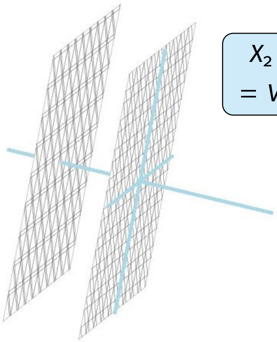
$X_3 = X \setminus V(z) = V(x, y) \setminus V(z)$



$X = V(xy, xz, yz), f = x(x-1)$

intersect(X, f)
syzcof$(X, f) = y$

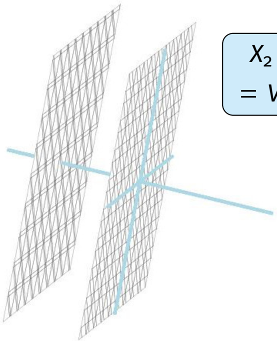
$X_1 = \overline{X \setminus V(y)}$
$= V(x, z)$

remove$(X, V(x, z))$

$X_2 = X \setminus V(x)$
$= V(y, z) \setminus V(x)$

$X_3 = X \setminus V(z)$
$= V(x, y) \setminus V(z)$

intersect(X_3, x)
returns X_3



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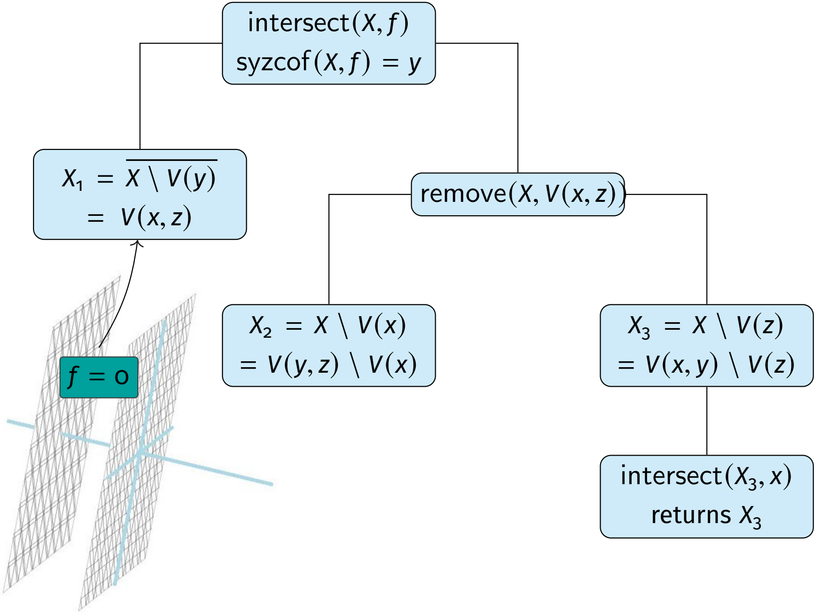

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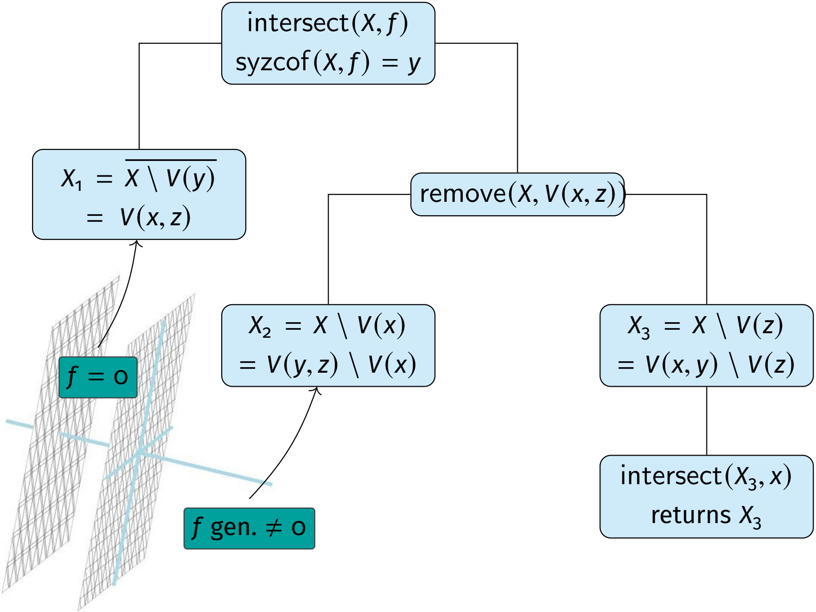

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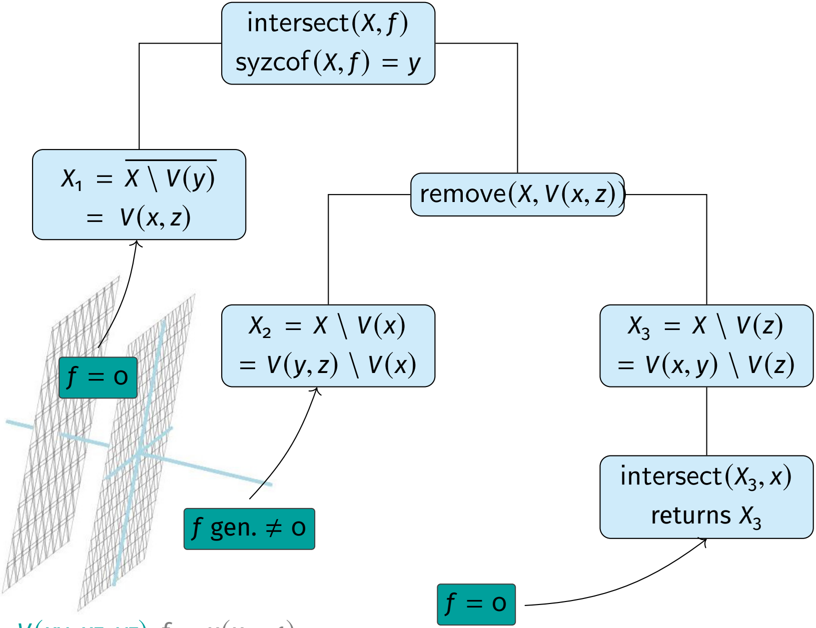


$X = V(xy, xz, yz), f = x(x-1)$

intersect(X, f)

syzcof$(X, f) = y$

$X_1 = \overline{X \setminus V(y)}$

$= V(x, z)$

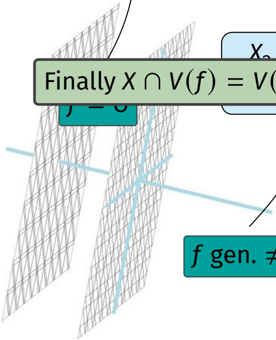
remove$(X, V(x, z))$

$X_2 = X \setminus V(x)$

$X_3 = X \setminus V(z)$

(z)

Finally $X \cap V(f) = V(x, z) \sqcup V(x, y) \setminus V(z) \sqcup V(y, z, x - 1) \setminus V(x)$



f gen. $\neq 0$

intersect(X_3, x)

returns X_3

$f = 0$

$X = V(xy, xz, yz), f = x(x - 1)$

“Naive” data structure:

- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbb{K}[\mathbf{x}]$ finite, $h \in \mathbb{K}[\mathbf{x}]$).
- Gröbner basis of $I := (\langle F \rangle : g^{\infty})$ (satisfies $\sqrt{I} = I(X)$).

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$X \cap V(f)$ *already equidimensional* if

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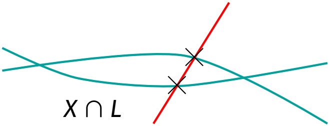

$X \cap V(f)$ *already equidimensional* if

$f|_X = 0$ or $f \neq 0$ generically on $X \Leftrightarrow$

this happens at randomly selected

points on X

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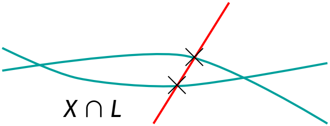

L general affine subspace of complementary dimension

$X \cap V(f)$ *already equidimensional* if

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$f\|_X = 0$ or gen. $\neq 0$ iff $f\|_{X \cap L} = 0$ or gen. $\neq 0$

$\Longrightarrow$ Avoid knowing a GB of I!

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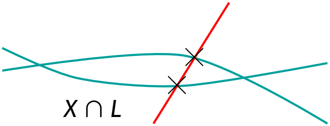

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$X \cap L$ called *lifting fiber* or *witness set*
(Lecerf 2003; Sommese, Verschelde, and Wampler 2005)

	nb. comps.	Our Algorithm	●	●	●	●
Cyclic(8)	6	381.20	∞	∞	∞	∞
KdV		∞	352.61	∞	∞	7108.56
Leykin-1	13	2.63	4.38	640.52	∞	1.37
C1	4	128.84	∞	∞	∞	∞
PS(12)	2	51.21	∞	∞	∞	2060.03
Sing(10)	2	0.36	∞	∞	∞	∞
SOS(6,4)	2	4.83	∞	∞	∞	∞
SOS(6,5)	2	13.72	∞	∞	∞	∞
Steiner	2	869.79	∞	∞	∞	∞
sys2161	33	7.54	28.76	∞	∞	7.57
sys2880	50	4.27	144.30	1.80	3.44	4.00

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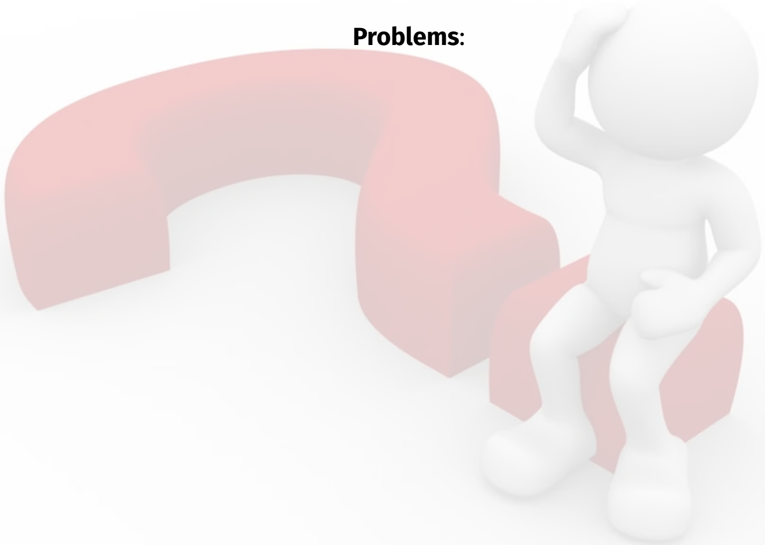
● Homological method in Oscar presented in (Eisenbud, Huneke, and W. Vasconcelos 1992).
● Prime decomposition in Magma.

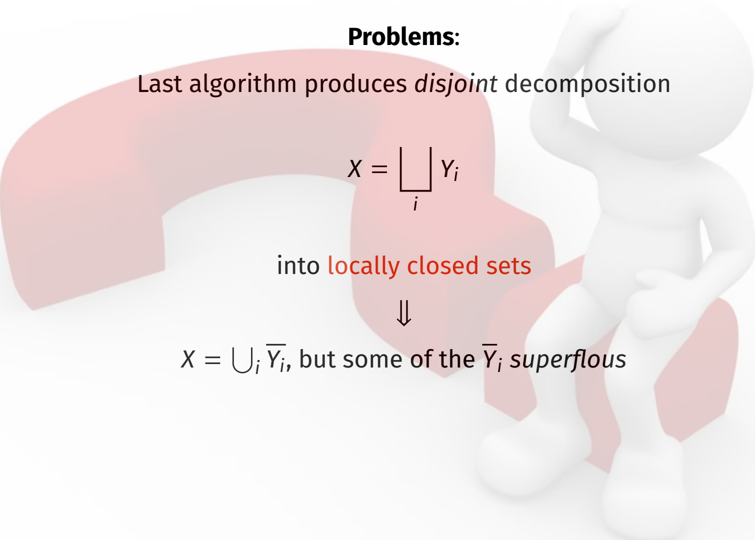
Reference: Eder, Lairez, Mohr, and Safey El Din 2023,
A Direttissimo Algorithm for Equidimensional Decomposition

A Non-Incremental Algorithm

Algorithm 3

Problems:





Problems:

Last algorithm produces *disjoint* decomposition

$$X = \bigsqcup_i Y_i$$

into locally closed sets

$$\Downarrow$$

$X = \bigcup_i \overline{Y_i}$, but some of the $\overline{Y}_i$ *superflous*

Problems:

Incremental structure $\rightsquigarrow$ computations tend to be hard "in the middle"

$\cong$

complexity bounds for Gröbner bases

(Bardet, J.-C. Faugère, and Salvy 2015; Hashemi and Seiler 2017)

Definition

$f_1, \ldots, f_c$ regular sequence iff for every i and every g with $gf_i \in \langle f_1, \ldots, f_{i-1} \rangle$ we have $g \in \langle f_1, \ldots, f_{i-1} \rangle$.

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$$\Downarrow$$

Non-incremental regularity criterion

$$X = V(f_1, \ldots, f_c)$$

This time suppose we have function

$$\text{syzcof}(X)\text{: returns } g, i \text{ s.t. } gf_i \in \langle f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_c \rangle, g \notin I(X)$$

$\rightsquigarrow$ done with signature-based computations

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Termination: when codim bound is matched

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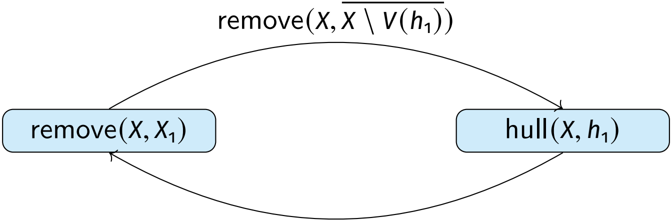

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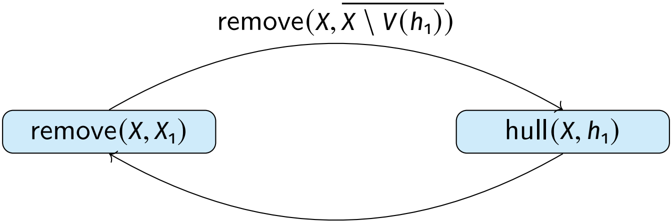

Output has no redundancies

	Non-Incremental Algorithm	Incremental Algorithm
Cyclic(8)	∞	381.2
PS(14)	248.2	3128.3
PS(16)	13666.2	∞
SOS(7,5)	112.2	∞
Steiner	404.9	870.0
sys2161	26.2	7.54
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Computed in characteristic 65521 on Intel Xeon Gold 6244.
Reference: Preprint to come!

Other strategy: (Berthomieu, Mohr 2024), Computing Generic Fibers of Polynomial Ideals using FGLM and Hensel Lifting

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Thank you!