# Random growth in half-space and solutions of integrable equations

Harriet Walsh (Université d'Angers)

Joint work with Mattia Cafasso, Alessandra Occelli and Daniel Ofner

Joint conference DRN + EFI

Anglet, 10 June 2024

# Plan

1. Polynuclear growth and a connection with the Painlevé II equation in a classical case

2. Variations: half-space, external sources

3. Polynuclear growth in half-space with external sources

4. Ideas of proof: Riemann–Hilbert problems

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# The polynuclear growth (PNG) model

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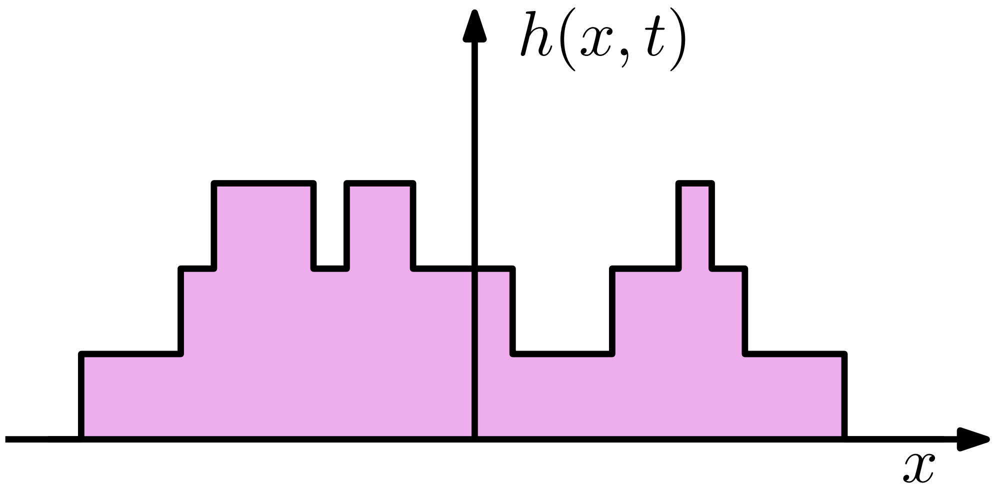

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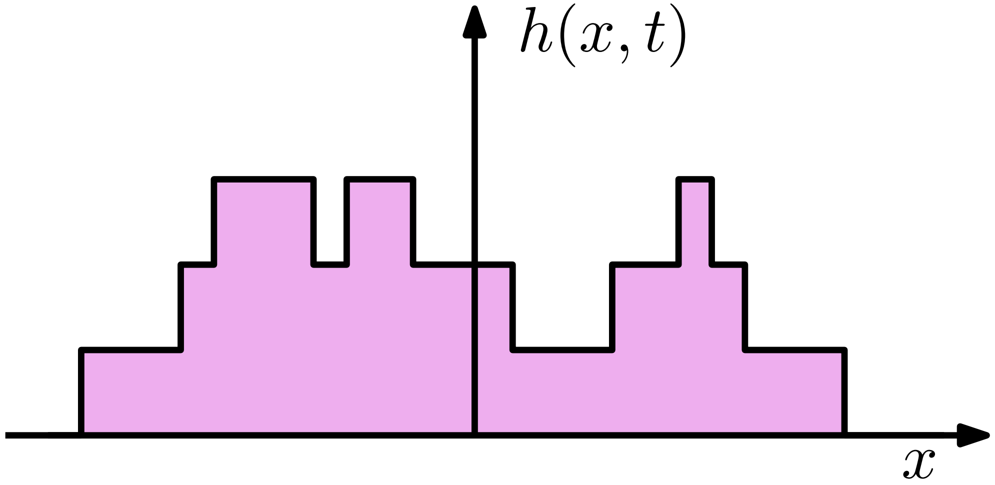


- At $t = 0$, $h(x,0) = 0$ for all $x$.
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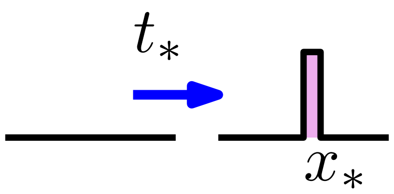
$$h(x_*, t_* + \delta) = h(x_*, t_*) + 1.$$



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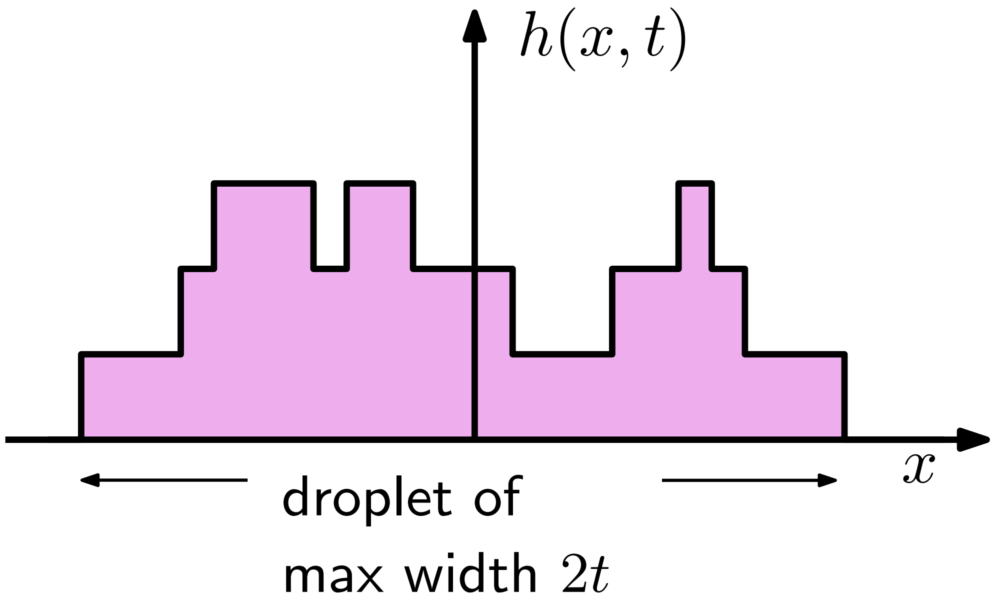


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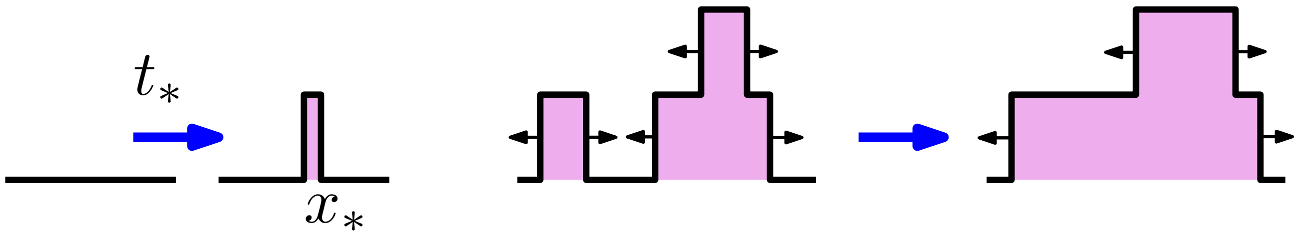

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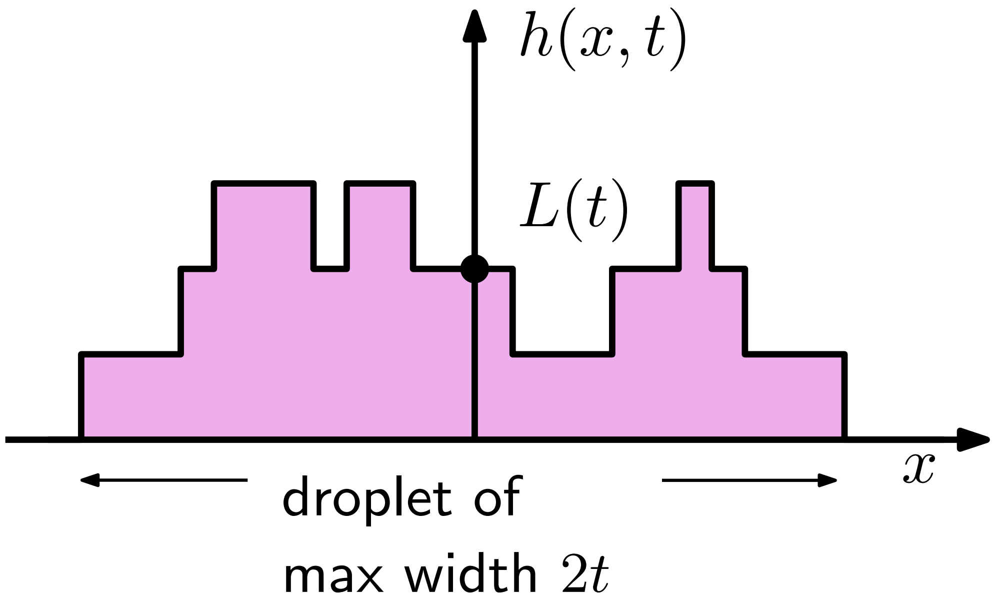


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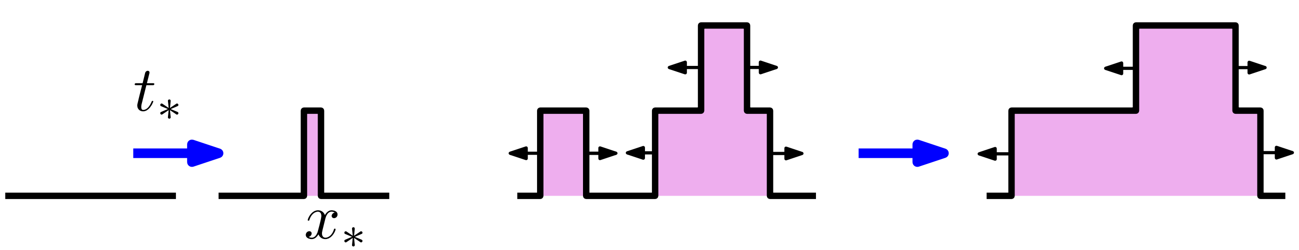


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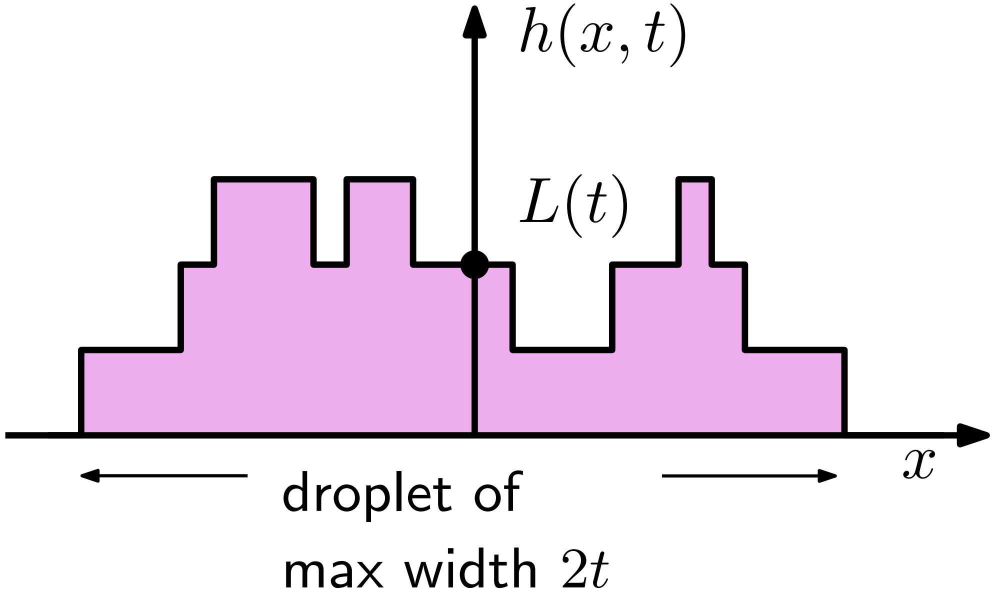


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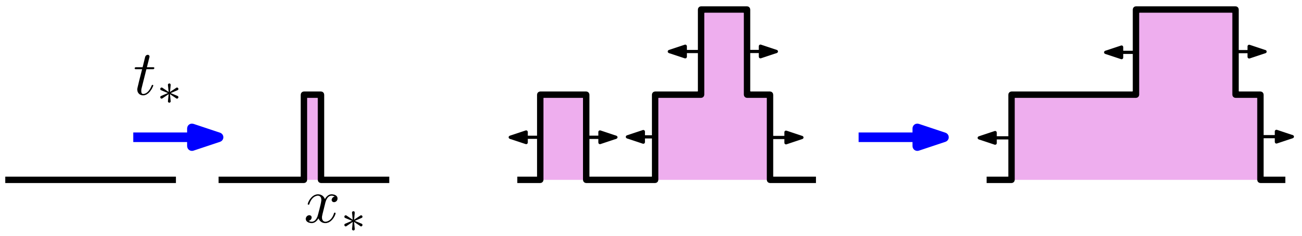


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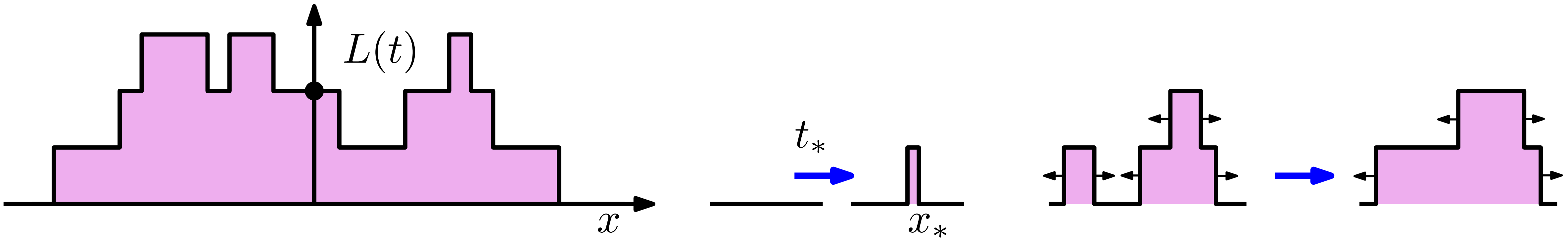
**Universality**: PNG has characteristics of **KPZ random growth**: local height, mechanism to fill gaps in, the right scaling exponents... (Prähofer & Spohn '00)

**Integrability**: We have exact expressions $\mathbb{P}[L(t) < \ell]$ and other marginal distributions in the model.

# PNG height and longest increasing subsequences

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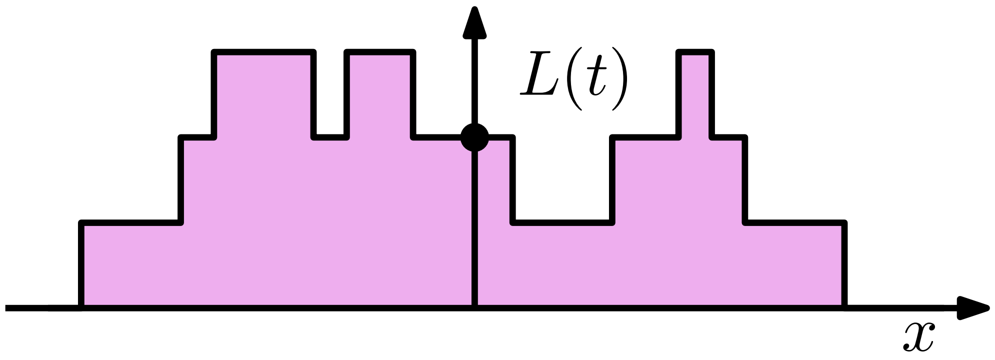


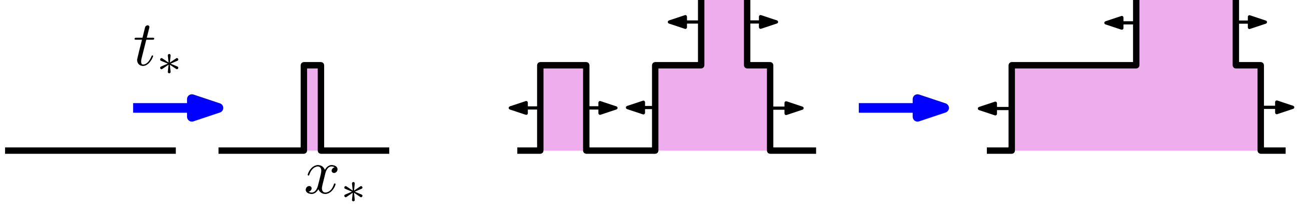


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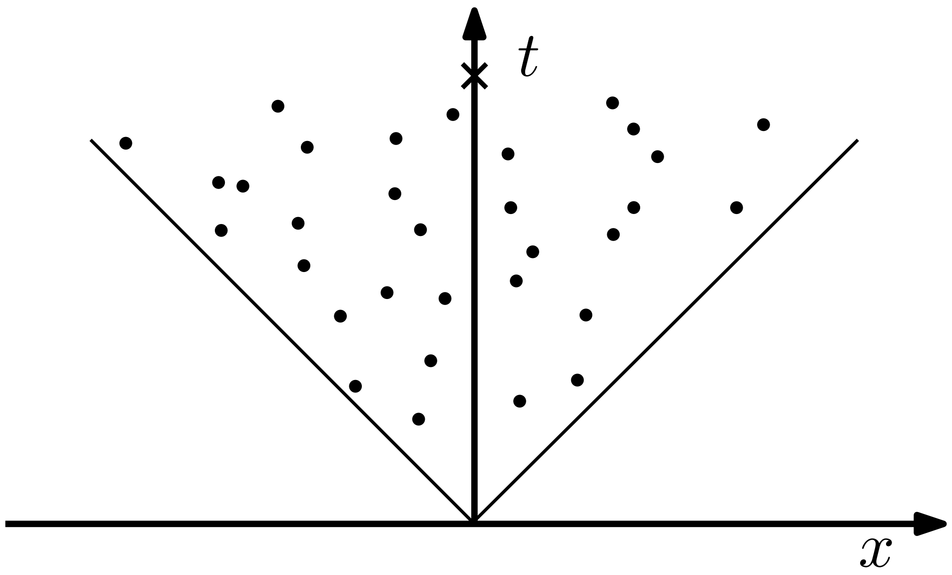

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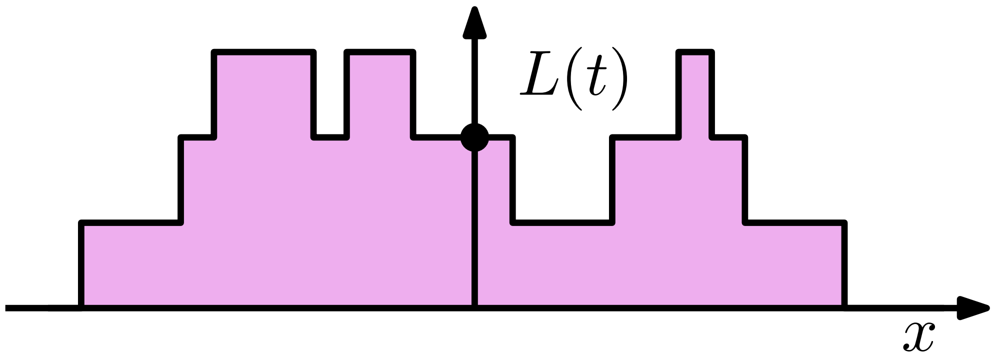


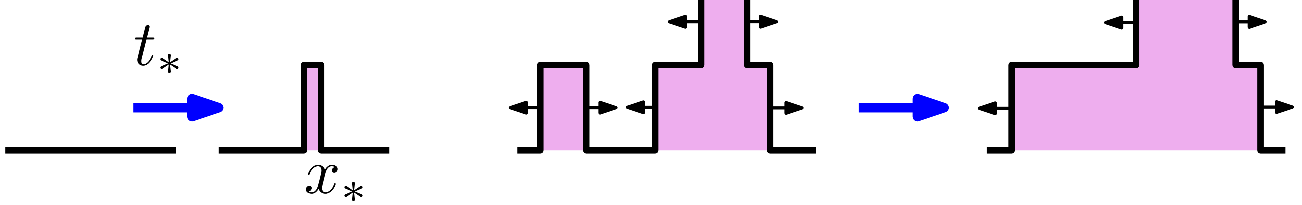


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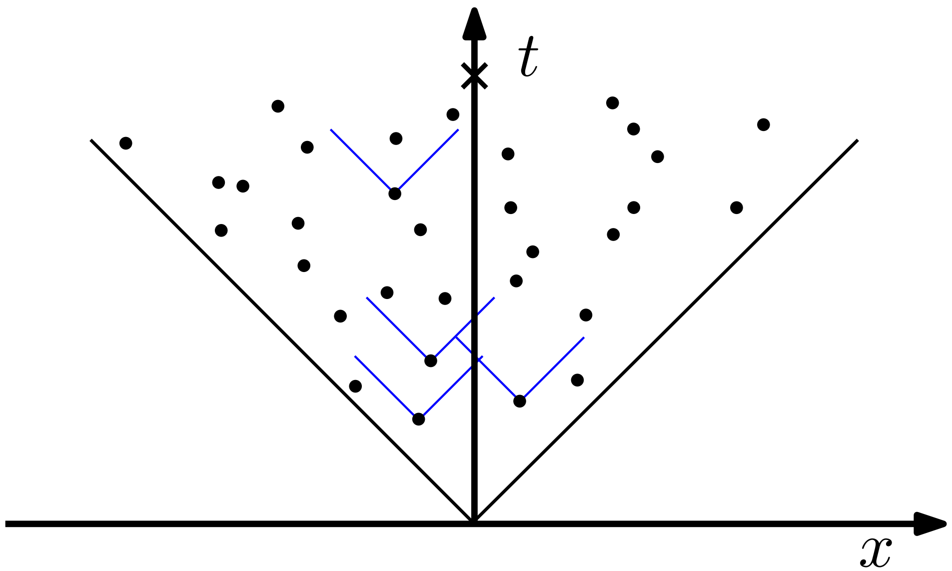

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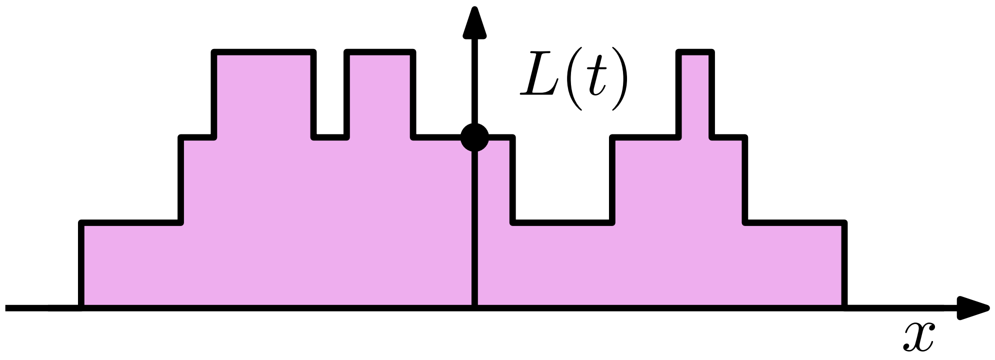


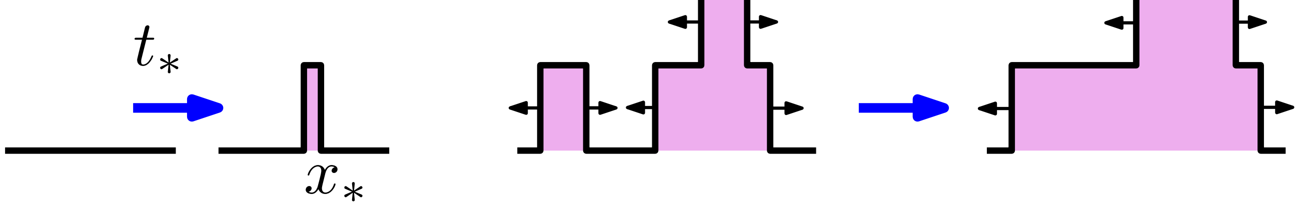


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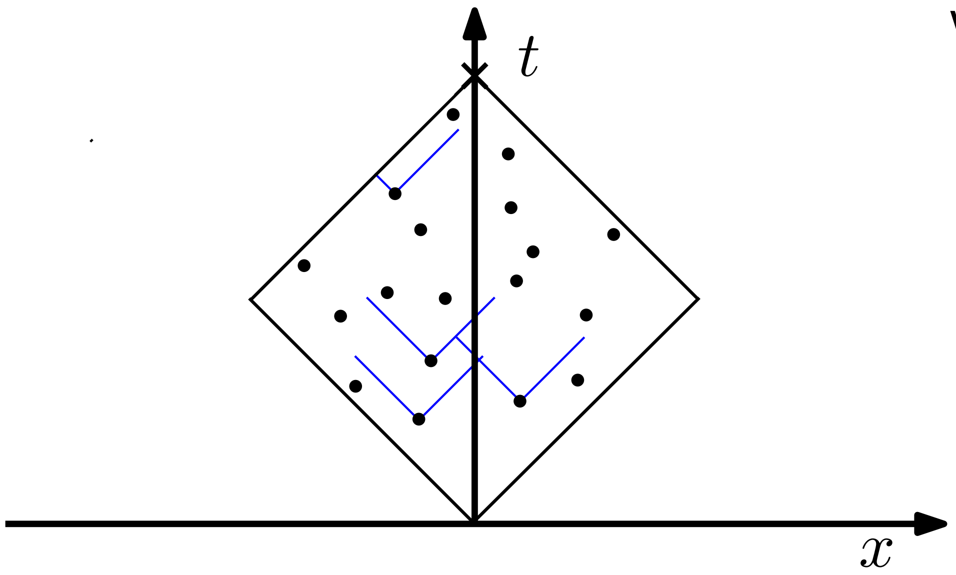

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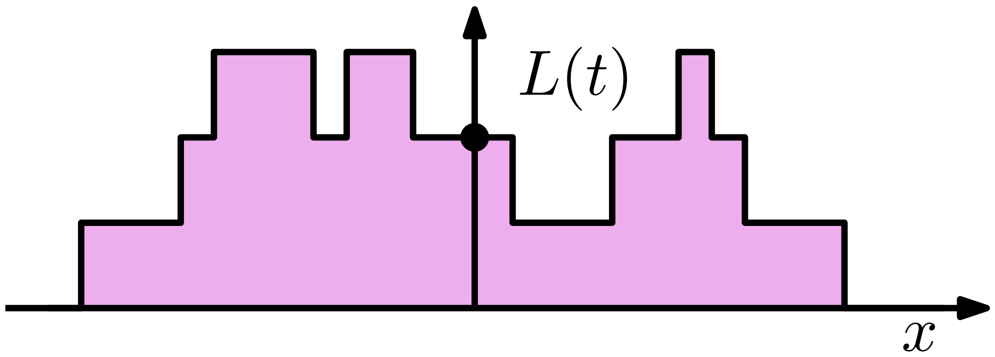


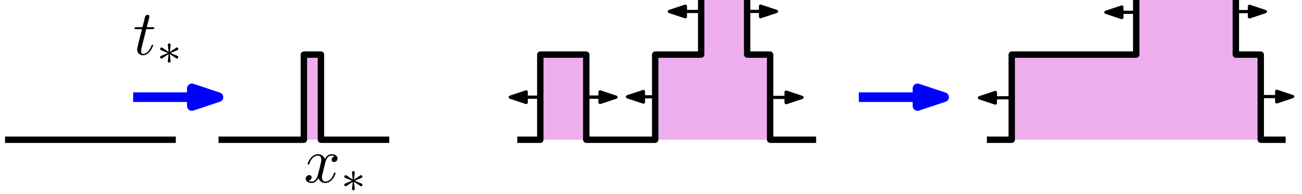


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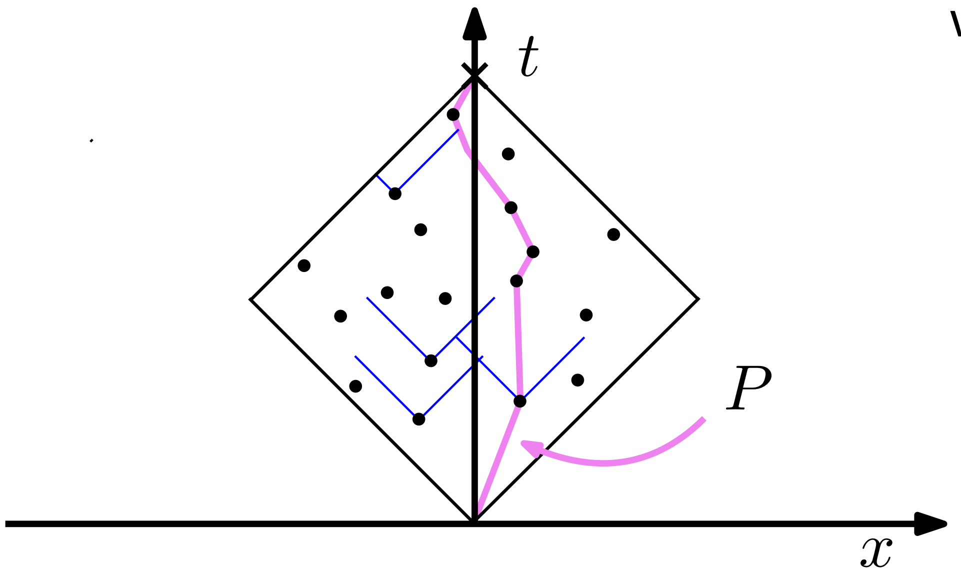

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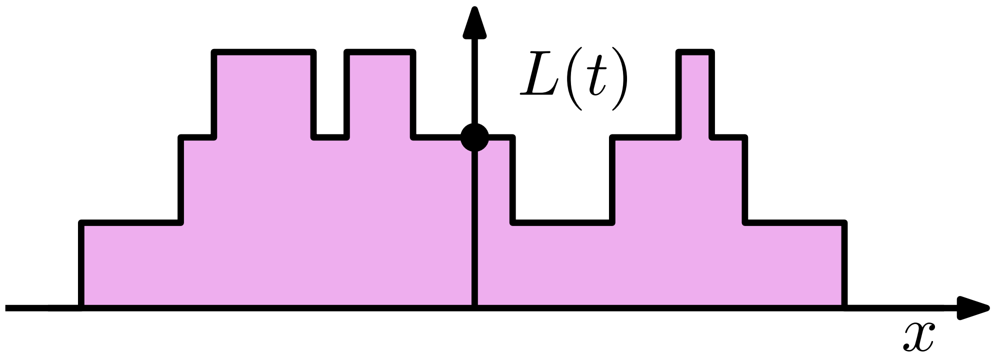


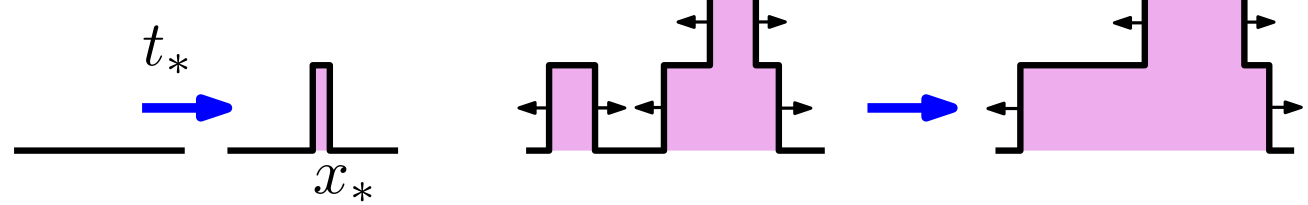


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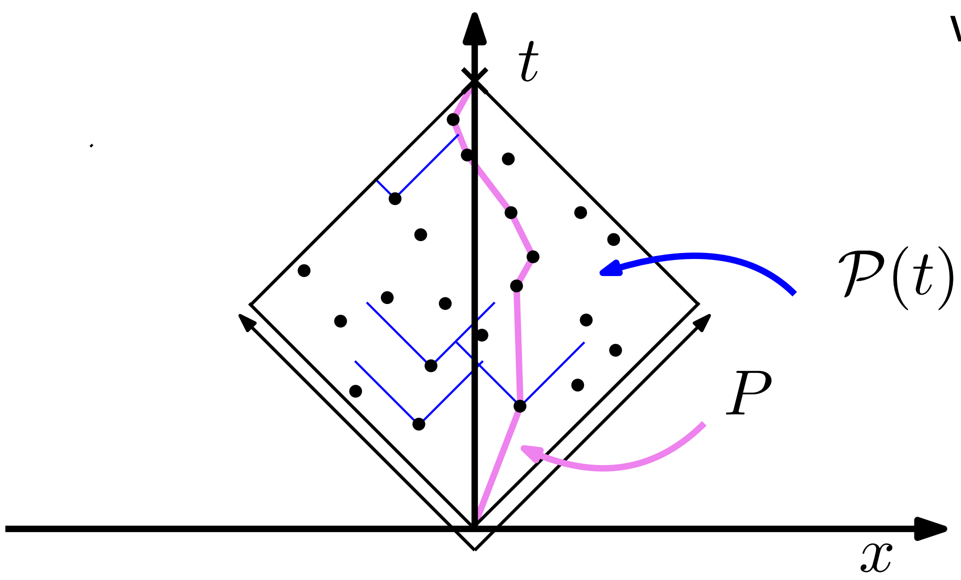


We have

$$L(t) = \max_{\substack{P:(0,0)\to(t,t) \\ P \text{ increasing}}} \#(P \cap \mathcal{P}(t))$$

where $\mathcal{P}(t)$ is this set of nucleation points in coordinates $(w_*, z_*) = (t_* + x_*, t_* - x_*)$.

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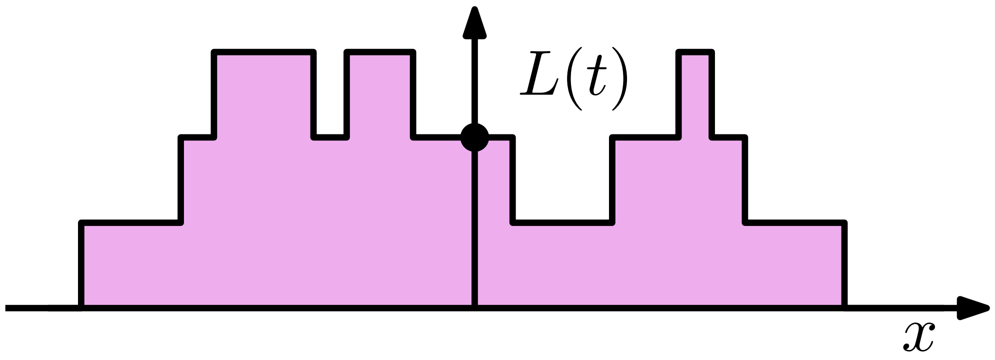


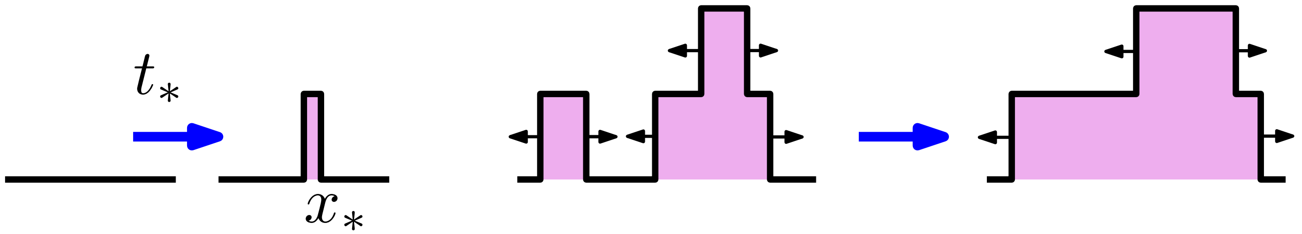


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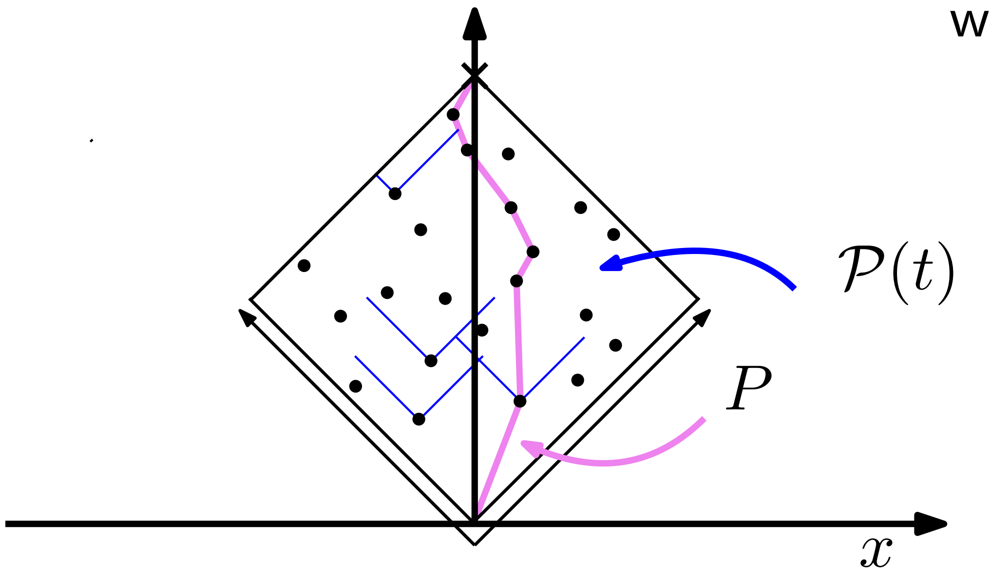


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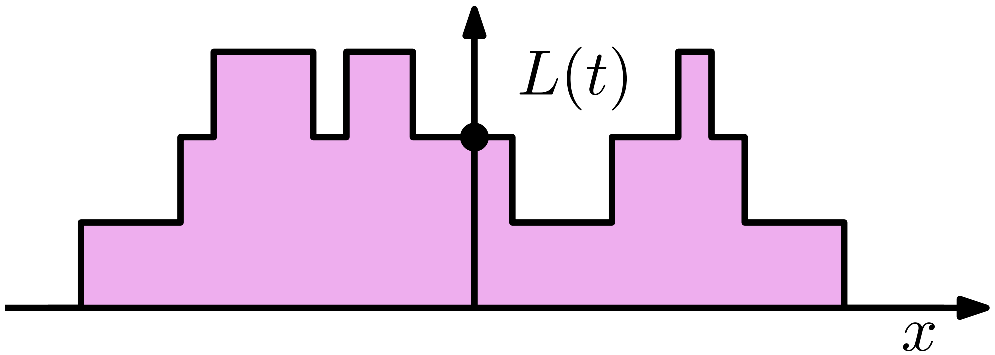


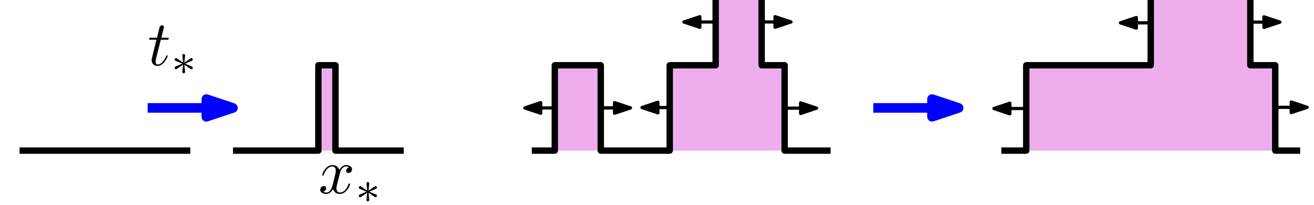


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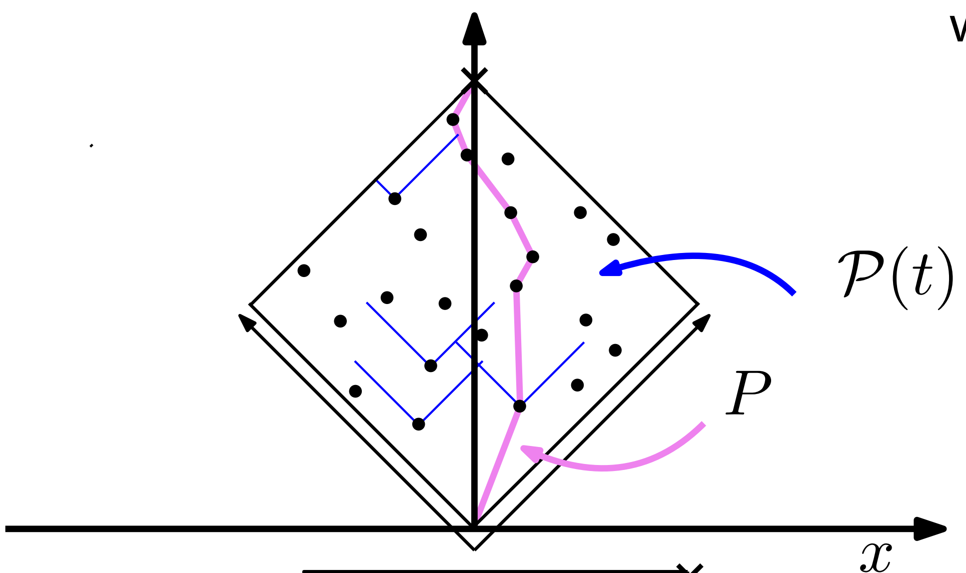


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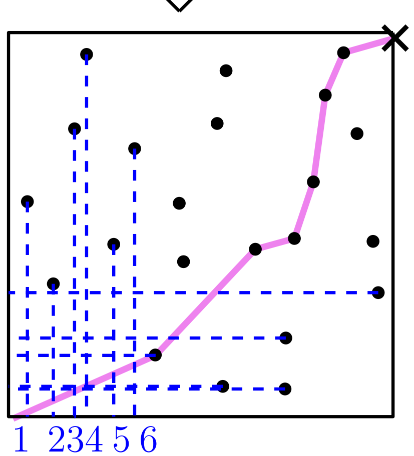
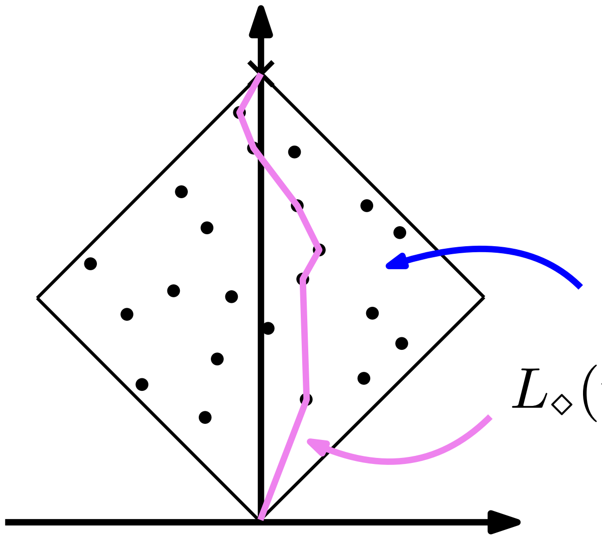



Ordering the coordinates of the points in $\mathcal{P}(t)$, we have $L(t) = \max|\text{inc. subseq.}(\sigma)|$ for some random permutation $\sigma$.

The Robinson–Schensted bijection associates each $\sigma \in S_n$ with a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \ldots)$ of $n$ along with two SYT of shape $\lambda$, such that $\max|\text{i. s.}(\sigma)| = \lambda_1$.

# PNG fluctuations, random matrix distributions and Painlevé II



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$$L_{\diamond}(t) = \max_{P:(0,0)\nearrow(1,1)} \#P \cap \mathcal{P}(t)$$

Consider "classical" droplet PNG with $\mathcal{P}(t)$ made up of $N \sim \mathrm{Poi}(t^2)$ points sampled independently and uniformly inside $(0,1) \times (0,1)$.

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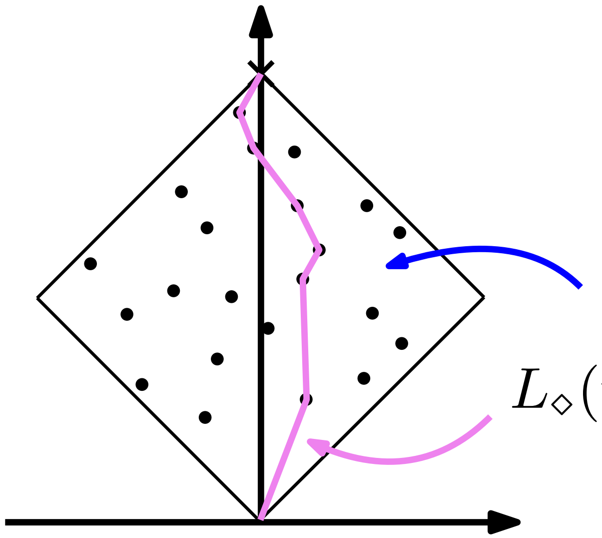

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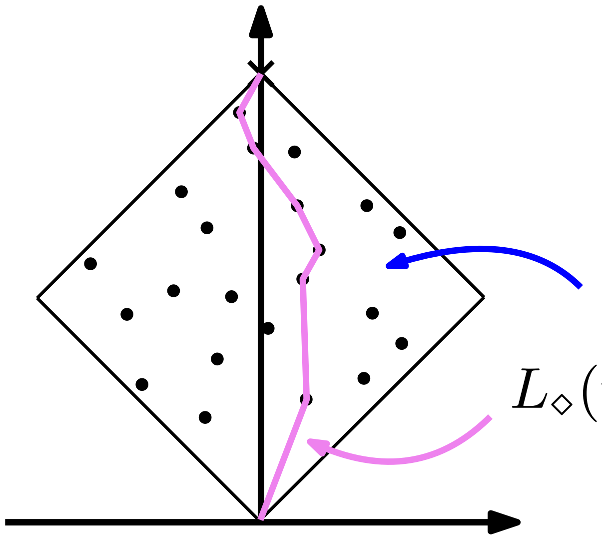

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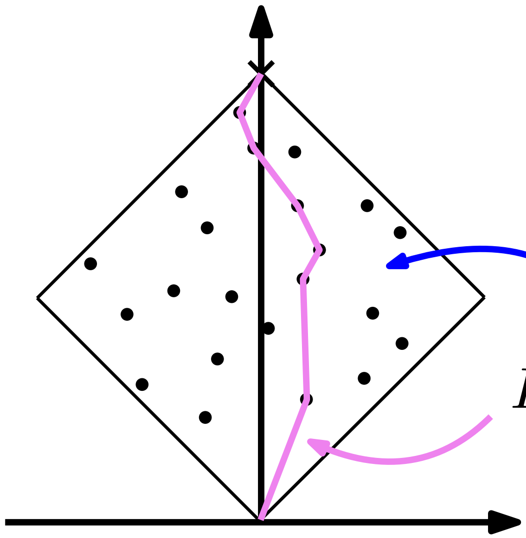

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$F_{\mathrm{GUE}}(s)$ is the limiting distribution of the fluctuations in the largest eigenvalue of a random Hermitian matrix in the Gaussian unitary ensemble.

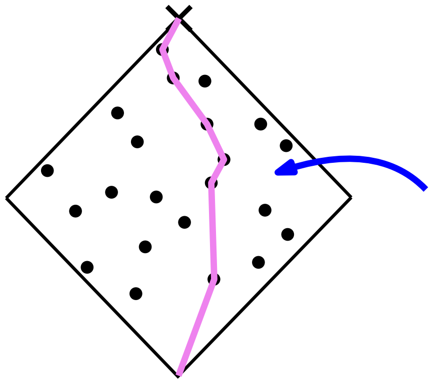
It can be written

$$F_{\mathrm{GUE}}(s) = \exp \int_s^\infty v(x)dx$$

where $v(x) = \int_{-\infty}^x u(y)^2 dy$ in terms of a solution $u$ of the **Painlevé II equation**

$$u''(x) = 2u(x)^3 + xu(x) \qquad \text{with } u(x) \sim -\mathrm{Ai}(x) \text{ as } x \to \infty.$$

# PNG fluctuations via Fredholm determinants



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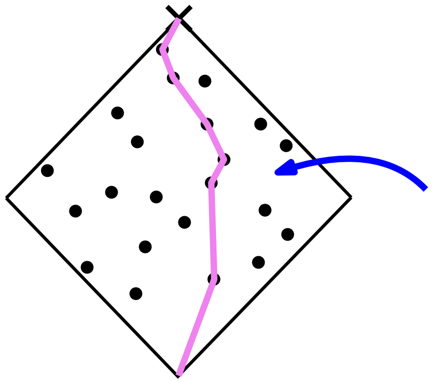

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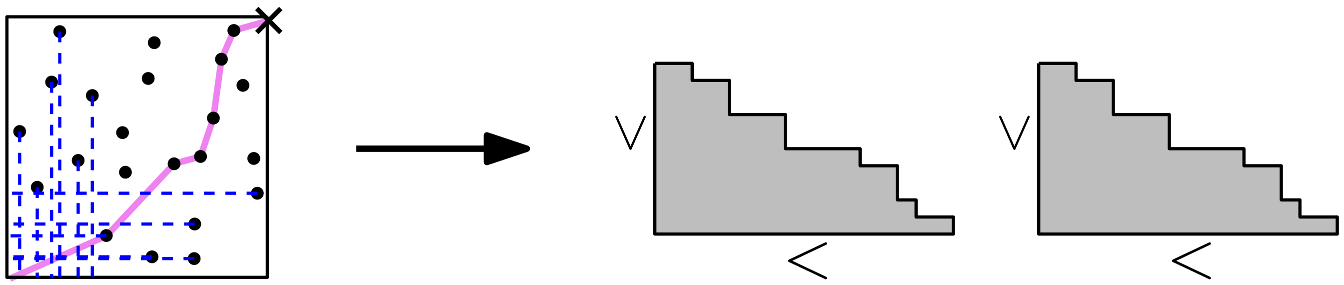
**One route to a proof**: We have $L_\diamond(t) = \max|\mathrm{i.s.}(\sigma)|$ where $\sigma$ is a uniform random permutation of $(1, \ldots, N)$ with $N \sim \mathrm{Poi}(t^2)$. By the Robinson–Schensted bijection, $L(t) \sim \lambda_1$ where $\lambda$ is a random partition of $N$ with $\mathbb{P}(\lambda) \propto \#\mathrm{SYT}(\lambda)^2$.



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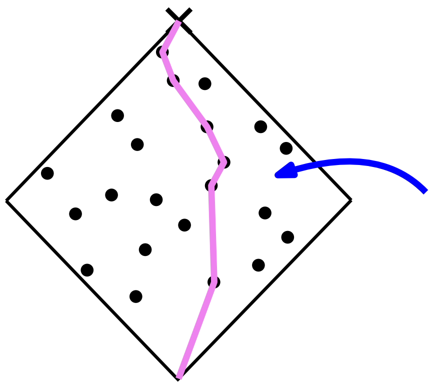

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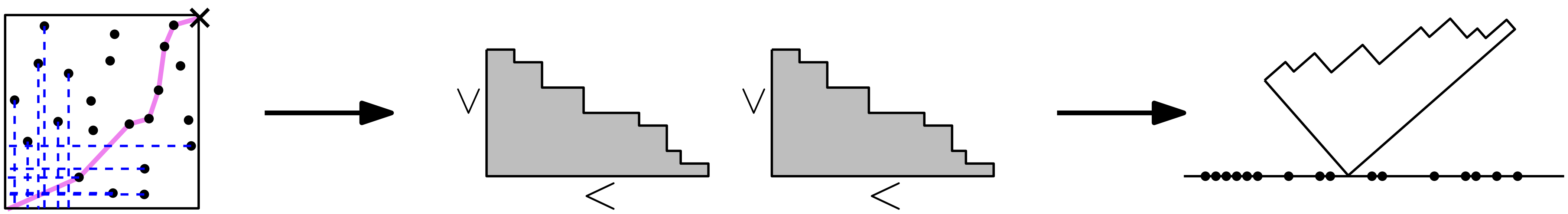

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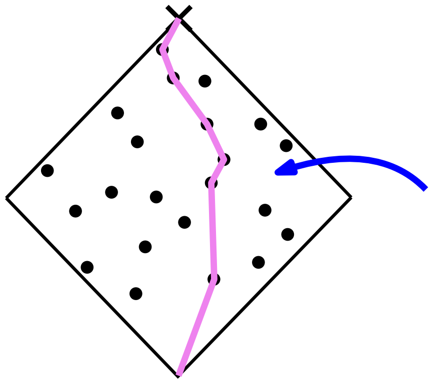

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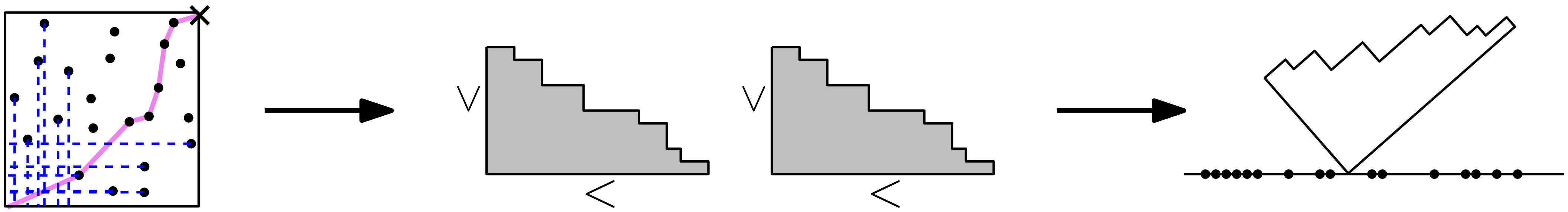

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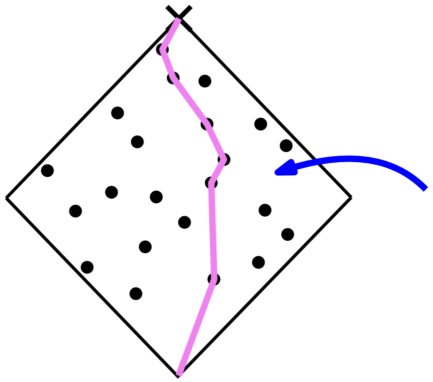
$$\xrightarrow[t\to\infty \text{ saddle point}]{k_i \sim 2t + x_i t^{1/3}} \frac{1}{(2\pi i)^2} \iint_{\Gamma'} \frac{e^{\frac{\zeta^3}{3} - x_i\zeta}}{e^{\frac{\omega^3}{3} - x_j\omega}} \frac{d\zeta d\omega}{\zeta - \omega} = \mathcal{A}(x_i, x_j).$$

# PNG fluctuations via Fredholm determinants

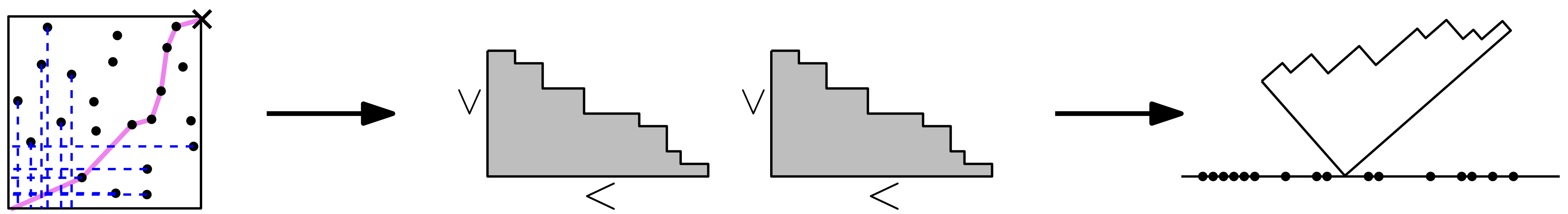


$\mathcal{P}(t) = \mathrm{Poi}(t^2, D_1)$

- **Baik–Deift–Johansson '98**:

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{L_\diamond(t) - 2t}{t^{1/3}} < s\right] = F_{\mathrm{GUE}}(s)$$

**One route to a proof**: We have $L_\diamond(t) = \max|\mathrm{i.s.}(\sigma)|$ where $\sigma$ is a uniform random permutation of $(1, \ldots, N)$ with $N \sim \mathrm{Poi}(t^2)$. By the Robinson–Schensted bijection, $L(t) \sim \lambda_1$ where $\lambda$ is a random partition of $N$ with $\mathbb{P}(\lambda) \propto \#\mathrm{SYT}(\lambda)^2$.



We have

$$\mathbb{P}(L_\diamond(t) < \ell) = \det(1 - K)_{l^2(\ell,\ell+1,\ldots)} \xrightarrow[t\to\infty]{\ell\sim 2t+st^{1/3}} \det(1 - \mathcal{A})_{L^2(s,\infty)} = F_{\mathrm{GUE}}(s)$$

where $\quad K(k_i, k_j) = \dfrac{1}{(2\pi i)^2} \displaystyle\iint_\Gamma e^{tS(z)-tS(w)+\ldots} \frac{dz dw}{z - w}$

$$\xrightarrow[t\to\infty \text{ saddle point}]{k_i\sim 2t+x_i t^{1/3}} \frac{1}{(2\pi i)^2} \iint_{\Gamma'} \frac{e^{\frac{\zeta^3}{3} - x_i\zeta}}{e^{\frac{\omega^3}{3} - x_j\omega}} \frac{d\zeta d\omega}{\zeta - \omega} = \mathcal{A}(x_i, x_j).$$

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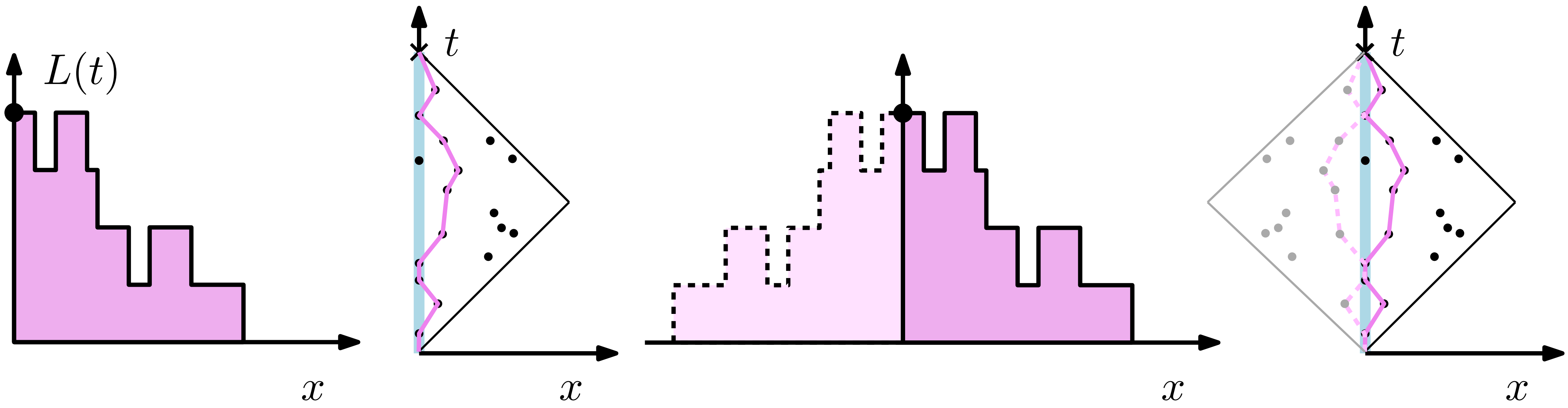
# PNG fluctuations in half-space

In **half-space** PNG, we take nucleation points only at $x_* \geq 0$ (or a symmetric set).



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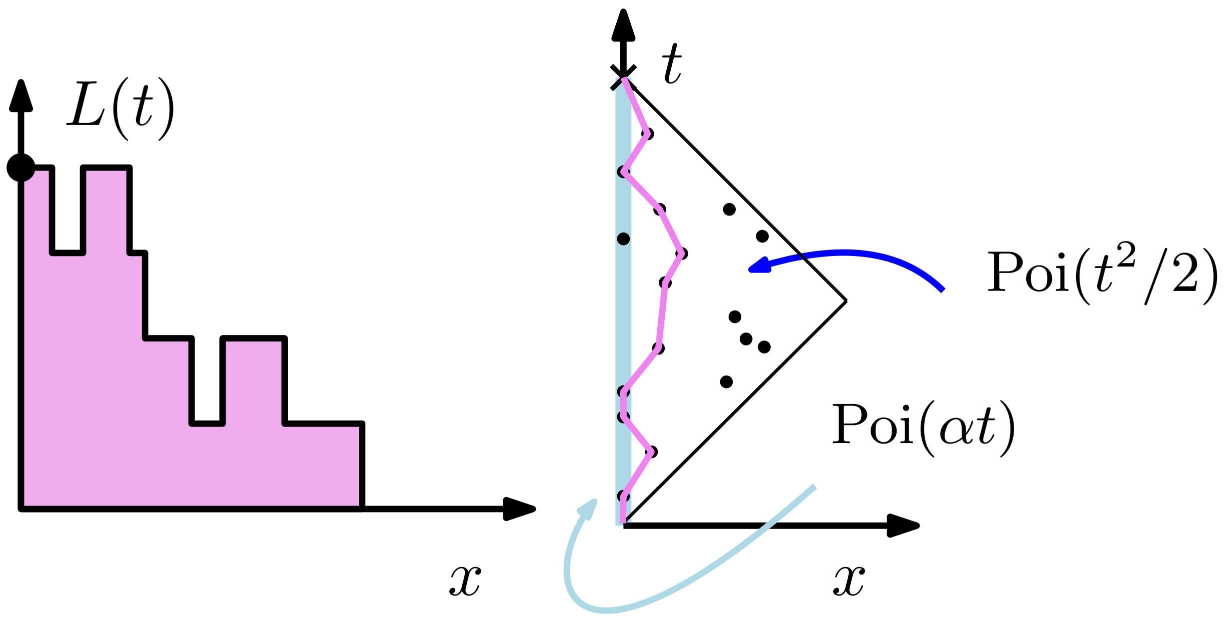


Take $\mathcal{P}(t)$ composed of $\mathrm{Poi}(t^2/2)$ independent points on $\{(x,y)|0 < y < x < 1\}$ and $\mathrm{Poi}(\alpha t)$ independent points on $\{(x,x)|0 < x < 1\}$.

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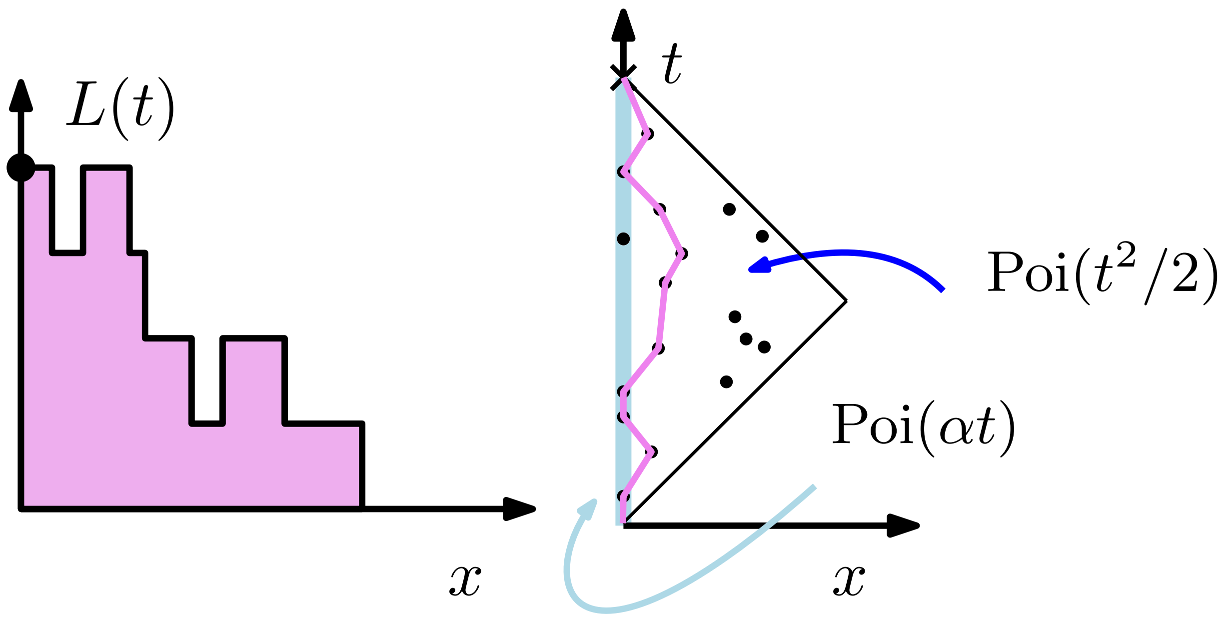


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- If $0 \leq \alpha \leq 1$,
$$\frac{L_{\triangleright}(t)}{t} \xrightarrow{p} 2 \text{ as } t \to \infty.$$

- **Baik–Rains '01**:
$$\lim_{t\to\infty} \mathbb{P}\left[\frac{L_{\triangleright}(t) - 2t}{t^{1/3}} < s\right] = \begin{cases} F_{\mathrm{GSE}}(s), & 0 \leq \alpha < 1 \\ F_{\mathrm{GOE}}(s), & \alpha = 1 \end{cases}$$

$F_{\mathrm{GOE}/\mathrm{GSE}}(s)$ gives the asymptotic fluctuations in the largest eigenvalue of a random symmetric/quaternionic matrix in the Gaussian orthogonal/symplectic ensemble, and

$$F_{\mathrm{GOE}}(s) = \exp \int_s^\infty \frac{v(x)+u(x)}{2} dx, \quad F_{\mathrm{GSE}}(s) = \frac{1}{2}\left[F_{\mathrm{GOE}}(s) + \exp \int_s^\infty \frac{v(x)-u(x)}{2} dx\right].$$

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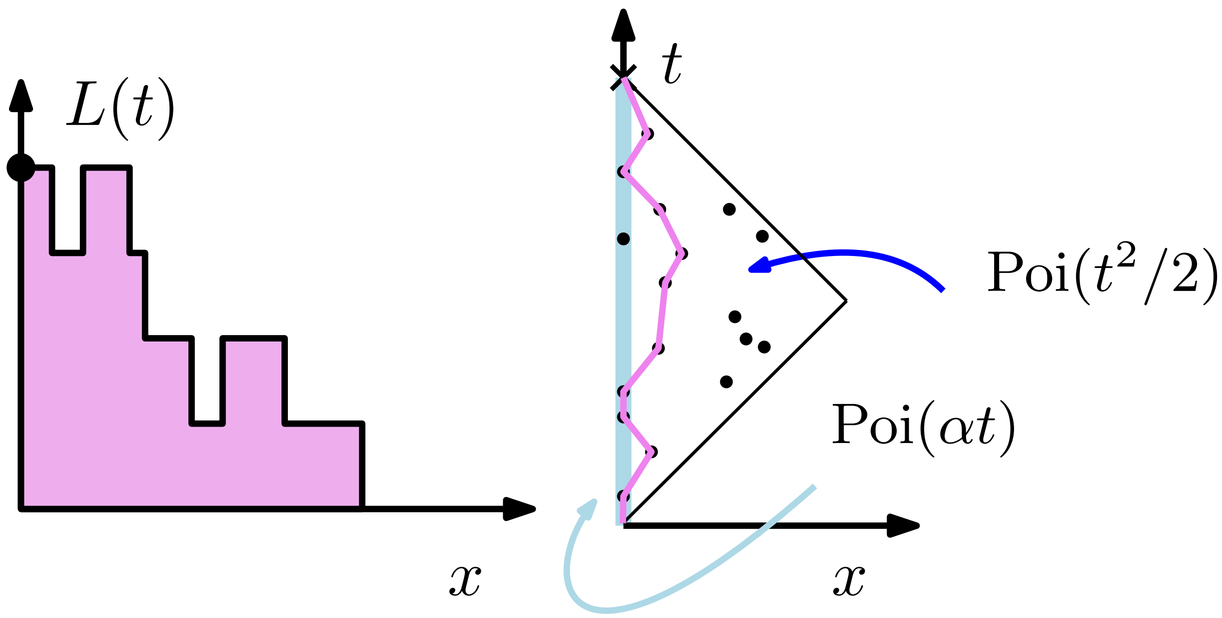


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$*$ For $\alpha > 1$, $L_{\triangleright}(t) \sim (1+\alpha)t$ with Gaussian fluctuations.

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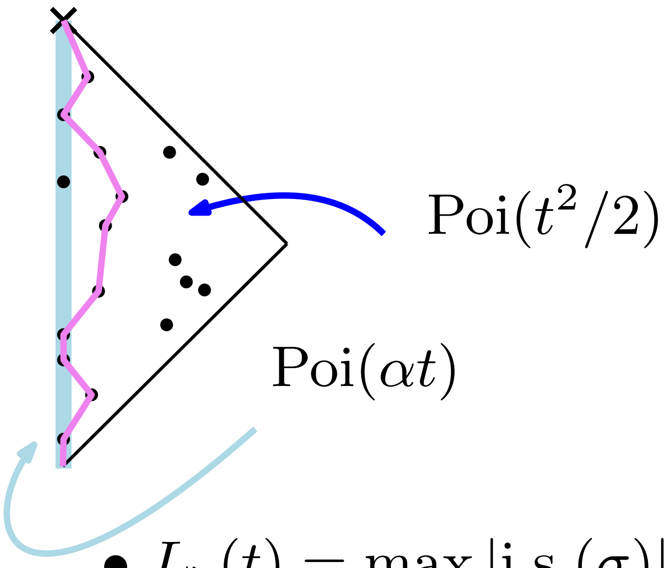


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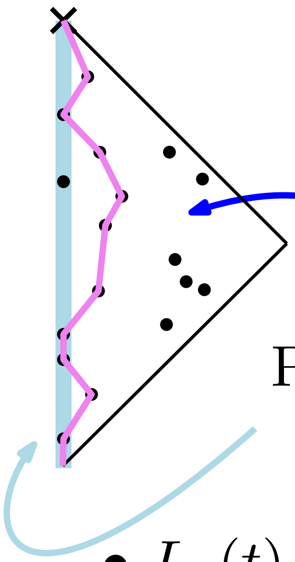
# PNG fluctuations in half-space

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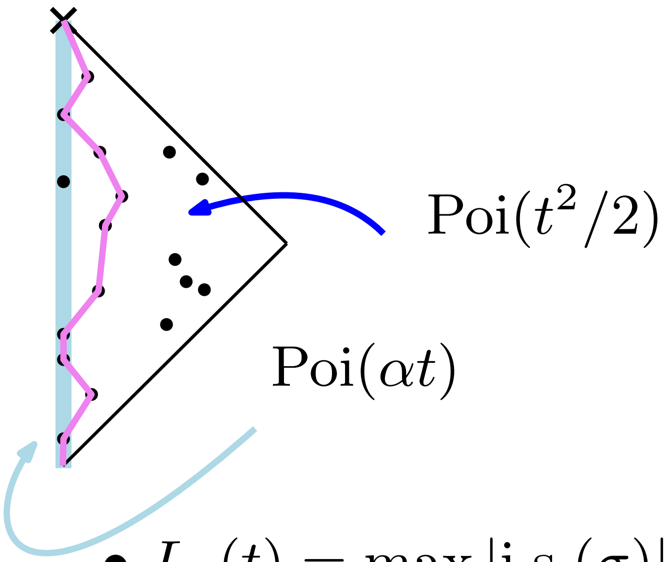


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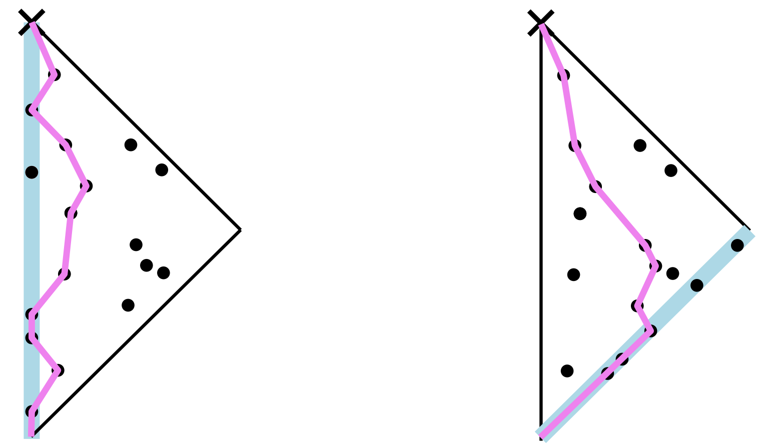
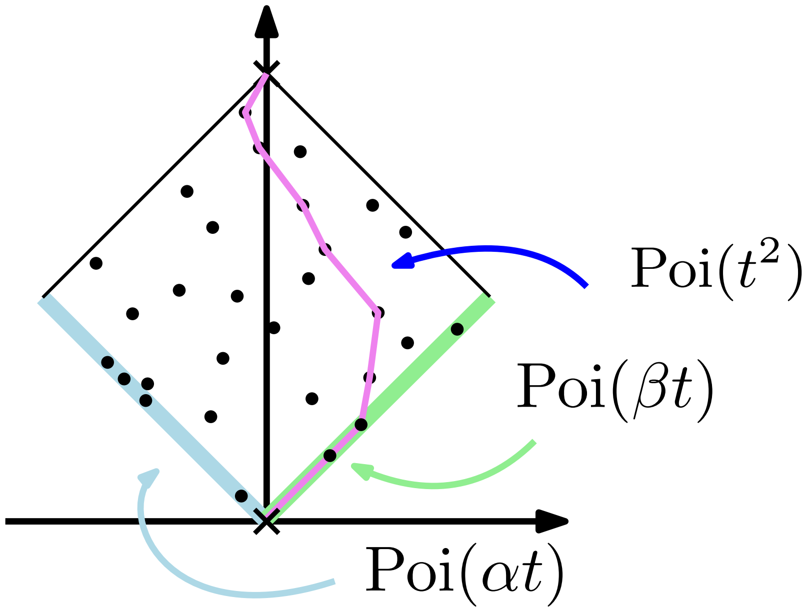
- Whereas $F_{\mathrm{GUE}}$ appears universally in random growth with **droplet** initial conditions, $F_{\mathrm{GOE}}$ appears with **flat** initial conditions. The $\alpha = 1$ case corresponds to a uniform involution.

An equivalence in law of $L_{\triangleright}(t)$:



# PNG fluctuations with external sources




In full-space PNG, different regimes are obtained by adding sources on the edges.

Take $\mathcal{P}(t)$ with $\mathrm{Poi}(t^2)$ independent points on $(0,1)\times(0,1)$, $\mathrm{Poi}(\alpha t)$ on $0\times(0,1)$ and $\mathrm{Poi}(\beta t)$ on $(0,1)\times 0$.

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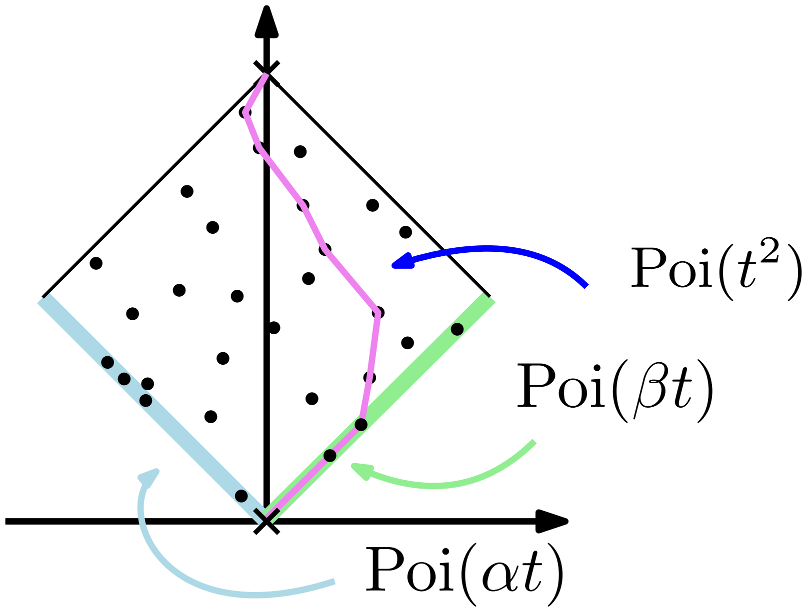


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- **Baik–Rains '00**:

$$\lim_{t\to\infty}\mathbb{P}\left[\frac{L_\diamond(t)-2t}{t^{1/3}}<s\right]=\begin{cases}F_{\mathrm{GUE}}(s), & 0\le\alpha,\beta<1.\\ F_{\mathrm{GOE}}(s)^2, & 0\le\alpha<1,\beta=1 \text{ or vice versa}\\ F_{\mathrm{BR}}(s), & \alpha=\beta=1\end{cases}$$

$F_{\mathrm{BR}}$ has not been observed in any matrix models. It can be written

$$F_{\mathrm{BR}}(s)=\left[1+\left(s+2u'(s)+2u(s)^2\right)v(s)\right]\exp\left[2\int_s^\infty u(x)dx\right]F_{\mathrm{GUE}}(s).$$

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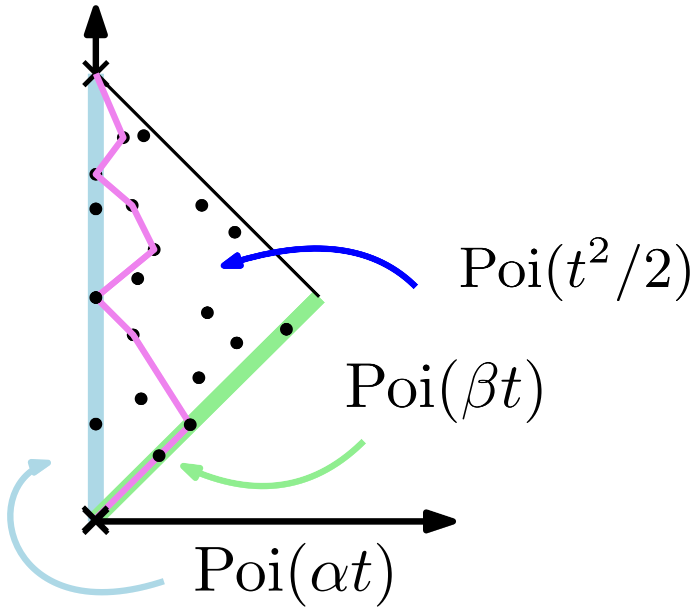
# PNG fluctuations in half-space with external sources




To find a third regime in half-space PNG, we add external sources.

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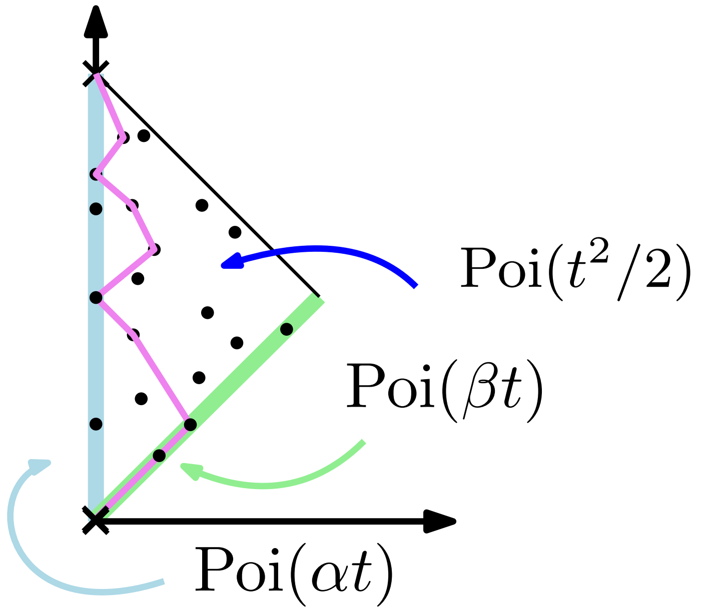


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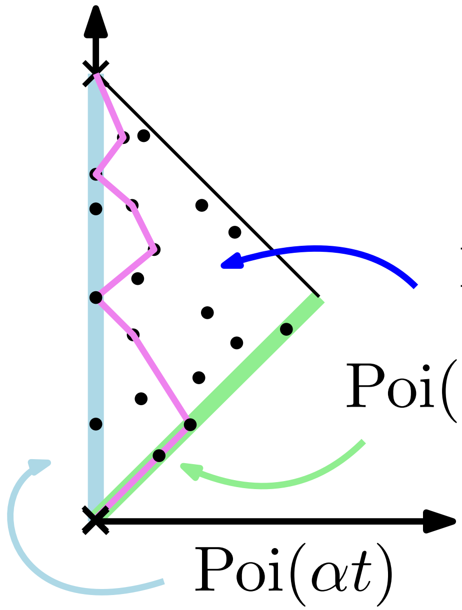


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- **Betea–Ferrari–Occelli '20**:
Limiting Fredholm pfaffian distribution for $L_{\triangleright}$.

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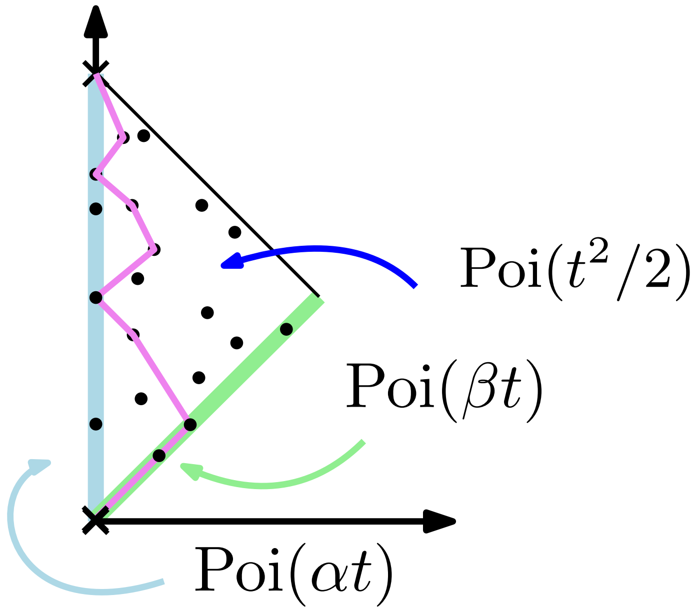


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- **Cafasso–Occelli–Ofner-W. '24+**:

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We find a half-space analogue of $F_{\mathrm{BR}}$ (not found elsewhere)

$$F_{\frac{1}{2}\mathrm{BR}}(s) = \left[1 + \left(s + 2u'(s) + 2u(s)^2\right)\frac{v(s)+u(s)}{2}\right] \exp\left[2\int_s^\infty u(x)dx\right] F_{\mathrm{GOE}}(s).$$

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# An expression for the Painlevé II solution

Can we use saddle point analysis? Is there an expression for the Painlevé II solution $u(s)$ (or $v(s)$) similar to

$$\mathcal{A}(x,y) = \frac{1}{(2\pi i)^2}\iint_\Gamma \frac{e^{\zeta^3 - x\zeta}}{e^{\omega^3 - y\omega}}\frac{d\zeta d\omega}{\zeta - \omega} \dots \text{ ?}$$

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Cauchy: $g(\zeta; x, y) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{e^{\zeta^3 - x\zeta}}{\zeta - \omega} d\omega,$

$$\implies g_+(\zeta; x, y) = g_-(\zeta; x, y) + e^{\zeta^3 - x\zeta} \quad \text{for } \zeta \in \Gamma.$$

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**Riemann–Hilbert problem (RHP):** Find a $2 \times 2$ complex matrix $m(z; s)$ such that

$$\begin{cases} m(z;s) \text{ is analytic in } z \in \mathbb{C} \setminus \mathbb{R} \\ m_+(z;s) = m_-(z;s) \begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3 + sz)} \\ e^{-2i(\frac{4}{3}z^3 + sz)} & 0 \end{pmatrix} \text{ for } z \in \mathbb{R} \\ m(z;s) = I + O(z^{-1}) \text{ as } z \to \infty. \end{cases}$$

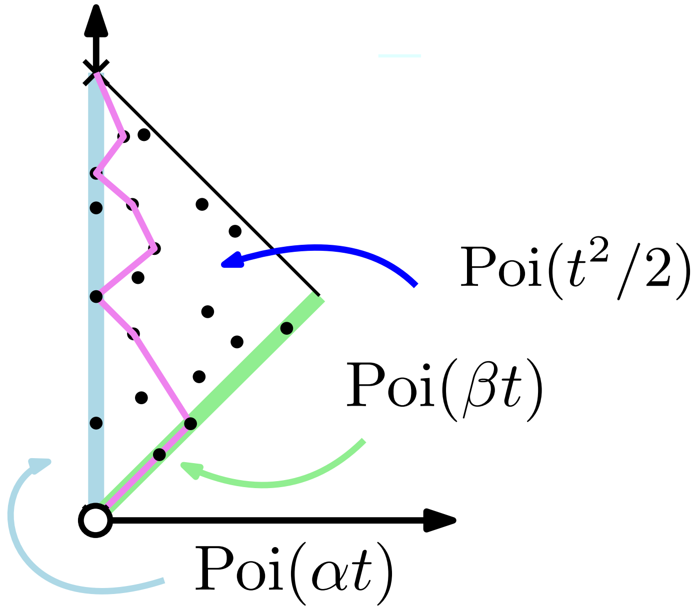
- $m(z; s)$ is unique
- expanding around $z = \infty$ as $m(z; s) = I + m_1(s) z^{-1} + O(z^{-2})$, we have

$$m_1(s) = \frac{i}{2} \begin{pmatrix} v(s) & -u(s) \\ u(s) & -v(s) \end{pmatrix}.$$

(Jimbo & Miwa; Flaschka & Newell '81)

# From PNG to RHP in half-space

To recover a limiting distribution in terms of $u, v$ we express $\mathbb{P}(L_{\triangleright}(t) < \ell)$ in terms of a RHP, at fixed $t, \ell$.



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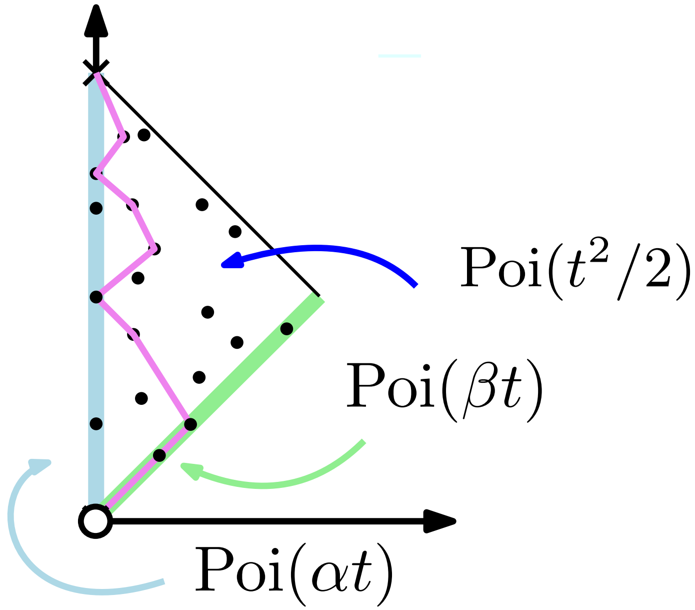


Set $L^{\mathrm{g}}_{\triangleright}(t) := L_{\triangleright}(t) + \mathrm{geom}(\alpha\beta)$. By Robinson–Schensted–Knuth, we have

$$\mathbb{P}(L^{\mathrm{g}}_{\triangleright}(t) < \ell) = \frac{1}{Z} \sum_{\lambda, \lambda_1 < \ell} \alpha^{\#\text{odd rows}(\lambda)} s_\lambda[\beta; t]$$

. $s_\lambda[\beta; t]$ an evaluated Schur function.

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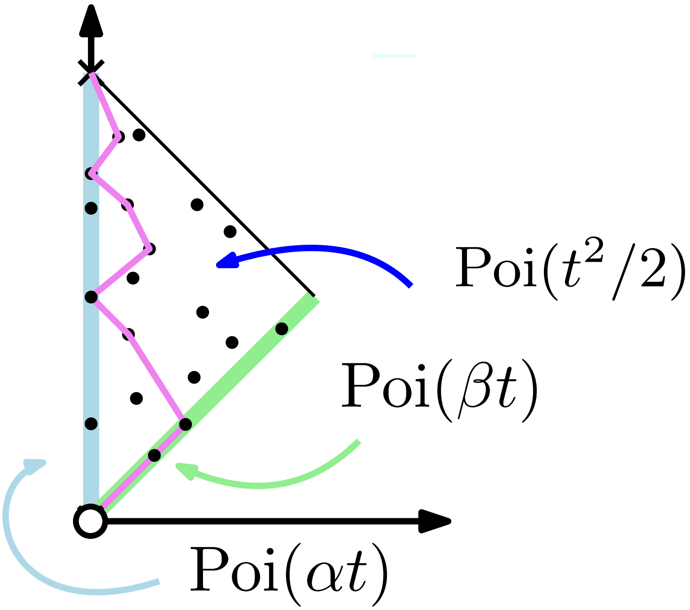


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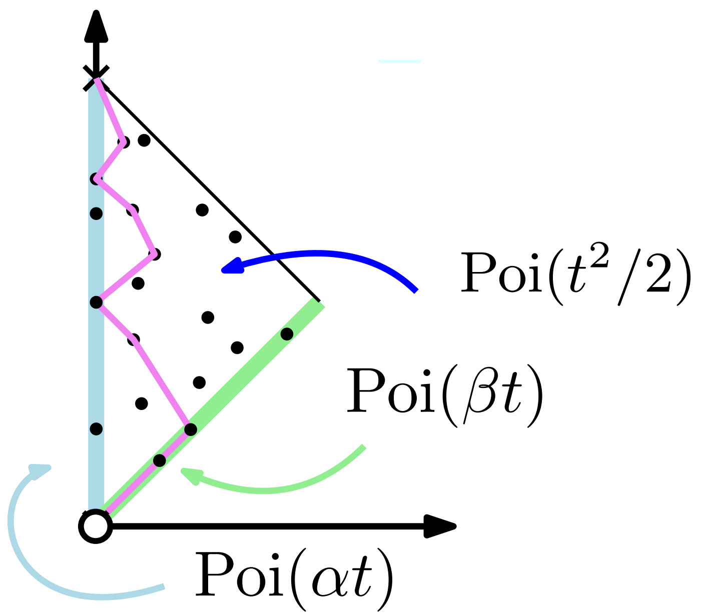


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$\{U \in O(\ell) : \det U = +1\}$ $\{U \in O(\ell) : \det U = -1\}$

$$= \frac{1}{Z} \left[ \det T^+_{\alpha,\beta,t}(\ell) + \det T^-_{\alpha,\beta,t}(\ell) \right]$$

$T^{\pm}_{\alpha,\beta,t}(\ell)$ has size $\begin{cases} \frac{\ell}{2} \times \frac{\ell}{2}, & \ell \text{ even} \\ \frac{\ell-1}{2} \times \frac{\ell-1}{2}, & \ell \text{ odd} \end{cases}$ and entries of the form $t_{i-j} \pm t_{i+j+a}$,

where $\sum_i t_i z^i = (1 + \alpha z)(1 + \alpha z^{-1})(1 + \beta z)(1 + \beta z^{-1}) e^{t(z+z^{-1})}$.

We can express $\det T^{\pm}_{\alpha,\beta,t}$ in terms of orthogonal polynomials on the unit circle.

# From PNG to RHP in half-space

To be precise, in terms of the monotone polynomials $\pi_\ell$ and numbers $N_\ell$ satisfying

$$\pi_\ell(z) = z^\ell + \text{lower order}, \quad \oint_{|z|=1} \pi_\ell(z) z^{-k} e^{t(z+z^{-1})} dz = \delta_{\ell k} N_\ell,$$

for $\ell$ even we have

$$\det T^{\pm}_{\alpha,\beta,t}(\ell) = \Bigg[ \frac{(\alpha^2\beta^2 \mp \alpha\beta\pi_\ell(0))}{\alpha\beta - 1} \pi_{\ell-1}(-\alpha)\pi_{\ell-1}(-\beta) - \frac{(1 \mp \alpha\beta\pi_\ell(0))}{\alpha\beta - 1} \pi^*_{\ell-1}(-\alpha)\pi^*_{\ell-1}(-\beta)$$
$$\mp \frac{(\alpha^2 \mp \alpha\beta\pi_\ell(0))}{\alpha - \beta} \pi_{\ell-1}(-\alpha)\pi^*_{\ell-1}(-\beta) - \frac{(\beta^2 \mp \alpha\beta\pi_\ell(0))}{\alpha - \beta} \pi^*_{\ell-1}(-\alpha)\pi_{\ell-1}(-\beta) \Bigg]$$
$$\cdot \frac{N_0 \cdot N_2 \cdot N_4 \cdots N_\ell}{1 \mp \pi_\ell(0)} \qquad \text{where } \pi^*_\ell(z) := \pi_\ell\left(\frac{1}{z}\right) z^\ell$$

and for odd $\ell$ we have

$$\det T^{\pm}_{\alpha,\beta,t}(\ell) = \Bigg[ \frac{(\alpha^2\beta^2 \pm \alpha\beta\pi_\ell(0))}{\alpha\beta - 1} \pi_{\ell-1}(-\alpha)\pi_{\ell-1}(-\beta) - \frac{(1 \pm \alpha\beta\pi_\ell(0))}{\alpha\beta - 1} \pi^*_{\ell-1}(-\alpha)\pi^*_{\ell-1}(-\beta)$$
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$$\cdot \frac{N_1 \cdot N_3 \cdot N_5 \cdots N_\ell}{1 \mp \pi_{\ell+2}(0)}.$$

# From PNG to RHP in half-space

To be precise, in terms of the monotone polynomials $\pi_\ell$ and numbers $N_\ell$ satisfying

$$\pi_\ell(z) = z^\ell + \text{lower order}, \quad \oint_{|z|=1} \pi_\ell(z) z^{-k} e^{t(z+z^{-1})} dz = \delta_{\ell k} N_\ell,$$

for $\ell$ even we have

$$\begin{aligned}\det T^{\pm}_{\alpha,\beta,t}(\ell) = \Big[ & \frac{(\alpha^2\beta^2 \mp \alpha\beta\pi_\ell(0))}{\alpha\beta - 1} \pi_{\ell-1}(-\alpha)\pi_{\ell-1}(-\beta) - \frac{(1 \mp \alpha\beta\pi_\ell(0))}{\alpha\beta-1} \pi^*_{\ell-1}(-\alpha)\pi^*_{\ell-1}(-\beta) \\ & \mp \frac{(\alpha^2 \mp \alpha\beta\pi_\ell(0))}{\alpha-\beta} \pi_{\ell-1}(-\alpha)\pi^*_{\ell-1}(-\beta) - \frac{(\beta^2 \mp \alpha\beta\pi_\ell(0))}{\alpha-\beta} \pi^*_{\ell-1}(-\alpha)\pi_{\ell-1}(-\beta) \Big] \\ & \cdot \frac{N_0 \cdot N_2 \cdot N_4 \cdots N_\ell}{1 \mp \pi_\ell(0)} \qquad \text{where } \pi^*_\ell(z) := \pi_\ell\left(\frac{1}{z}\right) z^\ell \end{aligned}$$

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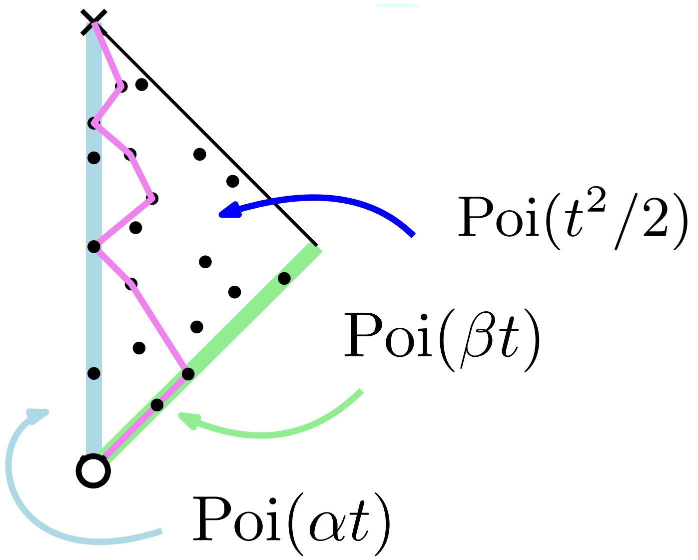
- An ugly but convenient expression! (via Baik, Deift & Johansson '99.)

# From PNG to RHP in half-space

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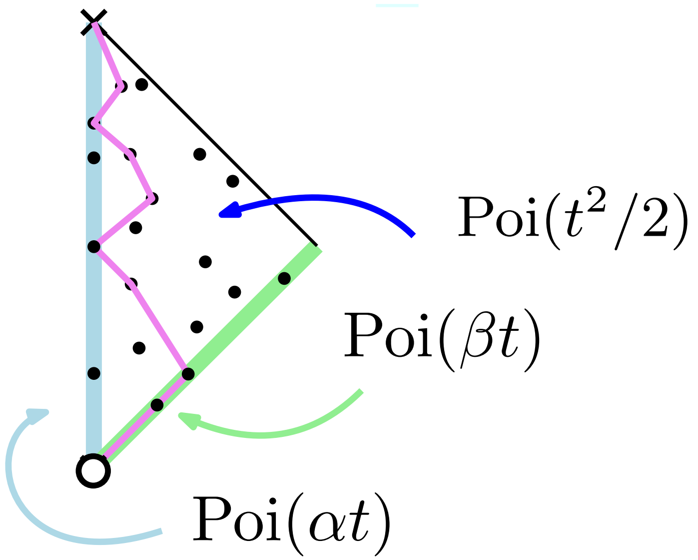


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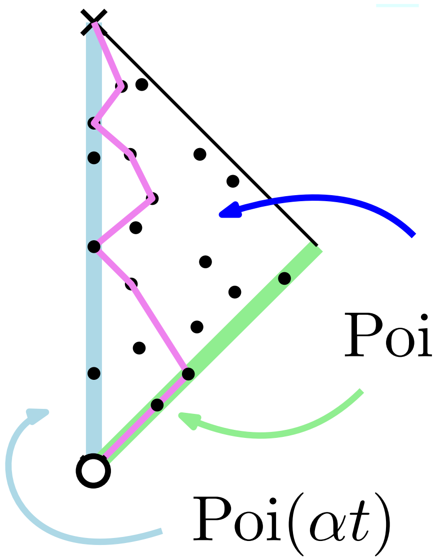


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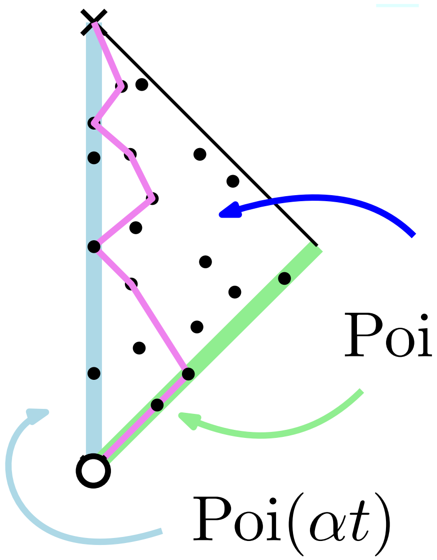


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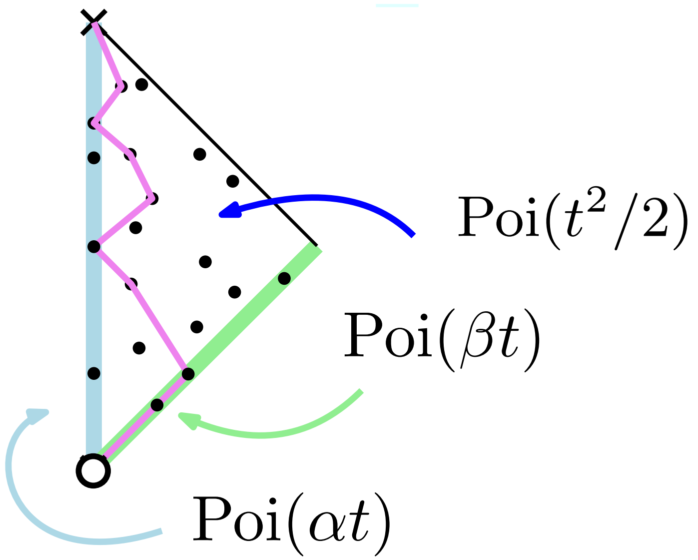


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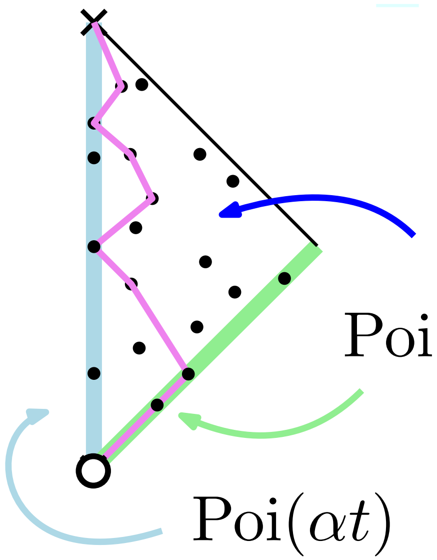


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• **Mini problem:** We can't compare $\det T^{\pm}_{\alpha,\beta,t}(\ell)$ and $\det T^{\pm}_{\alpha,\beta,t}(\ell - 1)$. But we can instead look at

$$\frac{\mathbb{P}(L_{\triangleright}(t) < \ell + 1) + \alpha\beta\,\mathbb{P}(L_{\triangleright}(t) < \ell)}{2} = \frac{\mathbb{P}(L^{\mathrm{g}}_{\triangleright}(t) < \ell + 1) - \alpha^2\beta^2\,\mathbb{P}(L^{\mathrm{g}}_{\triangleright}(t) < \ell - 1)}{2(1 - \alpha\beta)}.$$

# Critical scaling and interpolating distribution

We identify a critical window around in which we recover a parametrised limiting distribution.

**Intuition:** At $\alpha, \beta = 1$, $O_p(t^{1/3})$ points on the boundary contribute to $L_{\triangleright}(t)$.

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**Intuition:** At $\alpha, \beta = 1$, $O_p(t^{1/3})$ points on the boundary contribute to $L_\triangleright(t)$.

**Cafasso–Occelli–Ofner–W. '24+**: In a regime where $\alpha \sim 1 - \frac{2w}{t^{1/3}}$, $\beta \sim 1 - \frac{2y}{t^{1/3}}$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_\triangleright(t) - 2t}{t^{1/3}} < s\right) = H_\triangleright(w, y; s)$$

$$:= \Bigg[a(w,s)a(y,s) + v(s)\tfrac{b(w,s)b(y,s)-a(w,s)a(y,s)}{4(w+y)}$$

$$-u(s)\tfrac{a(y,s)b(w,s)-b(y,s)a(w,s)}{4(w-y)}\Bigg]F_{\mathrm{GSE}}(s)$$

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where $a(w,s) := m(-iw;s)_{22}$ and $b(w,s) := m(-iw;s)_{12}$, in terms of the entries of the solution $m(z;s)$ of the Painlevé II RHP.

We use the fact that

$$\tfrac{1}{2}\left[\mathbb{P}(L(t) < \ell) + \alpha\beta\mathbb{P}(L(t) < \ell - 1)\right] + O(t^{-2/3}) \leq \mathbb{P}(L(t) < \ell) \leq \tfrac{1}{2}\left[\mathbb{P}(L(t) < \ell + 1) + \alpha\beta\mathbb{P}(L(t) < \ell)\right] + O(t^{-2/3}).$$

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- Here, we see the full RHP solution $m$, not just the Painlevé II solution $u$ (and $v$).

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- The RHP formulation also allows complete asymptotics of $m$ to be found:

$$H_\triangleright(w,y;s) \to \begin{cases} F_{\mathrm{GSE}}(s), & w,y\to\infty \\ F_{\mathrm{GOE}}(s), & w\to\infty, y=0 \text{ or vice versa} \\ F_{\frac{1}{2}\mathrm{BR}}(s), & w=y=0. \end{cases}$$

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- **Baik–Rains '01**: In full space, in the same regime there is an analogous interpolating distribution

$$H_\diamond(w,y;s) = \Big[a(w,s)a(y,s) + v(s)\frac{b(w,s)b(y,s)-a(w,s)a(y,s)}{2(w+y)}\Big] F_{\mathrm{GUE}}(s)$$

which interpolates between $F_{\mathrm{GUE}}(s)$, $F_{\mathrm{GOE}}(s)^2$ and $F_{\mathrm{BR}}(s)$.

## Perspectives

- We don't know much about the new distribution $F_{\frac{1}{2}\mathrm{BR}}(s)$. How does it behave? Can we find it elsewhere? Can we write it as a Fredholm determinant?
- Can we find these distributions from the Fredholm pfaffian of Betea, Ferrari & Occelli?
- Can we find them from Fredholm determinant expressions found in Betea '18?
- **Next step:** we are studying the discrete time totally asymmetric simple exclusion process (TASEP) in a corresponding regime – we expect the same limit, but the analysis is different.

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**Thank you for your attention!**