Gaps in the growth of coefficients of Mahler functions

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Mahler functions

Definition

Let $k \geq 2$. A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a k-**Mahler function** (M-**function**) if

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0,$$

with polynomials $p_0(z), \ldots, p_d(z)$ and $p_0(z)p_d(z) \neq 0$.

- Coefficient recurrence: For $n \geq 0$ sufficiently large,

$$a_n = \sum_{j=1}^{s} -\alpha_j a_{n-j} + \sum_{i=1}^{d} \sum_{j=0}^{s} \beta_{i,j} a_{\frac{n-j}{k^i}}$$

  ($a_q = 0$ for $q \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$).
- Coefficients in a number field.
- Meromorphic in open unit disk (for every absolute value).

Logarithmic height

How to measure size of coefficients? 2^n vs. $1/2^n$.

- If $\alpha = a/b \in \mathbb{Q}\setminus\{0\}$ a reduced fraction, set

$$h(\alpha) = \log \max\{|a|, |b|\}.$$

- Local point of view: Archimedean absolute value $|\cdot| = |\cdot|_\infty$ and for each $p \in \mathbb{P}$ a non-archimedean one:

$$|\alpha|_p = p^{-v_p(\alpha)}.$$

$$\text{E.g., } \left|\frac{2^3 5^2}{7}\right|_p = \begin{cases} 1/8 & \text{if } p = 2, \\ 1/25 & \text{if } p = 5, \\ 7 & \text{if } p = 7, \\ 200/7 & \text{if } p = \infty, \\ 1 & \text{otherwise.} \end{cases}$$

$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |\alpha|_p = 1. \qquad \text{(Product formula).}$$

Logarithmic height

$$h(a/b) = \log \max\{|a|,|b|\} = \log \prod_{p \in \mathbb{P} \cup \{\infty\}} \max\{1, |a/b|_p\}.$$

Because (say $b = p_1^{e_1} \cdots p_r^{e_r}$, $\gcd(a,b) = 1$),

$$\begin{aligned}\prod_{p \in \mathbb{P} \cup \{\infty\}} &= \max\{1, |\tfrac{a}{b}|_p\} = \max\{1, |\tfrac{a}{b}|\} |\tfrac{a}{b}|_{p_1} \cdots |\tfrac{a}{b}|_{p_r} \\ &= \max\{1, |\tfrac{a}{b}|\} p_1^{e_1} \cdots p_r^{e_r} = \max\{1, |\tfrac{a}{b}|\} |b|.\end{aligned}$$

$$\max\{1, |\tfrac{a}{b}|\}\ |b| = \begin{cases} |\tfrac{a}{b}||b| = |a| & \text{if } |a| \geq |b| \\ |b| & \text{if } |a| \leq |b|. \end{cases}$$

If $\alpha \in K$, K a number field:

$$h(\alpha) \coloneqq \log \prod_{v \in M_K} \max\{1, |\alpha|_v\}.$$

- $h(\alpha^{-1}) = h(\alpha)$.
- Only finitely many $\alpha \in \overline{\mathbb{Q}}$ of bounded degree and height.
- $h(\alpha) = 0 \;\Leftrightarrow\; \alpha$ is a root of unity $\;\Leftrightarrow\; |\alpha| = 1$ for all $|\cdot|$.

Height Gap Theorem

$$(b_n) \in O \cap \Omega(g(n)) \quad \Leftrightarrow \quad (b_n) \in O(g(n)) \text{ and } (b_n) \notin o(g(n)).$$

Theorem (Adamczewski–Bell–S. 2020)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ be an M-function. Then $h(a_n)$ falls into one of the following classes.

1. $O \cap \Omega(n)$.
2. $O \cap \Omega(\log^2 n)$.
3. $O \cap \Omega(\log n)$.
4. $O \cap \Omega(\log\log n)$. *k*-regular
5. $O(1)$. *k*-automatic

So $h(a_n) \in o(n) \Rightarrow h(a_n) \in O(\log^2(n))$ etc.

$O \cap \Omega(n)$

$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$

$\prod_{n=0}^{\infty} \frac{1}{1-2z^{k^n}}$

$O \cap \Omega(n)$

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$O \cap \Omega(\log^2(n))$

$\prod_{n=0}^{\infty} \frac{1}{1-z^{k^n}} = \sum_{n=0}^{\infty} a_n z^n$

Partitions into k-powers:
$n = j_1 k^{n_1} + \cdots + j_r k^{n_r}$

$\log(a_n) \sim \log^2(n)/2\log(k)$
(Mahler '40, de Bruijn '48, Dumas-Flajolet '96)

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$O \cap \Omega(\log(n))$

$\frac{z}{(1-z)^2} = \sum_{n=0}^{\infty} n z^n$

$\sum_{n=0}^{\infty} v_p(n!) z^n \quad v_p(n!) \sim n/(p-1)$

$O \cap \Omega(n)$

$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$

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$O \cap \Omega(\log\log(n))$

$\sum_{n=0}^{\infty} s_n z^n$

s_n = digit sum in base k

$O \cap \Omega(n)$

$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$

$\prod_{n=0}^{\infty} \frac{1}{1-2z^{k^n}}$

$O \cap \Omega(\log^2(n))$

$\prod_{n=0}^{\infty} \frac{1}{1-z^{k^n}} = \sum_{n=0}^{\infty} a_n z^n$

Partitions into k-powers:
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$O \cap \Omega(\log\log(n))$

$\sum_{n=0}^{\infty} s_n z^n$

s_n = digit sum in base k

$t_n \in \{0,1\}$ = parity of no. of binary digits of n

k-automatic sequences $\quad O \cap \Omega(1)$

k-automatic/k-regular sequences

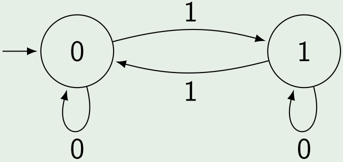
k-automatic sequences

Definition

$(a_n)_n$ is k-**automatic** if it can be produced a finite automaton (with outputs):
Input: Base k representation of the number $n = \langle b_m \cdots b_0 \rangle_k$.
Output: a_n.

Example

Thue-Morse sequence: t_n = parity of number of 1s in binary expansion of n.



k-regular sequences

Definition (Allouche–Shallit)

$(a_n)_n$ in $\overline{\mathbb{Q}}$ is k-**regular** if there exist $u \in \overline{\mathbb{Q}}^{1\times d}$, $v \in \overline{\mathbb{Q}}^{d\times 1}$ and $M(0), \ldots, M(k-1) \in \overline{\mathbb{Q}}^{d\times d}$ such that

$$s_n = uM(b_m)M(b_{m-1})\cdots M(b_0)v \quad \text{if } n = \langle b_m \cdots b_0 \rangle_k.$$

Example

- If a_n = number of 1s in binary representation of n, the sequence is 2-regular but not 2-automatic.
- Cantor sequence: numbers where 1 doesn't appear in ternary representation.

Theorem

k-regular sequence $(a_n)_n$: $\quad$ k-automatic $\Leftrightarrow$ $\{a_n : n\}$ finite.

Connection to M-functions

Theorem (Becker '94)

k-regular series $\Rightarrow$ M-function.

Idea: $s_{e,r} := (a_{k^e n+r})_n$, $0 \leq r < k^e$, $f_{e,r}(z) := \sum_{n=0}^{\infty} a_{k^e n+r} z^n$.

$s_{0,0} = (a_n)_n$ k-regular $\Leftrightarrow$
k-kernel $\{ s_{e,r} : e \geq 0, 0 \leq r < k^e \}$ spans fin-dim. $\overline{\mathbb{Q}}$-vector space.

$$f_{e,r}(z) \in \operatorname{span}_{\overline{\mathbb{Q}}(z)} \left\{ f_{e',r'}(z^{\widetilde{k}}) : e' \leq E,\ 0 \leq r' \leq k^{e'} - 1 \right\}, \quad \widetilde{k} = k^{E+1}.$$

$\Rightarrow$ $f(z)$ contained in fin.-dim. $\overline{\mathbb{Q}}(z)$-vector space in $\overline{\mathbb{Q}}[\![z]\!]$, closed under $z \mapsto z^{\widetilde{k}}$.

$\Rightarrow$ $f(z)$, $f(z^{\widetilde{k}})$, …, $f(z^{\widetilde{k}^j})$, … linearly dependent over $\overline{\mathbb{Q}}(z)$.

$\Rightarrow$ $f(z)$ k-Mahler.

Connection to M-functions

Conversely: M-function $\not\Rightarrow$ k-regular.

Theorem (Becker '94)

If $f(z)$ is k-Becker, i.e., satisfies an equation of the form

$$1 \cdot f(z) + p_1(z) f(z^k) + \cdots + p_d(z) f(z^{k^d}) = 0,$$

then $f(z)$ is k-regular.

Idea: $f(z) = \sum_{i=1}^{d} p_i(z) f(z^{k^i})$.

Consider $\overline{\mathbb{Q}}$-vector space V spanned by $z^j f(z^{k^i})$ $(i \leq d,\ j \leq D)$.

Cartier operators $\Delta_r(f(z)) \coloneqq \sum_{n=0}^{\infty} a_{nk+r} z^n$,

$$\Delta_r(z^j f(z^{k^i})) = \Delta(z^j) f(z^{k^{i-1}}), \quad (i \geq 1)$$

and functional equation $\Rightarrow$ V closed under Δ_r $\Rightarrow$ f is k-regular.

k-Becker $\Rightarrow$ k-regular, k-regular $\not\Rightarrow$ k-Becker,

But:

- If $f(z)$ is k-Mahler with $p_0(z)f(z) = \sum_{i=1}^{d} p_i(z)f(z^{k^i})$ and $p_0(0) = 1$, then
$$f(z) = \frac{g(z)}{\prod_{n=0}^{\infty} p_0(z^{k^n})},$$
with $g(z)$ being k-Becker (Dumas '93).
- If $f(z)$ is k-regular, then there exists a polynomial $q(z)$ such that
$$f(z) \cdot \frac{1}{z^{\gamma} q(z)} \in \overline{\mathbb{Q}}[\![z^{\pm 1}]\!]$$
is k-Becker. Here $q(0) = 1$ and $1/q(z)$ k-regular. (Becker's conjecture, proven by Bell, Chyzak, Coons, Dumas '19.)

Gaps

Height Gap Theorem

$$(b_n) \in O \cap \Omega(g(n)) \quad \Leftrightarrow \quad (b_n) \in O(g(n)) \text{ and } (b_n) \notin o(g(n)).$$

Theorem (Adamczewski–Bell–S. 2020)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function. Then $h(a_n)$ falls into one of the following classes.

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3. $O \cap \Omega(\log n)$.
4. $O \cap \Omega(\log \log n)$.
5. $O(1)$.

k-automatic

k-regular

The Mahler denominator

Definition

Let $f(z) \in K[\![z]\!]$ be a k-Mahler series, and let

$$\mathfrak{I} = \left\{ p(z) \in K[z] : p(z)f(z) \in \sum_{i=1}^{\infty} K[z]f(z^{k^i}) \right\}.$$

The k-**Mahler denominator** of $f(z)$ is the unique generator $\mathfrak{d}(z) \in K[z]$ of the ideal $\mathfrak{I}$, with the lowest non-zero coefficient of $\mathfrak{d}(z)$ being 1.

- $\mathfrak{d}(z)$ divides $p_0(z)$ in any Mahler equation for $f(z)$.
- $\mathfrak{d}(z)$ need **not** be $p_0(z)$ of a minimal homogeneous Mahler equation of $f(z)$.

Example

$$(z - 1/2)f(z) - (z - 1/8)(z^3 - 1/2)f(z^3) = 0$$

has a unique non-zero solution (up to scalars).

However $\mathfrak{d}(z) = 1$, because

$$1 \cdot f(z) = (z - 1/8)(z^2 + 1/2z + 1/4)(z^9 - 1/2)f(z^9).$$

First gap

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function.

Proposition

TFAE:

1. $h(a_n) \in o(n)$.
2. Every nonzero root of $\mathfrak{d}$ is a root of unity.
3. f is analytic in $B_{|\cdot|}(0,1)$ for all absolute values.
4. $h(a_n) \in O(\log^2 n)$.

Sketch of proof

Show: $h(a_n) \in o(n) \Rightarrow$ roots of $\mathfrak{d}(z)$ in $\{0\} \cup \underbrace{\{\text{root of unity}\}}_{=\mu(\overline{\mathbb{Q}})}$.

- $h(a_n) \in o(n) \Rightarrow f(z)$ has radius of convergence $= 1$ for all $|\cdot|$.
- Suppose λ root of $\mathfrak{d}(z)$, $\lambda \notin \{0\} \cup \mu(\overline{\mathbb{Q}})$
  $\Rightarrow$ $|\lambda| < 1$ for some $|\cdot|$.
- Take a minimal Mahler equation for $f(z)$:

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0,$$

  with $p_0(z)p_d(z) \neq 0$ and d minimal.
- $\Rightarrow f(z), \ldots, f(z^{k^{d-1}})$ are linearly independent over $\overline{K}(z)$.
- Transcendence result: (Adamczewski–Faverjon '17)
  $\Rightarrow$ for large m, also $f(\lambda^{k^m}), \ldots, f(\lambda^{k^{m+d-1}})$ are linearly independent over $\overline{K}$.

Sketch of proof

Show: $h(a_n) \in o(n) \Rightarrow$ roots of $\mathfrak{d}(z)$ in $\{0\} \cup \mu(\overline{\mathbb{Q}})$.

- Have: $f(\lambda^{k^m}), \ldots, f(\lambda^{k^{m+d-1}})$ linearly independent over $\overline{K}$.
- Iterating $p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_d(z)f(z^{k^d}) = 0$:

$$r_0(z)f(z) + r_1(z)f(z^{k^m}) + \cdots + r_d(z)f(z^{k^{m+d-1}}) = 0$$

  ($r_0(z), \ldots, r_d(z) \in \overline{\mathbb{Q}}[z]$ coprime).
- $\mathfrak{d}(\lambda) = 0 \Rightarrow r_0(\lambda) = 0$, so

$$r_1(\lambda)f(\lambda^{k^m}) + \cdots + r_d(\lambda)f(\lambda^{k^{m+d-1}}) = 0,$$

  a contradiction!

Now: Coefficient recursion (in matrix form) $\Rightarrow h(a_n) \in O(\log^2 n)$.

First gap

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ be an M-function.

Proposition

TFAE:

1. $h(a_n) \in o(n)$.
2. Every nonzero root of $\mathfrak{d}$ is a root of unity.
3. f is analytic in $B_{|\cdot|}(0, 1)$ for all absolute values.
4. $h(a_n) \in O(\log^2 n)$.

Second gap

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function.

Proposition

TFAE:

1. $h(a_n) \in o(\log^2 n)$.
2. Every nonzero root ζ of $\mathfrak{d}$ is a root of unity with order not coprime with k ($\zeta^{k^j} \neq \zeta$ for all $j \geq 1$).
3. f is k-regular.
4. $h(a_n) \in O(\log n)$.

Sketch of argument

Suppose $\zeta^k = \zeta \neq 0$ and $h(a_n) \in o(\log^2(n))$. **Show**: $\mathfrak{d}(\zeta) \neq 0$.

Strategy:

1. M-function structure and $\mathfrak{d}(\zeta) = 0 \;\Rightarrow\;$ lower bound for $f(t_j\zeta)$ along some sequence $(t_j)_j \to 1$, $0 < t_j < 1$.
2. Pigeonhole argument $\;\Rightarrow\;$ lower bound for subsequence of a_n, contradicting $h(a_n) \in o(\log^2 n)$.

(1) Suppose $\mathfrak{d}(\zeta) = 0$, $p_0 = \mathfrak{d}$,

$$p_0(z)f(z) = \sum_{i=1}^{d} p_i(z)f(z^{k^i}).$$

(A) *Try to get rid of p_0:*

$$g(z) \coloneqq f(\zeta z) \prod_{n=0}^{\infty} \frac{p_0(\zeta z^{k^n})}{(1 - z^{k^n})^s}, \quad \text{so that} \quad g(z) = \sum_{i=1}^{d} r_i(z) g(z^{k^i})$$

with suitable s:

$$s \coloneqq \min\left\{ \frac{\nu_i + (i-1)\nu_0}{i} : 1 \le i \le d \right\} \in \mathbb{Q}_{\ge 0},$$

with $\nu_i = \mathrm{v}_{z-\zeta}(p_i(z))$, so that $r_i(z) \in \overline{\mathbb{Q}}[z]$, $r_{i_0}(1) \ne 0$, $s < \nu_0$.

(B) *Linear system for g:* $w(z) = \left(g(z), \dots, g(z^{k^{d-1}})\right)^T$, and

$$w(z) = A(z)w(z^k)$$

where $A(z) \in \overline{\mathbb{Q}}[z]^{d\times d}$.

(C) Suitable $1-\varepsilon < t_0 < 1$ and $t_j := t_{j-1}^{1/k}$, $(t_j)_j \to 1$.

$$w(t_j) = A(t_j)A(t_{j-1})\cdots A(t_1)w(t_0).$$

Simplification: say $A(1)$ invertible, $c_1 := 2\|A(1)^{-1}\|$. So

$$0 < c_0 := \|w(t_0)\| \le c_1^j \|w(t_j)\|.$$

That is

$$\|w(t_j)\| \ge c_0(1/c_1)^j > 0.$$

(Adamczewski-Bell '17, in general: $A(1)$ not nilpotent)
$\Rightarrow$ a similar bound for $|g(t_j)|$.

Now: $|g(t_j)| \geq (1-t_j)^a$,

$$f(\zeta t_j) = g(t_j) \cdot \prod_{n=0}^{\infty} \frac{(1-t_j^{k^n})^{s}}{p_0(\zeta z^{k^n})},$$

Mahler:

$$\prod_{n=0}^{\infty}(1-t_j^{k^n})^{-1} \geq \sum_{n=m_0}^{\infty} \exp(c\log^2 n)t_j^n \geq \exp(c\log^2 m)t_j^m \quad (m \geq m_0).$$

Combining:

$$|f(\zeta t_j)| \geq (1-t_j)^{a'} \exp(cb\log^2 m)t_j^{mb}$$

(for some sequence $(t_j)_j \to 1$ and all $m \geq m_0$)

Have: Using $\mathfrak{d}(\zeta) = 0$, $\zeta^k = \zeta$,

$$|f(\zeta t_j)| \geq (1 - t_j)^{a'} \exp(cb \log^2 m) t_j^{mb}$$

(for some sequence $(t_j) \to 1$ and all $m \geq m_0$)

(2) Now split

$$f(\zeta t_j) = \underbrace{\sum_{n=0}^{M} a_n (\zeta t_j)^n}_{=S_M} + \underbrace{\sum_{n=M+1}^{\infty} a_n (\zeta t_j)^n}_{|\cdot| \leq 1}.$$

with sufficiently small $M = M(t_j)$ s.t. pigeonhole principle gives $\log|a_n| > c \log^2 n$ infinitely often for some $c > 0$.

Can take $M \approx \lceil m \log^2 m \rceil$ with $m \approx c'(1 - t_j)^{-1}$.

Contradicts $h(a_n) \in o(\log^2 n)$.

... on to regularity

Now we have $p_0(z)f(z) = \sum_{i=1}^{d} p_i(z)f(z^{k^i})$ where the roots ζ_r of p_0 are roots of unity with $\zeta^{k^j} \neq \zeta$ for $j \geq 0$.

Factoring $\mathfrak{d}(z^{k^n})$:

$$f(z) = \underbrace{g(z)}_{k\text{-Becker}} \cdot \prod_r \underbrace{\prod_{n=0}^{\infty} \frac{1}{1 - \zeta_r^{-1} z^{k^n}}}_{k\text{-regular}},$$

and products of k-regular sequences are k-regular.

So:

$$h(a_n) \in o(\log^2 n) \quad \Rightarrow \quad \mathfrak{d} \text{ “harmless”} \quad \Rightarrow \quad f(z)\ k\text{-regular.}$$

Now $h(a_n) \in O(\log(n))$ is elementary.

Second gap

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function.

Proposition

TFAE:

1. $h(a_n) \in o(\log^2 n)$.
2. Every nonzero root ζ of $\mathfrak{d}$ is a root of unity with order not coprime with k ($\zeta^{k^j} \neq \zeta$ for all $j \geq 1$).
3. f is k-regular.
4. $h(a_n) \in O(\log n)$.

Third gap

Now f is k-regular; let $\mathcal{S}$ be the matrix semigroup of a minimal linear representation, i.e.,

$$f(n) = uM(b_k)\cdots M(b_0)v \quad \text{with} \quad n = \langle b_m \cdots b_0 \rangle_k$$

and $\mathcal{S} = \langle M(i) : 0 \le i \le k-1 \rangle \subseteq K^{d\times d}$.

Proposition

TFAE:

1. $h(a_n) \in o(\log n)$.
2. $\mathcal{S}$ is tame (every nonzero eigenvalue is a root of unity).
3. The sequence $(a_n)_{n\ge 0}$ is a $\overline{\mathbb{Q}}$-linear combination of word-convolution products of k-automatic sequences.
4. $h(a_n) \in O(\log\log n)$.

$a_n = uM(w)v$ with $M\colon \{0,\dots,k-1\}^* \to K^{d\times d}$, $u \in K^{1\times d}$, $v \in K^{d\times 1}$ and w a base-k representation of n.

"$h(a_n) \in o(\log n) \Rightarrow \mathcal{S}$ tame" follows by pumping argument ($uM(w_0 w_1^k w_2)v$ blows up in some absolute value).

Matrix semigroup $\mathcal{S}$ is tame if and only if

$$T^{-1}\mathcal{S}T \subseteq \begin{pmatrix} \mathcal{S}_1 & \overline{\mathbb{Q}}^{d_1\times d_2} & \overline{\mathbb{Q}}^{d_1\times d_3} & \dots & \overline{\mathbb{Q}}^{d_1\times d_r} \\ 0 & \mathcal{S}_2 & \overline{\mathbb{Q}}^{d_2\times d_3} & \dots & \overline{\mathbb{Q}}^{d_2\times d_r} \\ 0 & 0 & \mathcal{S}_3 & \dots & \overline{\mathbb{Q}}^{d_3\times d_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathcal{S}_r \end{pmatrix}.$$

with $\mathcal{S}_i$ finite semigroups.

Fourth gap

Let $\mathcal{S}$ be the matrix semigroup of a minimal linear representation.

Proposition

TFAE:

1. We have $h(a_n) \in o(\log\log n)$.
2. $\mathcal{S}$ is finite.
3. f is k-automatic.
4. We have $h(a_n) = O(1)$. Equivalently, $\{ a_n : n \geq 0 \}$ is finite.

Height Gap Theorem

$$(b_n) \in O \cap \Omega(g(n)) \quad \Leftrightarrow \quad (b_n) \in O(g(n)) \text{ and } (b_n) \notin o(g(n)).$$

Theorem (Adamczewski–Bell–S. 2020)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ be an M-function. Then $h(a_n)$ falls into one of the following classes.

1. $O \cap \Omega(n)$.
2. $O \cap \Omega(\log^2 n)$.
3. $O \cap \Omega(\log n)$.
4. $O \cap \Omega(\log \log n)$. k-regular
5. $O(1)$. k-automatic

Corollaries for other fields

Corollary

Let K be a field of characteristic 0 and $f(z)$ a k-Mahler series over K. Then $f(z)$ is k-automatic if and only if it has finitely many distinct coefficients.

In positive characteristic:

- holds if k is a power of p;
- false if k and p are coprime (Becker 1994).

Corollary

Let K be a field of characteristic 0 and $f(z)$ a k-Mahler series over K. Then $f(z)$ is k-regular if and only if all nonzero roots of the Mahler denominator in $\overline{K}$ are roots of unity with order not coprime to k.

Decidability

Decidability

Theorem (Adamczewski–Bell–S. 2020)

The cases in the main theorem are decidable.

Contrast with:

Theorem (Krenn–Shallit 2020)

If $f(n)$ is k-regular with values in $\mathbb{Q}$, it is undecidable whether $|f(n)|$ is bounded (in the archimedean absolute value).

Reason: It is

- decidable if a f.g. matrix semigroup is **tame** (eigenvalues in $\{0\} \cup \mu(\overline{\mathbb{Q}})$)
- undecidable if it has **joint spectral radius** ≤ 1 ($\limsup_{n\to\infty} \|X_1 \cdots X_n\|^{1/n} \leq 1$) (Blondel–Tsitsiklis '00).