Refined enumeration of planar Eulerian orientations

Andrew Elvey Price
Joint work with Mireille Bousquet-Mélou

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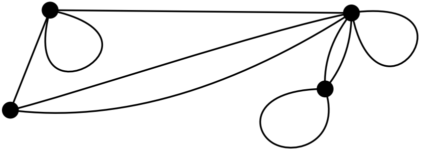
10/06/2024

PLANAR MAPS



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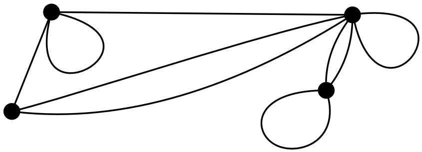
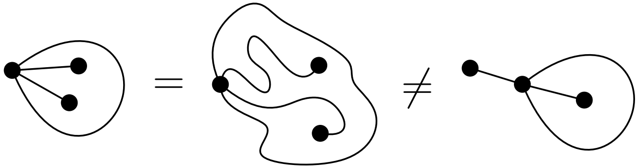
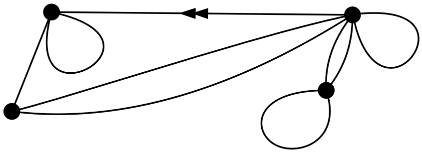
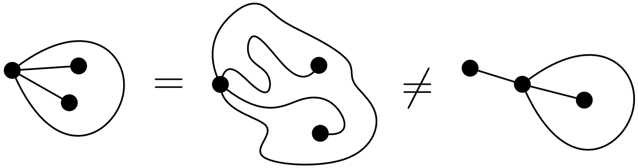


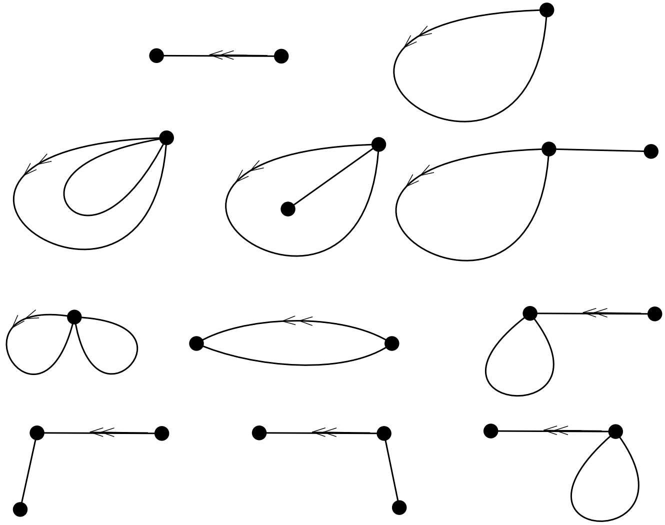


ROOTED PLANAR MAPS

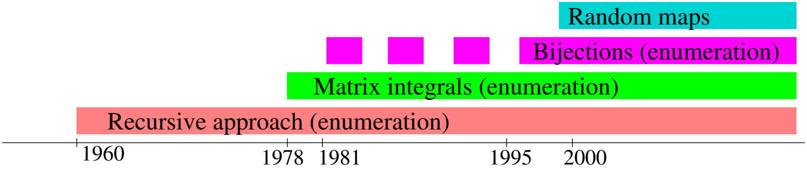




SMALL PLANAR MAPS



A chronology of planar maps



- **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless, Bonzom...
- **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne, Holden, Budzinski, Louf, Carrance

MAPS EQUIPPED WITH AN ADDITIONAL STRUCTURE

How many maps equipped with...

- a spanning tree [Mullin 67, Bernardi]
- a spanning forest? [Bouttier et al., Sportiello et al., Bousquet-Mélou & Courtiel]
- a self-avoiding walk? [Duplantier & Kostov; Gwynne & Miller]
- a proper q-colouring? [Tutte 74-83, Bouttier et al.]
- a bipolar orientation? [Kenyon, Miller, Sheffield, Wilson, Fusy, Bousquet-Mélou...]

Maps equipped with an additional structure

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Additional structures in this talk:

- Maps equipped with an *height function* (H-maps)
- Maps equipped with an *Eulerian orientation* (EO-maps)
- Quadrangulations equipped with a *height function* (H-quads)
- Quartic maps equipped with an *Eulerian orientation* (EO-quarts)

Background

- 2000: EO-quarts problem non-rigorously "solved" with weight ω [Kostov]
- 2013: Bijective link between H-quads and H-maps [Ambjørn and Budd]
- 2017: EO-maps enumeration problem posed [Bousquet-Mélou, Bonichon, Dorbec, Pennarun]
- 2018: Bijective link H-maps to EO-maps and H-quads to EO-quarts [E.P., Guttmann], conjectured Asymptotics
- 2020: Exact solution for $\omega = 0, 1$ [E.P., Bousquet-Mélou] (using guess and check of functional equations)
- 2023: Exact solution for all ω [E.P., Zinn-Justin] (using complex analysis, following Kostov)

BACKGROUND

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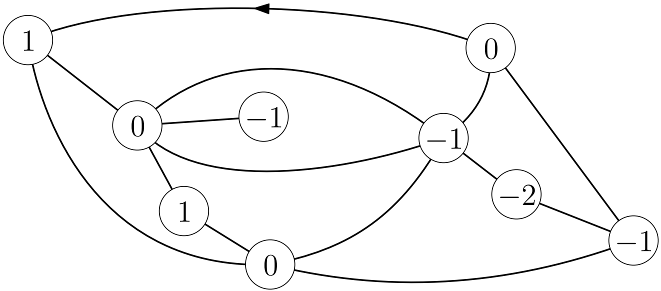
This work:

- Exact solution for all ω (using algebraic methods)
- Exact solution for $\omega = 0, 1$ with new weight v
- Functional equations for all ω, v.

Height-labelled quadrangulations:

- Each face has degree 4
- Adjacent labels differ by 1
- Root edge labelled from 0 to 1



Height-labelled quadrangulations:

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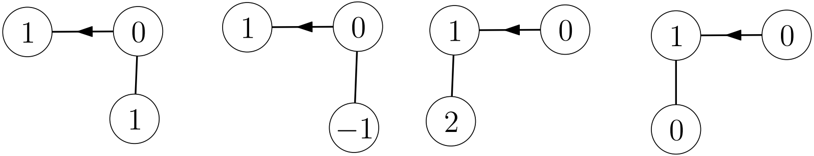
Aim: determine the generating function $\mathsf{Q}(t) = 4t + 35t^2 + \ldots$ that counts height-labelled quadrangulations by faces.



Exact solution [E.P., Bousquet-Mélou, 2020]

Let $\mathsf{R}(t) \in t\mathbb{Z}[[t]]$ be the unique series satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}(t)^{n+1}.$$

Theorem: The generating function of height-labelled quadrangulations is given by

$$\mathsf{Q}(t) := q_0 + q_1 t + q_2 t^2 + \cdots = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}(t)).$$

Asymptotically,

$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2 (\log n)^2},$$

where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

Recall: Height-labelled quadrangulations:

- Each face has degree 4
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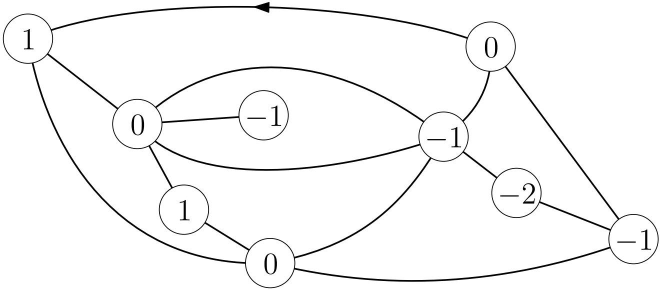
Weights:

- A weight v per local minimum
- A weight ω per alternating face



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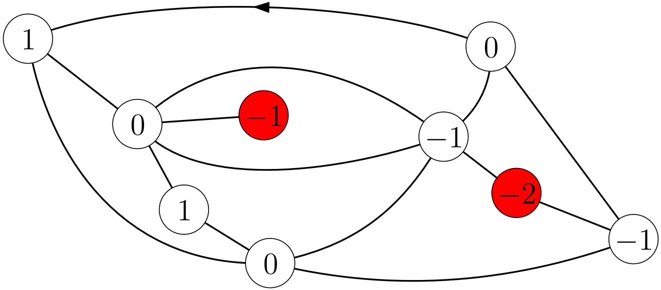

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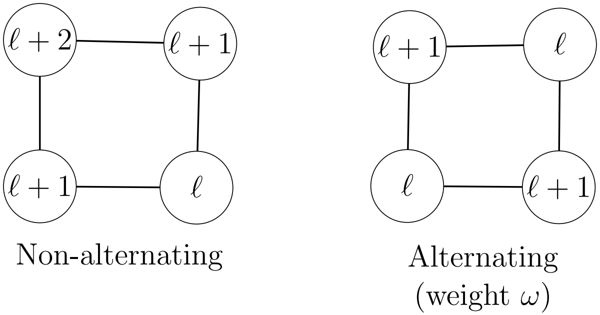

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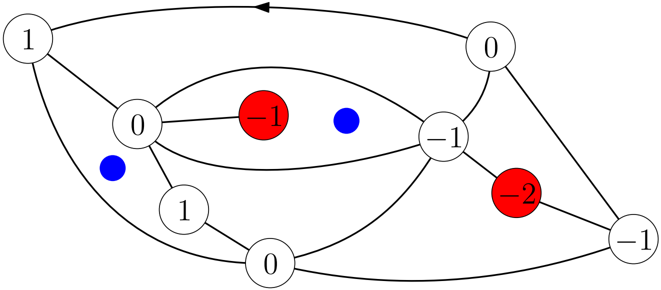

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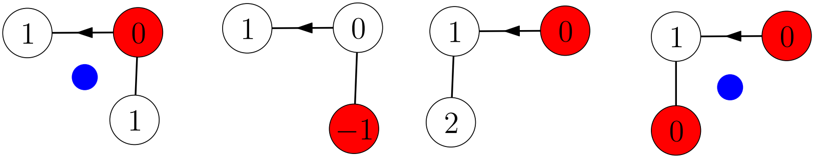
Aim: determine the refined generating function

$$\mathsf{Q}(t,\omega,v) = \left(2v + \omega v + \omega v^2\right) t + \cdots$$



TALK OUTLINE

- **Part 1:** Combinatorics $\rightarrow$ Functional equations for $\mathsf{Q}(t, \omega, v)$
- **Part 2:** Solution for $\mathsf{Q}(t, 0, v)$ and $\mathsf{Q}(t, 1, v)$
- **Part 3:** Complex analytic version of functional equations, solution to $\mathsf{Q}(t, \omega, 1)$
- **Bonus (if time permits):** Bijections to Eulerian orientations and six vertex model

Part 1: Combinatorics $\rightarrow$ Functional equations

Functional equations preview

Theorem: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(x)) = x,$$

The series $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q}(t, \omega, v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

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$$y\mathcal{M}\left(y\right)\left(1-\mathcal{M}\left(y\right)-\frac{(1-v)t}{y}-\omega y\right) \in K[[y]],$$

$$\mathcal{M}\left(\mathcal{M}(x)\right) = x,$$

Meaning of $\mathcal{M}(\mathcal{M}(x))$:

Writing $\mathcal{M}(x) = \sum_{n=1}^{\infty}\sum_{j=-n}^{\infty} m_{n,j}(\omega, v)x^j t^n$, we have

$$\mathcal{M}(x)^j t^n \in x^n (t/x)^{j+n}\mathbb{Z}(\omega, v)[[x, t/x]],$$

so

$$\mathcal{M}(\mathcal{M}(x)) := \sum_{n=1}^{\infty}\sum_{j=-n}^{\infty} m_{n,j}(\omega, v)\mathcal{M}(x)^j t^n \in \mathbb{Z}(\omega, v)[[x, t/x]]$$

is well defined.

Characterisation 1: There are series $\mathsf{P}(y) \in \mathbb{Z}[[y, \omega, v, t]]$ and $\mathsf{D}(x, y), \mathsf{E}(x, y) \in \mathbb{Z}[[x, y, \omega, v, t]]$, uniquely defined by:

$$\mathsf{D}(x,y) = v + \frac{y}{v}\mathsf{D}(x,y)[z^1]\mathsf{D}(x,z) + y[x^{\geq 0}]\left(\frac{1}{x}\mathsf{D}(x,y)\mathsf{P}\left(\frac{t}{x}\right)\right),$$

$$(1-x)(\mathsf{D}(x,y)-v) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y}, z\right)\right).$$

$$\mathsf{E}(x,y) = \mathsf{E}(y,x) = \frac{1}{v}[x^{\geq 0}]\left(\mathsf{D}\left(\frac{t}{x}, y\right)\mathsf{P}(x)\right)$$

The generating function $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q} = [y^1]\mathsf{P}(y) - v.$$

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I will show one element of the proof.

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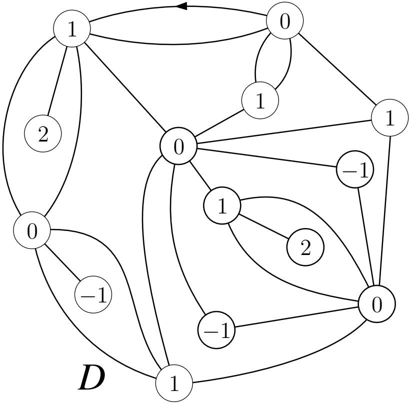
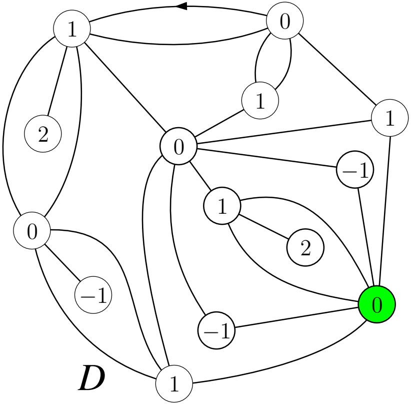
D-patch: Digons are allowed next to the root vertex and the outer face may have any degree.



Restrictions:
- outer labels must be 0 or 1.
- vertices adjacent to the root must be labelled 1.

In $\mathsf{D}(x, y)$:
- x counts digons.
- y counts the degree of the outer face (halved)
- t, ω, v same as before.

Colour the vertex two places clockwise from the root vertex around the outer face.



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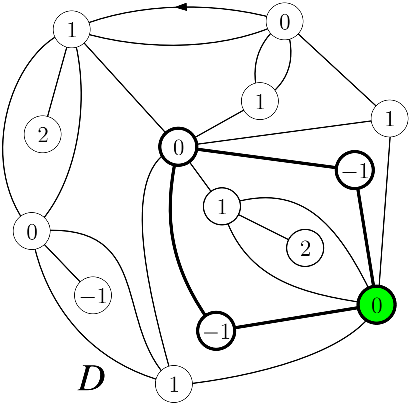
DECOMPOSITION OF D-PATCHES

Highlight the maximal connected subgraph of nonpositive labels, containing the coloured vertex.



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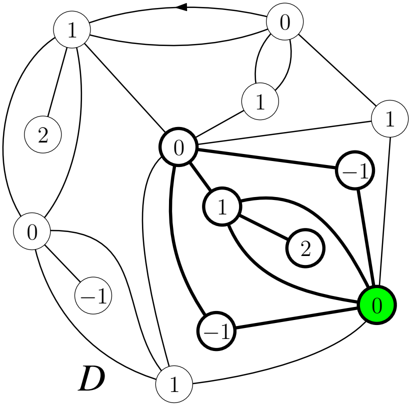
Add to the subgraph all vertices and edges contained in its inner face(s).



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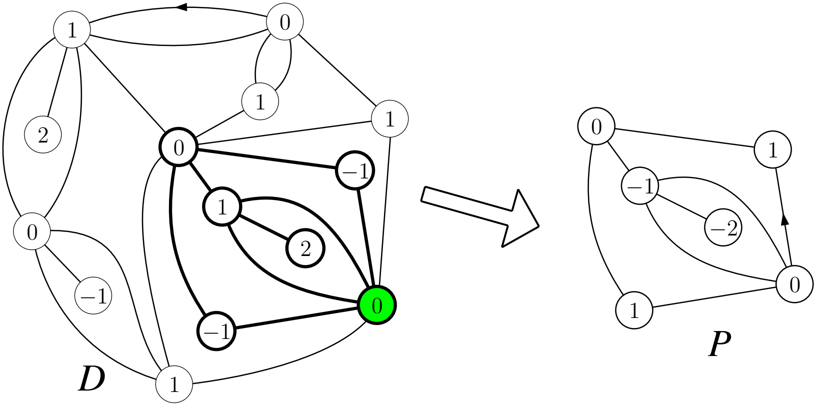
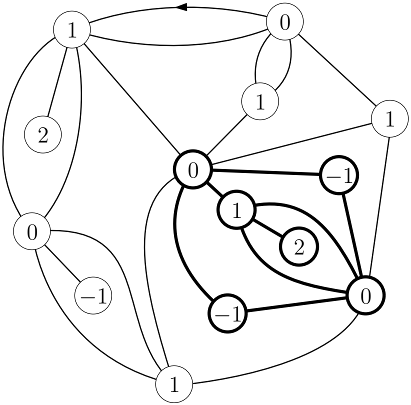
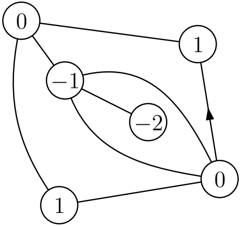
Record the subgraph with inverted labels.



Contract the highlighted map to a single vertex (labelled 0).





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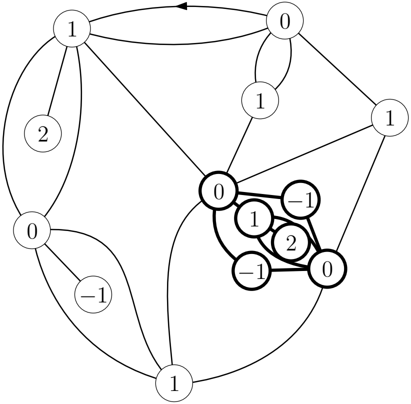

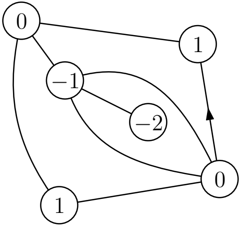

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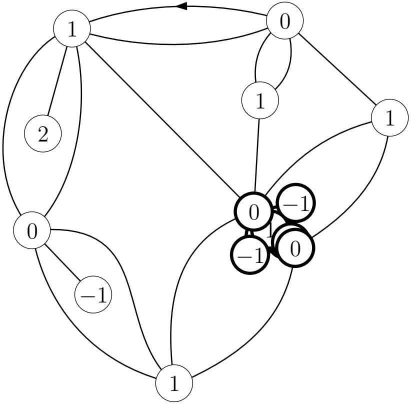

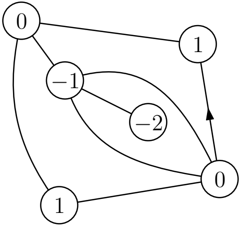

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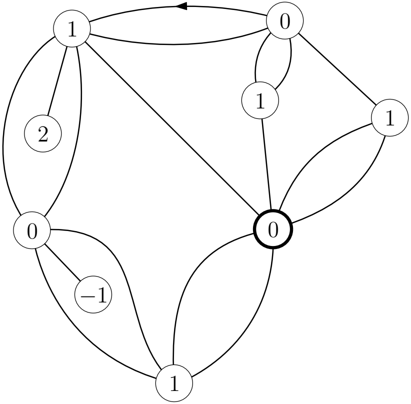

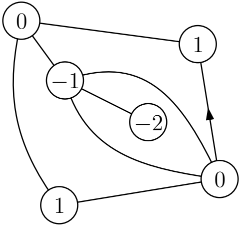

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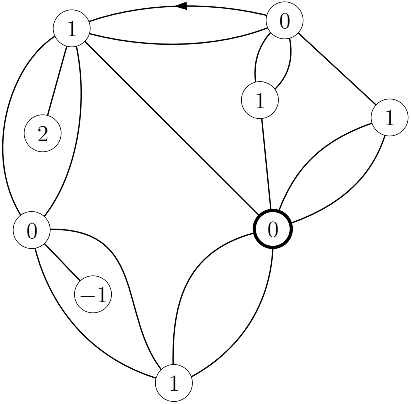
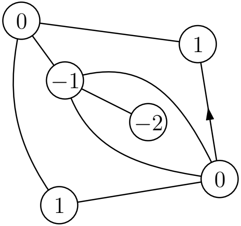
Contract the highlighted map to a single vertex (labelled 0). The new vertex may be adjacent to digons.





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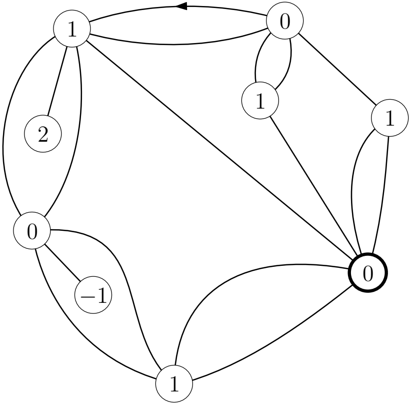
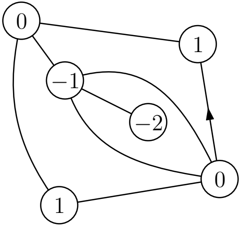
DECOMPOSITION OF D-PATCHES

Merge the new vertex with the root vertex.





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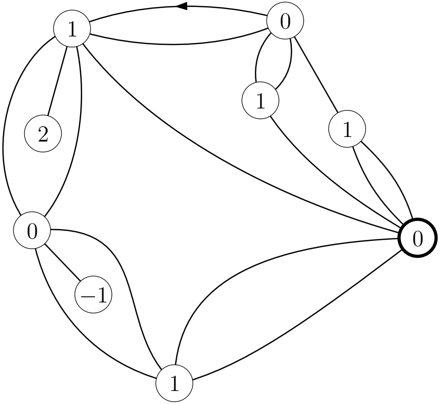

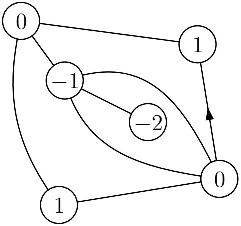

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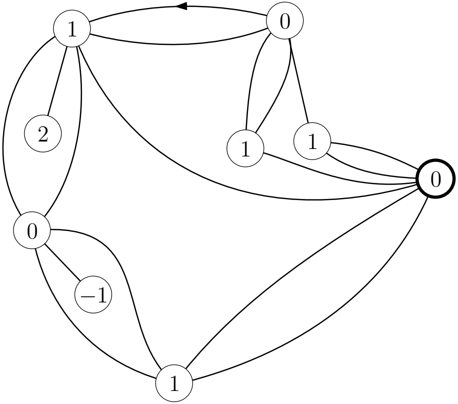

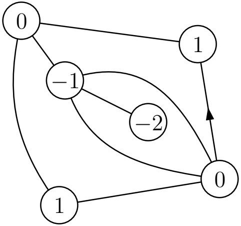

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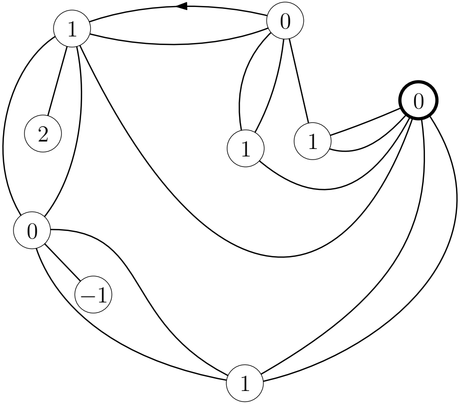

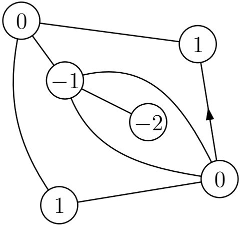

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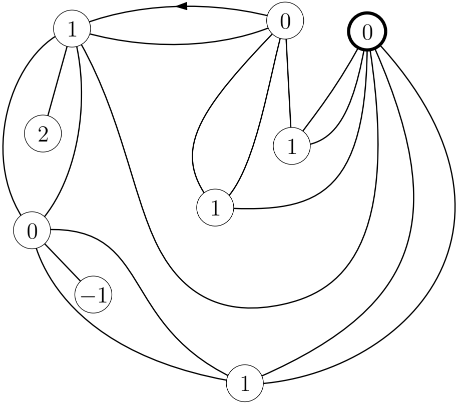
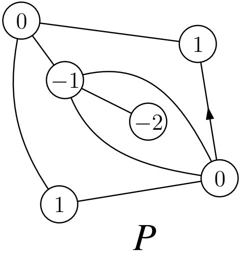
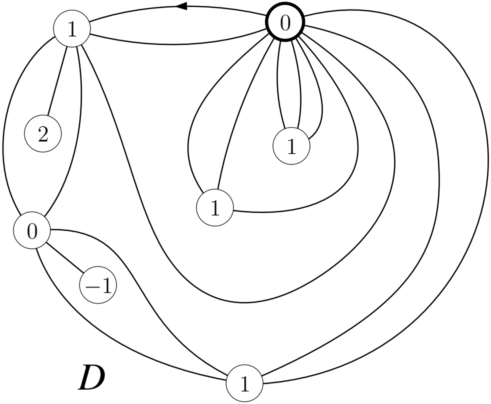
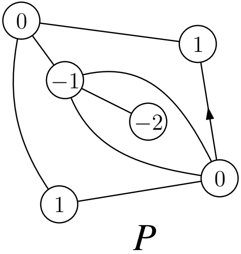




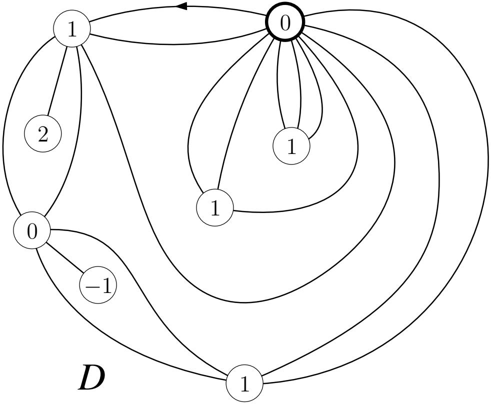
DECOMPOSITION OF D-PATCHES

Merge the new vertex with the root vertex.

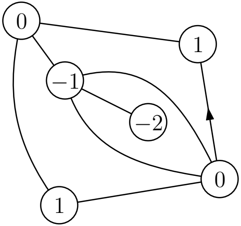




Merge the new vertex with the root vertex. This new map is a D-patch!



D



P

SIMPLIFYING EQUATIONS

Equations:

$$\mathsf{D}(x,y) = v + \frac{y}{v}\mathsf{D}(x,y)[z^1]\mathsf{D}(x,z) + y[x^{\geq 0}]\left(\frac{1}{x}\mathsf{D}(x,y)\mathsf{P}\left(\frac{t}{x}\right)\right)$$

$$[y^{>0}](1-x)\mathsf{D}(x,y) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\mathsf{E}(x,y) = \mathsf{E}(y,x) = \frac{1}{v}[x^{\geq 0}]\left(\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}(x)\right)$$

Equations:

$$\mathsf{D}\left(\frac{t}{x},y\right)=v+\frac{y}{v}\mathsf{D}\left(\frac{t}{x},y\right)[z^1]\mathsf{D}\left(\frac{t}{x},z\right)+y[x^{\leq 0}]\left(\frac{x}{t}\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}\left(x\right)\right)$$

$$[y^{>0}](1-x)\mathsf{D}(x,y)=[y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y)+y-vy+\omega\frac{t}{y}+\frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\mathsf{E}(x,y)=\mathsf{E}(y,x)=\frac{1}{v}[x^{\geq 0}]\left(\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}(x)\right)$$

SIMPLIFYING EQUATIONS

Equations:

$$\frac{t}{x}\mathsf{D}\left(\frac{t}{x},y\right)=\frac{tv}{x}+\frac{ty}{vx}\mathsf{D}\left(\frac{t}{x},y\right)[z^1]\mathsf{D}\left(\frac{t}{x},z\right)+y[x^{<0}]\left(\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}(x)\right)$$

$$[y^{>0}](1-x)\mathsf{D}(x,y)=[y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y)+y-vy+\omega\frac{t}{y}+\frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

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Equations:

$$\frac{t}{x}\mathsf{D}\left(\frac{t}{x},y\right)=\frac{tv}{x}+\frac{ty}{vx}\mathsf{D}\left(\frac{t}{x},y\right)[z^1]\mathsf{D}\left(\frac{t}{x},z\right)+y[x^{<0}]\left(\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}(x)\right)$$

$$[y^{>0}](1-x)\mathsf{D}(x,y)=[y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y)+y-vy+\omega\frac{t}{y}+\frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\mathsf{E}(x,y)=\mathsf{E}(y,x)=\frac{1}{v}[x^{\geq 0}]\left(\mathsf{D}\left(\frac{t}{x},y\right)\mathsf{P}(x)\right)$$

$$\frac{t}{xy}\mathsf{D}\left(\frac{t}{x},y\right)+v\mathsf{E}(x,y)=\frac{tv}{xy}+\left(\mathsf{D}\left(\frac{t}{x},y\right)\left(\frac{t}{xv}[z^1]\mathsf{D}\left(\frac{t}{x},z\right)+\mathsf{P}(x)\right)\right)$$

Equations:

$$[y^{>0}](1-x)\mathsf{D}(x,y) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\frac{t}{xy}\mathsf{D}\left(\frac{t}{x},y\right) + v\mathsf{E}\left(x,y\right) = \frac{tv}{xy} + \left(\mathsf{D}\left(\frac{t}{x},y\right)\left(\frac{t}{xv}[z^1]\mathsf{D}\left(\frac{t}{x},z\right) + \mathsf{P}\left(x\right)\right)\right)$$

SIMPLIFYING EQUATIONS

Equations:

$$[y^{>0}](1-x)\mathsf{D}(x,y) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\frac{t}{xy}\mathsf{D}\left(\frac{t}{x},y\right) + v\mathsf{E}\left(x,y\right) = \frac{tv}{xy} + \left(\mathsf{D}\left(\frac{t}{x},y\right)\left(\frac{t}{xv}[z^1]\mathsf{D}\left(\frac{t}{x},z\right) + \mathsf{P}\left(x\right)\right)\right)$$

Define $\mathcal{M}(x) \in \frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x) = \frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right) + \frac{t}{v}[z^1]\mathsf{D}(x,z),$$

that is

$$\mathcal{M}\left(\frac{t}{x}\right) = x\mathsf{P}\left(x\right) + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{x},z\right).$$

Equations:

$$[y^{>0}](1-x)\mathsf{D}(x,y) = [y^{>0}]\mathsf{D}(x,y)\left(y\mathsf{P}(y) + y - vy + \omega\frac{t}{y} + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{y},z\right)\right)$$

$$\frac{t}{xy}\mathsf{D}\left(\frac{t}{x},y\right) + v\mathsf{E}\left(x,y\right) = \frac{tv}{xy} + \mathsf{D}\left(\frac{t}{x},y\right)\frac{1}{x}\mathcal{M}\left(\frac{t}{x}\right)$$

Define $\mathcal{M}(x) \in \frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x) = \frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right) + \frac{t}{v}[z^1]\mathsf{D}(x,z),$$

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$$\mathcal{M}\left(\frac{t}{x}\right) = x\mathsf{P}\left(x\right) + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{x},z\right).$$

Equations:

$$[y^{>0}](1-x)\mathsf{D}(x,y) = [y^{>0}]\mathsf{D}(x,y)\left(\mathcal{M}\left(\frac{t}{y}\right) + y - vy + \omega\frac{t}{y}\right)$$

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SIMPLIFYING EQUATIONS

Equations:

$$0 = [y^{>0}]\mathsf{D}(x,y)\left(-1 + x + \mathcal{M}\left(\frac{t}{y}\right) + y - vy + \omega\frac{t}{y}\right)$$

$$v\mathsf{E}\left(x,y\right) = \frac{tv}{xy} + \mathsf{D}\left(\frac{t}{x},y\right)\left(\frac{1}{x}\mathcal{M}\left(\frac{t}{x}\right) - \frac{t}{xy}\right)$$

Define $\mathcal{M}(x) \in \frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x) = \frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right) + \frac{t}{v}[z^1]\mathsf{D}(x,z),$$

that is

$$\mathcal{M}\left(\frac{t}{x}\right) = x\mathsf{P}\left(x\right) + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{x},z\right).$$

Equations:

$$0 = [y^{<0}]\mathsf{D}\left(x, \frac{t}{y}\right)\left(-1 + x + \mathcal{M}\left(y\right) + \frac{t}{y} - \frac{vt}{y} + \omega y\right)$$

$$v - \frac{vt}{xy}\mathsf{E}\left(\frac{t}{x}, \frac{t}{y}\right) = \mathsf{D}\left(x, \frac{t}{y}\right)\left(1 - \frac{1}{y}\mathcal{M}\left(x\right)\right)$$

Define $\mathcal{M}(x) \in \frac{t}{x}\mathbb{Z}[\omega, v][[\frac{t}{x}, x]]$ by

$$\mathcal{M}(x) = \frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right) + \frac{t}{v}[z^1]\mathsf{D}(x, z),$$

that is

$$\mathcal{M}\left(\frac{t}{x}\right) = x\mathsf{P}\left(x\right) + \frac{t}{v}[z^1]\mathsf{D}\left(\frac{t}{x}, z\right).$$

Equations:

$$0=[y^{<0}]\mathsf{D}\left(x,\frac{t}{y}\right)\left(-1+x+\mathcal{M}\left(y\right)+\frac{t}{y}-\frac{vt}{y}+\omega y\right)$$

$$v-\frac{vt}{xy}\mathsf{E}\left(\frac{t}{x},\frac{t}{y}\right)=\mathsf{D}\left(x,\frac{t}{y}\right)\left(1-\frac{1}{y}\mathcal{M}\left(x\right)\right)$$

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$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

Equations:

$$\mathsf{D}\left(x,\frac{t}{y}\right)\left(x-1+\mathcal{M}\left(y\right)+\frac{(1-v)t}{y}+\omega y\right)\in K[[y]]$$

$$v-\frac{vt}{xy}\mathsf{E}\left(\frac{t}{x},\frac{t}{y}\right)=\mathsf{D}\left(x,\frac{t}{y}\right)\left(1-\frac{1}{y}\mathcal{M}\left(x\right)\right)$$

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$$\mathsf{D}\left(y,\frac{t}{x}\right)\left(1-\frac{1}{x}\mathcal{M}(y)\right)=\mathsf{D}\left(x,\frac{t}{y}\right)\left(1-\frac{1}{y}\mathcal{M}(x)\right)$$

Define $\mathcal{M}(x)\in\frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

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$$y\mathsf{D}\left(y,\frac{t}{x}\right)\left(x-\mathcal{M}\left(y\right)\right)=x\mathsf{D}\left(x,\frac{t}{y}\right)\left(y-\mathcal{M}\left(x\right)\right)$$

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Equations:

$$\mathsf{D}\left(x,\frac{t}{y}\right)\left(x-1+\mathcal{M}(y)+\frac{(1-v)t}{y}+\omega y\right)\in K[[y]]$$

$$y\mathsf{D}\left(y,\frac{t}{x}\right)(x-\mathcal{M}(y))=x\mathsf{D}\left(x,\frac{t}{y}\right)(y-\mathcal{M}(x))$$

$$\mathcal{M}(x)\mathsf{D}\left(\mathcal{M}(x),\frac{t}{x}\right)(x-\mathcal{M}(\mathcal{M}(x)))=0$$

Define $\mathcal{M}(x)\in\frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

Equations:

$$\mathsf{D}\left(x,\frac{t}{y}\right)\left(x-1+\mathcal{M}\left(y\right)+\frac{(1-v)t}{y}+\omega y\right)\in K[[y]]$$

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$$\mathcal{M}\left(\mathcal{M}(x)\right)=x$$

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$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

SIMPLIFYING EQUATIONS

Equations:

$$\mathsf{D}\left(x,\frac{t}{y}\right)\left(x-1+\mathcal{M}\left(y\right)+\frac{(1-v)t}{y}+\omega y\right)\in K[[y]]$$

$$y\left(x-\mathcal{M}\left(y\right)\right)/\mathsf{D}\left(x,\frac{t}{y}\right)=x\left(y-\mathcal{M}\left(x\right)\right)/\mathsf{D}\left(y,\frac{t}{x}\right)\in K[[y]]$$

$$\mathcal{M}\left(\mathcal{M}(x)\right)=x$$

Define $\mathcal{M}(x)\in\frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

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Equations:

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SIMPLIFYING EQUATIONS

Equations:

$$y\left(x-\mathcal{M}\left(y\right)\right)\left(x-1+\mathcal{M}\left(y\right)+\frac{(1-v)t}{y}+\omega y\right)\in K[[y]]$$

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$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

Equations:

$$x^2y-x\left(y-(1-v)t-\omega y^2\right)+y\mathcal{M}\left(y\right)\left(1-\mathcal{M}\left(y\right)-\frac{(1-v)t}{y}-\omega y\right)\in K[[y]]$$

$$\mathcal{M}\left(\mathcal{M}(x)\right)=x$$

Define $\mathcal{M}(x)\in\frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

Equations:

$$y\mathcal{M}(y)\left(1-\mathcal{M}(y)-\frac{(1-v)t}{y}-\omega y\right)\in K[[y]]$$

$$\mathcal{M}(\mathcal{M}(x))=x$$

Define $\mathcal{M}(x)\in\frac{t}{x}\mathbb{Z}[\omega,v][[\frac{t}{x},x]]$ by

$$\mathcal{M}(x)=\frac{t}{x}\mathsf{P}\left(\frac{t}{x}\right)+\frac{t}{v}[z^1]\mathsf{D}(x,z),$$

Characterisation of $\mathcal{M}(x)$

Theorem: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(x)) = x,$$

The series $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q}(t, \omega, v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

Characterisation of $\mathcal{M}(x)$

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Next section: Solution for $\omega = 0, 1$
Following section: Solution for $v = 1$
Still open: General solution

Part 2: Solution for $\omega = 0, 1$

(Eulerian (partial) orientations by edges and vertices).

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}\left(y\right)\left(1 - \mathcal{M}\left(y\right) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}\left(\mathcal{M}(y)\right) = y,$$

The series $\mathsf{Q}(t, \omega, v)$ is given by

$$\mathsf{Q}(t, \omega, v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y}\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

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SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}\left(y\right)\left(1 - \mathcal{M}\left(y\right) - \frac{(1-v)t}{y}\right) \in K[[y]],$$

$$\mathcal{M}\left(\mathcal{M}(y)\right) = y,$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$(1 - \mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$R(y) := (1-y)(1-\mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$
$$R(y) := (1-y)(1-\mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

So, $R(y) \in K[[y]]$ satisfies $R(\mathcal{M}(y)) = R(y) \in K[[y]]$, which is only possible if $R(y)$ doesn't depend on y.

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$(1-y)(1-\mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) = R \in t\mathbb{Z}[v][[t]],$$

$$\mathcal{M}(\mathcal{M}(y)) = y,$$

So, $R(y) \in K[[y]]$ satisfies $R(\mathcal{M}(y)) = R(y) \in K[[y]]$, which is only possible if $R(y)$ doesn't depend on y.

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$(1-y)(1-\mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) = R \in t\mathbb{Z}[v][[t]],$$
$$\mathcal{M}(\mathcal{M}(y)) = y,$$

Solution for $\mathcal{M}(y)$:

$$M(y) = \frac{y + t(v-1)}{2y}\left(1 - \sqrt{1 - 4y\frac{t(v-1) + R/(1-y)}{(y+t(v-1))^2}}\right)$$
$$= \frac{tv - t}{y} + \sum_{n,k,j\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{n+j}{n} t^k (v-1)^k R^{n+1} y^{j-n-k-1}$$

SOLUTION FOR $\omega = 0$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y) - y\mathcal{M}(y)^2 - (1-v)t\mathcal{M}(y) \in K[[y]],$$

$$(1-y)(1-\mathcal{M}(y))(y\mathcal{M}(y) - t(v-1)) = R \in t\mathbb{Z}[v][[t]],$$
$$\mathcal{M}(\mathcal{M}(y)) = y,$$

Solution for $\mathcal{M}(y)$:

$$M(y) = \frac{y + t(v-1)}{2y}\left(1 - \sqrt{1 - 4y\frac{t(v-1) + R/(1-y)}{(y + t(v-1))^2}}\right)$$
$$= \frac{tv - t}{y} + \sum_{n,k,j \geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{n+j}{n} t^k (v-1)^k R^{n+1} y^{j-n-k-1}$$
$$tv = [y^{-1}]\mathcal{M}(y) = \sum_{n,k \geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}$$

Solution for $\mathcal{M}(y)$:

$$M(y) = \frac{y + t(v-1)}{2y}\left(1 - \sqrt{1 - 4y\frac{t(v-1) + R/(1-y)}{(y + t(v-1))^2}}\right)$$

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$$tv = [y^{-1}]\mathcal{M}(y) = \sum_{n,k \geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}$$

Solution for $\mathcal{M}(y)$:

$$M(y) = \frac{y + t(v-1)}{2y}\left(1 - \sqrt{1 - 4y\frac{t(v-1) + R/(1-y)}{(y+t(v-1))^2}}\right)$$

$$= \frac{tv - t}{y} + \sum_{n,k,j\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{n+j}{n} t^k (v-1)^k R^{n+1} y^{j-n-k-1}$$

$$tv = [y^{-1}]\mathcal{M}(y) = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}$$

Solution for generating function Q:

$$\mathsf{Q}(t,0,v) = t^{-2}[y^{-2}]\mathcal{M}(y) - v.$$

$$= -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

Solution for $\omega = 0$

Theorem: Let $R(t,v) \in \mathbb{Z}[v][[t]]$ be the unique series with constant term 0 satisfying

$$tv = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t,0,v)$ for height-labelled quadrangulations (with no alternating faces) counted by faces and local minima is given by

$$\mathsf{Q}(t,0,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

Corollary: $\mathsf{Q}(t,0,v)$ and $R(t,v)$ D-algebraic in t, v.

Solution for $\omega = 0$

Theorem: Let $R(t,v) \in \mathbb{Z}[v][[t]]$ be the unique series with constant term 0 satisfying

$$tv = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t,0,v)$ for height-labelled maps counted by edges and faces is given by

$$\mathsf{Q}(t,0,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

Corollary: $\mathsf{Q}(t,0,v)$ and $R(t,v)$ D-algebraic in t, v.

Solution for $\omega = 0$

Theorem: Let $R(t,v) \in \mathbb{Z}[v][[t]]$ be the unique series with constant term 0 satisfying

$$tv = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k}{n} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t,0,v)$ for Eulerian orientations counted by edges and vertices is given by

$$\mathsf{Q}(t,0,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{2n+k-1}{n} t^k (v-1)^k R^{n+1}.$$

Corollary: $\mathsf{Q}(t,0,v)$ and $R(t,v)$ D-algebraic in t, v.

Solution for $\omega = 1$

Theorem: Let $R(t, v) \in \mathbb{Z}[v][[t]]$ be the unique series with constant term 0 satisfying (in some domain)

$$t = \sum_{n,k \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{2n+k}{k} \binom{3n+2k}{n+k} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t, 1, v)$ for height-labelled quadrangulations counted by faces and local minima is given by

$$\mathsf{Q}(t, 1, v) = -v + \frac{1}{t^2} \sum_{n,k} \frac{1}{n+1} \binom{2n}{n} \binom{2n+k}{k} \binom{3n+2k-1}{2n+k} t^k (v-1)^k \mathsf{R}_1^{n+1}.$$

Corollary: $\mathsf{Q}(t, 1, v)$ and $R(t, v)$ D-algebraic in t, v.

SOLUTION FOR $\omega = 1$

Theorem: Let $R(t,v) \in \mathbb{Z}[v][[t]]$ be the unique series with constant term 0 satisfying (in some domain)

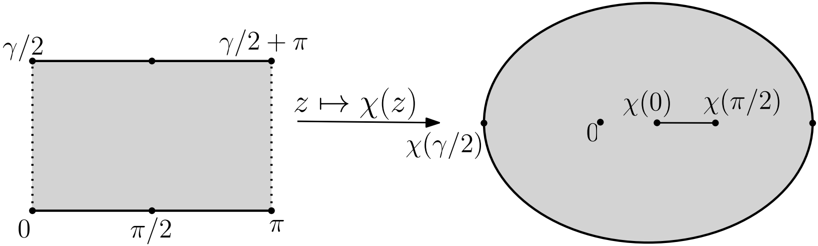
$$t = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k}{n+k} t^k (v-1)^k R^{n+1}.$$

The generating function $\mathsf{Q}(t,1,v)$ for Eulerian partial orientations counted by edges and vertices is given by

$$\mathsf{Q}(t,1,v) = -v + \frac{1}{t^2}\sum_{n,k} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k-1}{2n+k} t^k (v-1)^k \mathsf{R}_1^{n+1}.$$

Corollary: $\mathsf{Q}(t,1,v)$ and $R(t,v)$ D-algebraic in t, v.

Part 3: Analytic functional equations



Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}\left(y\right)\left(1 - \mathcal{M}\left(y\right) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}\left(\mathcal{M}(x)\right) = x,$$

Claim: For sufficiently small t, there is an even meromorphic function χ on $\mathbb{C}$ and some $\gamma \in i\mathbb{R}_{>0}$ satisfying

$$\mathcal{M}(\chi(z)) = \chi(\gamma - z),$$

and

$$1 + \frac{t(v-1)}{\chi(z)} = \chi(\gamma + z) + \omega\chi(z) + \chi(z - \gamma).$$

Recall: There is a unique series $\mathcal{M}(y) \in \frac{t}{y}\mathbb{Z}[\omega, v][[y, t/y]]$ with $[y^{-1}]\mathcal{M}(y) = tv$ satisfying

$$y\mathcal{M}(y)\left(1 - \mathcal{M}(y) - \frac{(1-v)t}{y} - \omega y\right) \in K[[y]],$$

$$\mathcal{M}(\mathcal{M}(x)) = x,$$

Claim: For sufficiently small t, there is an even meromorphic function χ on $\mathbb{C}$ and some $\gamma \in i\mathbb{R}_{>0}$ satisfying

$$\mathcal{M}(\chi(z)) = \chi(\gamma - z),$$

and

$$1 + \frac{t(v-1)}{\chi(z)} = \chi(\gamma + z) + \omega\chi(z) + \chi(z - \gamma).$$

Last section: Solved for $\omega = 0, 1$.
Next section: Solution for $v = 1$.
Still open: All other values ω, v.

Part 4: Six vertex model $(\nu = 1)$

(Previous solution: Kostov (2000)/EP and Zinn-Justin (2019)).

Recall: Solutions at $\omega = 0, 1$

The generating function $\mathsf{Q}(t, 0, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t, 0, 1) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t, 1, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t, 1, 1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

SOLUTION FOR $\mathsf{Q}(t, \omega, 1)$

Define

$$\vartheta(z,q) = \sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz} - e^{-(2n+1)iz})q^{(2n+1)^2/8}.$$

Let $q = q(t,\alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

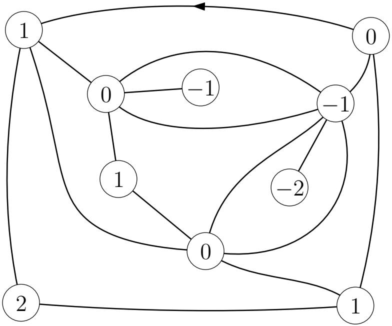
$$\mathsf{Q}(t,\gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t,\gamma)\right).$$

Thank you!

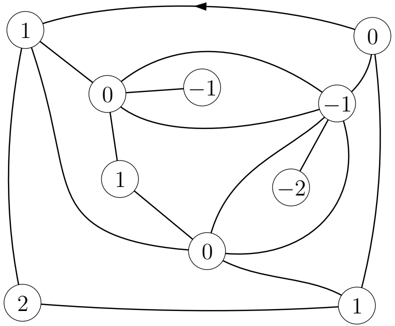
Bijection 1: height-labelled quadrangulations to weakly height-labelled maps

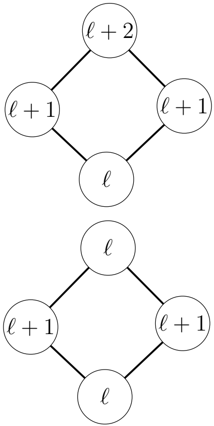
(Miermont (2009)/Ambjørn and Budd (2013)).

Start with a height-labelled quadrangulation.

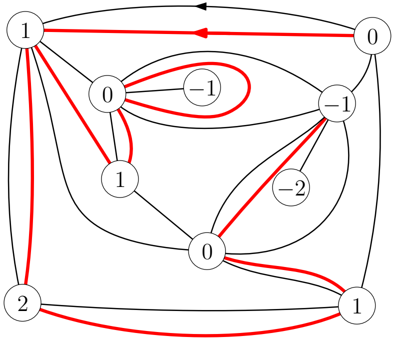


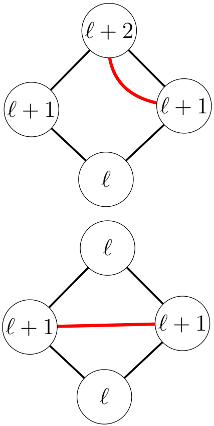
Start with a height-labelled quadrangulation.



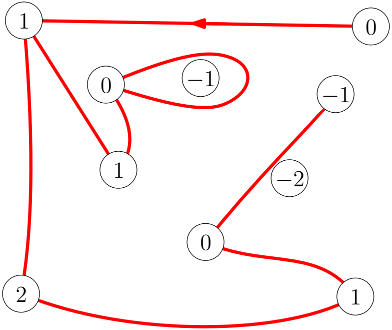


Draw a red edge in each face according to the rule.



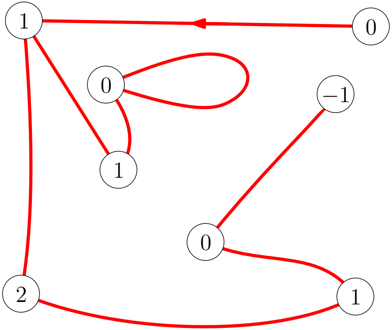


Remove all of the original edges.



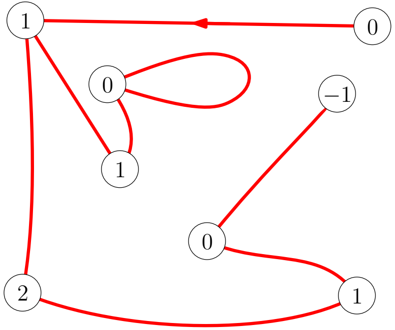
H-QUADRANGULATIONS TO H-MAPS

Remove any isolated vertices.



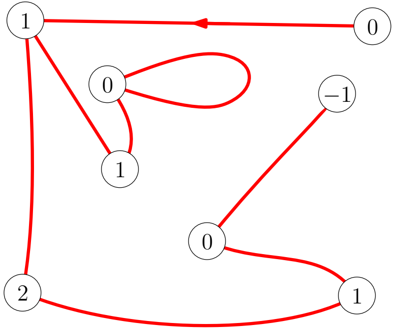
H-QUADRANGULATIONS TO H-MAPS

The new map is a weakly height-labelled map (adjacent labels differ by *at most* 1).



H-QUADRANGULATIONS TO H-MAPS

The new map is a weakly height-labelled map (adjacent labels differ by *at most* 1).



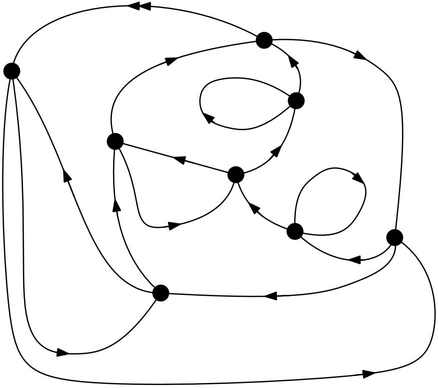
These are counted by edges (t), mono-value edges (ω) and faces (v).

Bijection 2: H-maps to Eulerian orientations (EO-maps)

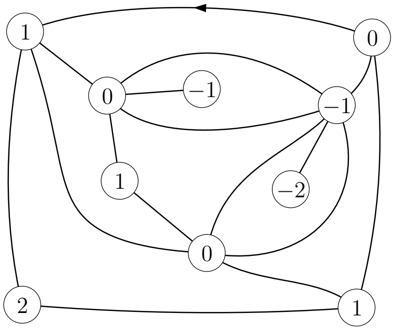
Same Bijection: H-quads to EO-quarts

(EP and Gutmann (2018)).

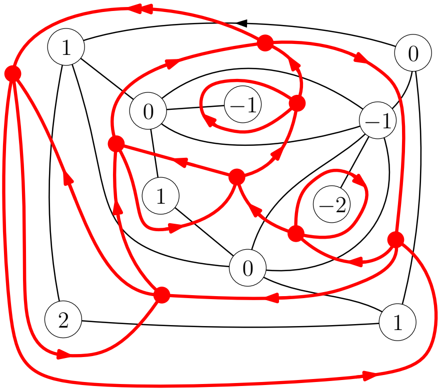
EO-quarts: each vertex has two incoming and two outgoing edges. Counted by vertices (t), alternating vertices (ω) and clockwise faces (v)

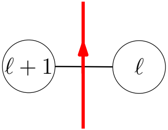


Start with a height-labelled quadrangulation.



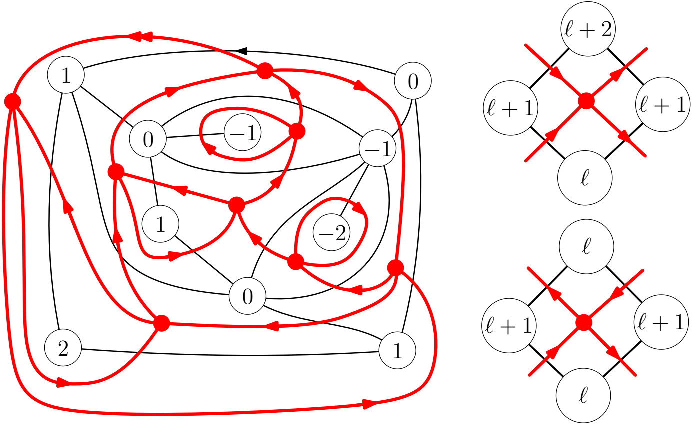
Draw the dual with edges oriented according to the rule.





Bijection to the ice model

Each red vertex has two incoming and two outgoing edges.



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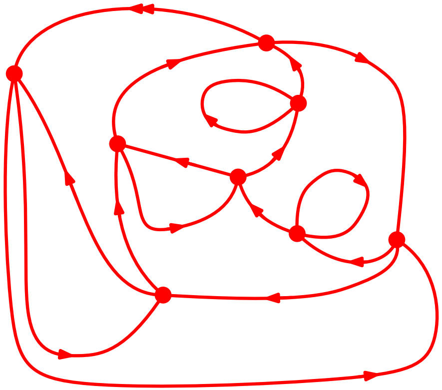
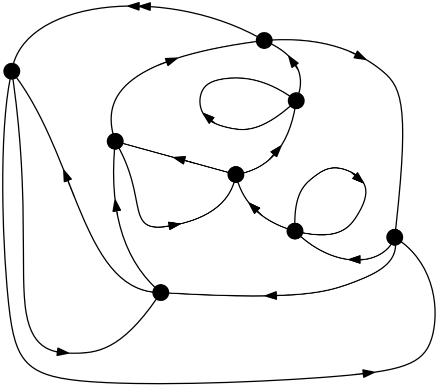
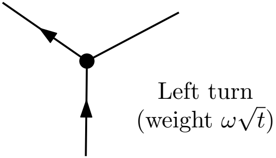
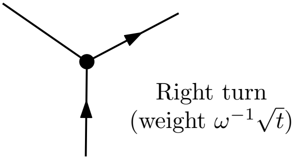


Each vertex has two incoming and two outgoing edges.

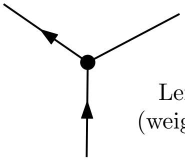


Let $\mathsf{C}(t,\omega)$ be the generating function for partially oriented cubic maps in which each vertex is one of the following types.

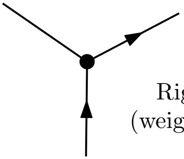




Let $\mathsf{C}(t,\omega)$ be the generating function for partially oriented cubic maps in which each vertex is one of the following types.



Left turn
(weight $\omega\sqrt{t}$)



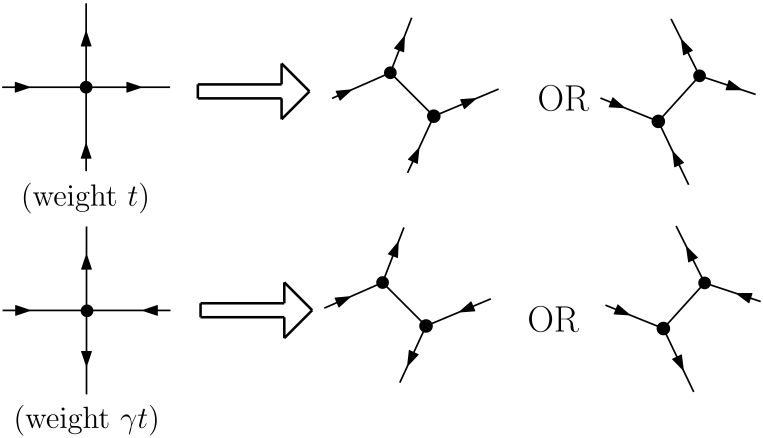
Right turn
(weight $\omega^{-1}\sqrt{t}$)

Theorem: $\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{C}(t,\omega)$.

Bijection 3: A loop model

(Kostov (2000)).

Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$



BONUS SLIDE: BIJECTION TO A LOOP MODEL

Theorem: $\mathsf{Q}(t, \omega^2 + \omega^{-2}) = \mathsf{C}(t, \omega)$